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NONTRIVIAL SOLUTIONS FOR NONLINEAR ALGEBRAIC SYSTEMS VIA A LOCAL MINIMUM THEOREM FOR FUNCTIONALS

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Abstract. In this article, we use a critical point theorem (local minimum result) for differentiable functionals to prove the existence of at least one nontrivial solution for a nonlinear algebraic system with a parameter. Our goal is achieved by requiring an appropriate asymptotic behavior of the nonlinear term at zero. Some applications to discrete equations are also presented.

1. Introduction

In this article we study the nonlinear algebraic system

$$Au = \lambda f(u), \tag{1.1}$$

where $u = (u_1, \dots, u_n)^t \in \mathbb{R}^n$ is a column vector in $\mathbb{R}^n$, $A = (a_{ij})_{n\times n}$ is a given positive definite matrix, $f(u) := (f_1(u_1), \dots, f_n(u_n))^t$, with $f_k : \mathbb{R} \to \mathbb{R}$ is a continuous function for every $k \in \mathbb{Z}[1, n] := \{1, \dots, n\}$, and λ is a positive parameter.

Discrete problems involving functions with two or more discrete variables are very relevant and have been deeply investigated. Such great interest is undoubtedly due to the advance of modern digital computing devices.

Indeed, since these relations can be simulated in a relatively easy manner by means of such devices and since such simulations often reveal important information about the behavior of complex systems, a large number of recent investigations related to image processing, population models, neural networks, social behaviors, digital control systems, are described in terms of such functional relations.

Moreover, a large number of problems can be formulated as special cases of the nonlinear algebraic system (1.1). For a survey on these topics we cite the recent paper [21]. A similar approach has also been used in others works (see for instance, the papers [17, 18, 19] and [20, 22, 23]).

Here, motivated by the interest on the subject, by using variational methods in finite dimensional setting and a local minimum theorem for differentiable functionals due to Ricceri [15], we prove the existence of at least one nontrivial solution for (1.1).

We also emphasize that if the functions f_k are nonnegative, for every $k \in \mathbb{Z}[1, n]$, our results guarantee a positive solution (see Remark 3.2 for more details). For

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instance, we can assume that A has the tridiagonal form

$$\operatorname{trid}_n(-1,2,-1) := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{n\times n}.$$

A direct application of our result to second-order discrete equations reads as follows.

Theorem 1.1. *Let $f(u) = (f_1(u_1), \dots, f_n(u_n))^t$, with $f_k : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function such that $f_k(0) = 0$, for every $k \in \mathbb{Z}[1, n]$. Assume also that*

$$\lim_{s\to 0^+} \frac{f_k(s)}{s} = +\infty, \quad \forall k \in \mathbb{Z}[1, n].$$

Then, there exists an open interval $\Lambda \subseteq (0, +\infty)$ such that for each parameter $\lambda \in \Lambda$, the problem

$$\begin{gathered} -\Delta^2 u_{k-1} = \lambda f_k(u_k), \quad \forall k \in \mathbb{Z}[1, n] \\ u_0 = u_{n+1} = 0, \end{gathered} \tag{1.2}$$

admits at least one positive solution u^λ. Moreover, the real function

$$\lambda \mapsto \frac{(u^\lambda)^t \operatorname{trid}_n(-1, 2, -1)u^\lambda}{2} - \lambda \sum_{k=1}^{n} \int_0^{u_k^\lambda} f_k(s)\, ds$$

is negative and strictly decreasing on the set Λ.

For completeness, we mention the recent papers [5, 10, 11, 12, 13] where existence and multiplicity of solutions for non-linear discrete problems were studied by using variational arguments. For a complete and exhaustive overview of variational methods we refer the reader to the monographs [1, 8, 14].

The plan of the paper is as follows. In Section 2 we introduce some basic notations. In Section 3 we obtain our existence result (see Theorem 3.1). Finally, applications to discrete equations involving certain tridiagonal matrices and fourth-order discrete equations are presented.

2. Preliminaries

We shall prove our results applying the following version of Ricceri's variational principle [15, Theorem 2.1].

Theorem 2.1. *Let X be a reflexive real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two Gâteaux differentiable functionals such that Φ is strongly continuous, sequentially weakly lower semicontinuous and coercive in X and Ψ is sequentially weakly upper semicontinuous in X. Let J_λ be the functional defined as $J_\lambda := \Phi - \lambda\Psi$, $\lambda \in \mathbb{R}$, and for any $r > \inf_X \Phi$ let φ be the function defined as*

$$\varphi(r) := \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)}.$$

Then, for any $r > \inf_X \Phi$ and any $\lambda \in (0, 1/\varphi(r))$, the restriction of the functional J_λ to $\Phi^{-1}((-\infty, r))$ admits a global minimum, which is a critical point (precisely a local minimum) of J_λ in X.

As the ambient space X, we consider the n-dimensional Banach space $\mathbb{R}^n$ endowed by the norm
$$\|u\|_2 := \Big(\sum_{k=1}^{n} u_k^2\Big)^{1/2}.$$
More generally, we set
$$\|u\|_r := \Big(\sum_{k=1}^{n} |u_k|^r\Big)^{1/r}, \quad (r \geq 1)$$
for every $u \in X$.

Let $\mathfrak{X}_n$ denote the class of all symmetric and positive definite matrices of order n. Further, we denote by $\lambda_1, \dots, \lambda_n$ the eigenvalues of A ordered as follows $0 < \lambda_1 \leq \dots \leq \lambda_n$.

It is well-known that if $A \in \mathfrak{X}_n$, for every $u \in X$, then one has
$$\lambda_1 \|u\|_2^2 \leq u^t A u \leq \lambda_n \|u\|_2^2, \tag{2.1}$$
$$\|u\|_\infty \leq \frac{1}{\sqrt{\lambda_1}} (u^t A u)^{1/2}, \tag{2.2}$$
where $\|u\|_\infty := \max_{k\in\mathbb{Z}[1,n]} |u_k|$.

For the rest of this article, we assume that $A \in \mathfrak{X}_n$. Set
$$\Phi(u) := \frac{u^t A u}{2}, \quad \Psi(u) := \sum_{k=1}^{n} F_k(u_k), \quad J_\lambda(u) := \Phi(u) - \lambda\Psi(u), \tag{2.3}$$
for $u \in X$, where $F_k(t) := \int_0^t f_k(s)\,ds$, for $(k,t) \in \mathbb{Z}[1,n] \times \mathbb{R}$.

Standard arguments show that $J_\lambda \in C^1(X,\mathbb{R})$ as well as that the critical points of J_λ are exactly the solutions of problem (1.1).

Indeed, a column vector $\overline{u} = (\overline{u}_1, \dots, \overline{u}_n)^t \in X$ is a critical point of the functional J_λ if the gradient of J_λ at $\overline{u}$ is zero, i.e.,
$$\frac{\partial J_\lambda(u)}{\partial u_1}\Big|_{u=\overline{u}} = 0, \ \frac{\partial J_\lambda(u)}{\partial u_2}\Big|_{u=\overline{u}} = 0, \dots, \frac{\partial J_\lambda(u)}{\partial u_n}\Big|_{u=\overline{u}} = 0.$$
Moreover, for every $k \in \mathbb{Z}[1,n]$, one has that
$$\frac{\partial u^t A u}{\partial u_k} = 2(Au)_k,$$
where $(Au)_k := \sum_{j=1}^{n} a_{kj} u_j$. Thus
$$\frac{\partial J_\lambda(u)}{\partial u_k} = (Au)_k - \lambda f_k(u_k), \quad \forall k \in \mathbb{Z}[1,n]$$
which yields our assertion.

3. Main Results

In this section we prove our existence result that reads as follows.

Theorem 3.1. *Let $f(u) = (f_1(u_1), \dots, f_n(u_n))^t$, with $f_k : \mathbb{R} \to \mathbb{R}$ be a continuous function for every $k \in \mathbb{Z}[1,n]$. In addition, if $f_k(0) = 0$ for every $k \in \mathbb{Z}[1,n]$, assume also that*
$$\lim_{s\to 0^+} \frac{F_k(s)}{s^2} = +\infty, \quad \forall k \in \mathbb{Z}[1,n].$$

Then, there exists an open interval $\Lambda \subseteq (0, +\infty)$ such that for each parameter $\lambda \in \Lambda$, problem (1.1) admits at least one nontrivial solution $u^\lambda \in X$. Moreover, the real function

$$\lambda \mapsto J_\lambda(u^\lambda) \tag{3.1}$$

is negative and strictly decreasing on Λ.

Proof. Our aim is to apply Theorem 2.1 to problem (1.1). To this end, let $X := \mathbb{R}^n$, and consider the functionals Φ and Ψ defined in (2.3). Note that $J_\lambda := \Phi - \lambda\Psi$. From (2.1) we know that the functional Φ is coercive. Also, Φ and Ψ satisfy all regularity assumptions in Theorem 2.1, because X is finite dimensional.

Let $\bar{c} > 0$ and set

$$r := \frac{\lambda_1}{2}\bar{c}^2.$$

Then, for all $u \in X$ with $\Phi(u) < r$, taking (2.2) into account one has $\|u\|_\infty \leq \bar{c}$. Hence,

$$\Psi(u) = \sum_{k=1}^{n} F_k(u_k) \leq \sum_{k=1}^{n} \max_{|\xi|\leq\bar{c}} F_k(\xi),$$

for every $u \in X$ such that $\Phi(u) < r$. Then

$$\sup_{\Phi(u)<r} \Psi(u) \leq \sum_{k=1}^{n} \max_{|\xi|\leq\bar{c}} F_k(\xi).$$

Taking into account the above computations, one has

$$\begin{aligned}
\varphi(r) &= \inf_{u\in\Phi^{-1}((-\infty,r))} \frac{\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v) - \Psi(u)}{r - \Phi(u)} \\
&\leq \frac{\sup_{v\in\Phi^{-1}((-\infty,r))} \Psi(v)}{r} \\
&\leq \frac{2}{\lambda_1} \frac{\sum_{k=1}^{n} \max_{|\xi|\leq\bar{c}} F_k(\xi)}{\bar{c}^2}.
\end{aligned}$$

Hence, we put

$$\lambda^\star := \frac{\lambda_1}{2} \frac{\bar{c}^2}{\sum_{k=1}^{n} \max_{|\xi|\leq\bar{c}} F_k(\xi)} \in (0, +\infty].$$

At this point, thanks to Theorem 2.1, for every $\lambda \in (0, \lambda^\star) \subseteq (0, 1/\varphi(r))$, the functional J_λ admits at least one critical point (local minima) $u^\lambda \in \Phi^{-1}((-\infty, r))$.

Now, we prove that for any fixed $\lambda \in (0, \lambda^\star)$ the solution u^λ found above is not the trivial function. If $f_k(0) \neq 0$ for some $k \in \mathbb{Z}[1, n]$, then it easily follows that $u^\lambda \not\equiv 0_X$, since the trivial vector does not solve problem (1.1).

Let us consider the case when $f_k(0) = 0$ for every $k \in \mathbb{Z}[1, n]$. In this setting, in order to prove that $u^\lambda \not\equiv 0_X$, first we claim that there exists a sequence $\{w_j\}_{j\in\mathbb{N}}$ in X such that

$$\limsup_{j\to+\infty} \frac{\Psi(w_j)}{\Phi(w_j)} = +\infty. \tag{3.2}$$

Because of our assumptions at zero, we can fix a sequence $\{\xi_j\} \subset \mathbb{R}^+$ converging to zero and two constants σ, κ (with $\sigma > 0$) such that

$$\lim_{j\to+\infty} \frac{F_k(\xi_j)}{\xi_j^2} = +\infty,$$

$$F_k(\xi) \geq \kappa\xi^2,$$

for every $\xi \in [0, \sigma]$ and $k \in \mathbb{Z}[1, n]$.

Now, fix $1 \leq l < n$ and a vector $v = (v_1, \dots, v_n) \in X$ such that:

(i) $v_k = 1$, for every $1 \leq k \leq l$;

(ii) $v_k \in [0, 1]$, for every $l + 1 \leq k \leq n$.

Finally, let $w_j := \xi_j v$ for any $j \in \mathbb{N}$. It is easily seen that $w_j \in X$ for any $j \in \mathbb{N}$. Fix $M > 0$ and consider a real positive number η with

$$M < \frac{l\eta + \kappa \sum_{k=l+1}^{n} v_k^2}{\Phi(v)}.$$

Then there is $\nu \in \mathbb{N}$ such that $\xi_j < \sigma$ and

$$\int_0^{\xi_j} f_k(s)\, ds \geq \eta \xi_j^2,$$

for every $j > \nu$ and $k \in \mathbb{Z}[1, n]$.

Now, for every $j > \nu$, bearing in mind the properties of the vector v ($0 \leq \xi_j v_k < \sigma$ for j sufficiently large and every $k \in \mathbb{Z}[1, n]$), one has

$$\begin{aligned} \frac{\Psi(w_j)}{\Phi(w_j)} &= \frac{\sum_{k=1}^{l} \big(\int_0^{\xi_j} f_k(s)\, ds \big) + \sum_{k=l+1}^{n} F_k(\xi_j v_k)}{\xi_j^2 \Phi(v)} \\ &\geq \frac{l\eta + \kappa \sum_{k=l+1}^{n} v_k^2}{\Phi(v)} > M. \end{aligned}$$

Since M could be taken arbitrarily large, (3.2) clearly follows.

Now, note that

$$\|w_j\|_2 = |\xi_j| \|v\|_2 \to 0,$$

as $j \to +\infty$, so that for j large enough,

$$\|w_j\|_2 < \sqrt{\frac{\lambda_1}{\lambda_n}}\, \bar{c}.$$

As a consequence of this and taking into account (2.1),

$$w_j \in \Phi^{-1}((-\infty, r)), \tag{3.3}$$

provided j is large enough. Also, by (3.2) and the fact that $\lambda > 0$,

$$J_\lambda(w_j) = \Phi(w_j) - \lambda \Psi(w_j) < 0, \tag{3.4}$$

for j sufficiently large.

Since u^λ is a global minimum of the restriction of J_λ to $\Phi^{-1}((-\infty, r))$, by (3.3) and (3.4) we conclude that

$$J_\lambda(u^\lambda) \leq J_\lambda(w_j) < 0 = J_\lambda(0), \tag{3.5}$$

so that $u^\lambda \not\equiv 0_X$. Thus, u^λ is a nontrivial solution of problem (1.1). Moreover, from (3.5) we get that for every $\lambda \in (0, \lambda^\star)$ the map (3.1) is negative.

Finally, we show that the map (3.1) is strictly decreasing in $(0, \lambda^\star)$. For our goal we observe that for any $u \in X$, one has

$$J_\lambda(u) = \lambda \Big(\frac{\Phi(u)}{\lambda} - \Psi(u) \Big). \tag{3.6}$$

Now, let us fix $0 < \lambda_1 < \lambda_2 < \lambda^\star$ and let u^{λ_i} be the global minimum of the functional J_{λ_i} restricted to $\Phi\big((-\infty, r)\big)$ for $i = 1, 2$.

Also, let

$$m_{\lambda_i} := \Big(\frac{\Phi(u^{\lambda_i})}{\lambda_i} - \Psi(u^{\lambda_i})\Big) = \inf_{v\in\Phi^{-1}((-\infty,r))} \Big(\frac{\Phi(v)}{\lambda_i} - \Psi(v)\Big),$$

for $i = 1, 2$.

Clearly, (3.1) together (3.6) and the positivity of λ imply that

$$m_{\lambda_i} < 0, \quad \text{for} \quad i = 1, 2. \tag{3.7}$$

Moreover,

$$m_{\lambda_2} \leq m_{\lambda_1}, \tag{3.8}$$

thanks to the fact that $0 < \lambda_1 < \lambda_2$. Then, by (3.6)–(3.8) and again by the fact that $0 < \lambda_1 < \lambda_2$, we get that

$$J_{\lambda_2}(u^{\lambda_2}) = \lambda_2 m_{\lambda_2} \leq \lambda_2 m_{\lambda_1} < \lambda_1 m_{\lambda_1} = J_{\lambda_1}(u^{\lambda_1}),$$

so that the map $\lambda \mapsto J_\lambda(u^\lambda)$ is strictly decreasing in $\lambda \in (0, \lambda^\star)$. The arbitrariness of $\lambda < \lambda^\star$ shows that $\lambda \mapsto J_\lambda(u^\lambda)$ is strictly decreasing in $(0, \lambda^\star)$. This concludes the proof. □

Remark 3.2. A vector $\overline{u} := (\overline{u}_1, \dots, \overline{u}_n)^t \in \mathbb{R}^n$ is said to be *positive* (*nonnegative*) if $\overline{u}_k > 0$ ($\overline{u}_k \geq 0$) for every $k \in \mathbb{Z}[1, n]$. Now, let $A \in \mathfrak{X}_n$ and consider the following conditions:

(A1) If $i \neq j$, $a_{ij} \leq 0$;
(A2) for every $i \in \mathbb{Z}[2, n]$, there exists $j_i < i$ such that $a_{ij_i} < 0$.

Assuming that (A1) holds and $\overline{u} := (\overline{u}_1, \dots, \overline{u}_n)^t \in X$ is a solution of

$$\sum_{j=1}^{n} a_{ij} u_j \geq 0, \quad \forall i \in \mathbb{Z}[1, n]\,, \tag{3.9}$$

then $\overline{u}_i \geq 0$, for every $i \in \mathbb{Z}[1, n]$ (see [4, 24] and [3, Proposition 2.1]). If, (A1) and (A2) hold, then any solution of (3.9) is trivial or otherwise is positive (see [3, Proposition 2.2]). Hence, if f_k are nonnegative, for every $k \in \mathbb{Z}[1, n]$, our results guarantee the existence of two nonnegative solutions if A satisfies hypothesis (A1). Finally, if (A1) and (A2), then the obtained solutions are positive.

Here, we present some direct applications to discrete equations.

3.1. **Tridiagonal matrices.** Let $n > 1$ and $(a, b) \in \mathbb{R}^- \times \mathbb{R}^+$ be such that

$$\cos(\frac{\pi}{n+1}) < -\frac{b}{2a}.$$

Set

$$\operatorname{trid}_n(a, b, a) = \begin{pmatrix} b & a & 0 & \dots & 0 \\ a & b & a & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & a & b & a \\ 0 & \dots & 0 & a & b \end{pmatrix}_{n\times n} \in \mathfrak{X}_n.$$

Note that $\operatorname{trid}_n(a, b, a)$ is a symmetric and positive definite matrix whose first eigenvalue is given by

$$\lambda_1 = b + 2a\cos\left(\frac{\pi}{n+1}\right),$$

see, for instance, [16, Example 9; page 179]. This matrix verifies conditions (A1) and (A2). Taking into account Theorem 3.1 and Remark 3.2, we have the following theorem.

Theorem 3.3. *In addition to the assumptions of Theorem 3.1, let f_k be nonnegative, for every $k \in \mathbb{Z}[1,n]$. Then, there exists an open interval $\Lambda \subseteq (0,+\infty)$ such that for each parameter $\lambda \in \Lambda$, the problem*

$$\operatorname{trid}_n(a,b,a)u = \lambda f(u) \tag{3.10}$$

admits at least one positive solution $u^\lambda \in X$. Moreover, the real function

$$\lambda \mapsto \frac{(u^\lambda)^t \operatorname{trid}_n(a,b,a)u^\lambda}{2} - \lambda \sum_{k=1}^{n} \int_0^{u_k^\lambda} f_k(s)\,ds$$

is negative and strictly decreasing on the set Λ.

An important case is given by the matrix

$$\operatorname{trid}_n(-1,2,-1) := \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ & & \ddots & & \\ 0 & \dots & -1 & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}_{n\times n} \in \mathfrak{X}_n,$$

which is associated to the second-order discrete boundary value problem

$$\begin{gathered} -\Delta^2 u_{k-1} = \lambda f_k(u_k), \quad \forall k \in \mathbb{Z}[1,n] \\ u_0 = u_{n+1} = 0, \end{gathered} \tag{3.11}$$

where $\Delta^2 u_{k-1} := \Delta(\Delta u_{k-1})$, and, as usual, $\Delta u_{k-1} := u_k - u_{k-1}$ denotes the forward difference operator. We point out that the matrix $\operatorname{trid}_n(-1,2,-1)$ was considered in order to study the existence of nontrivial solutions of nonlinear second-order difference equations [2, 6, 7, 9].

According to the above discussion, Theorem 1.1 in the Introduction immediately follows by Theorem 3.3 and Remark 3.2.

3.2. **Fourth-order difference equations.** As it is well-known, boundary value problems involving fourth-order difference equations such as

$$\begin{gathered} \Delta^4 u_{k-2} = \lambda f_k(u_k), \quad \forall k \in \mathbb{Z}[1,n] \\ u_{-2} = u_{-1} = u_0 = 0, \\ u_{n+1} = u_{n+2} = u_{n+3} = 0, \end{gathered} \tag{3.12}$$

can also be expressed as problem (1.1), where A is the real symmetric and positive definite matrix of the form

$$A^\star := \begin{pmatrix} 6 & -4 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ -4 & 6 & -4 & 1 & \dots & 0 & 0 & 0 & 0 \\ 1 & -4 & 6 & -4 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 6 & \dots & 0 & 0 & 0 & 0 \\ & & & & \ddots & & & & \\ 0 & 0 & 0 & 0 & \dots & 6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 & \dots & -4 & 6 & -4 & 1 \\ 0 & 0 & 0 & 0 & \dots & 1 & -4 & 6 & -4 \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 & -4 & 6 \end{pmatrix} \in \mathfrak{X}_n.$$

A direct application of our result to fourth-order difference equations yields the following result.

Theorem 3.4. *Let f satisfy all the assumptions of Theorem 3.1. Then, there exists an open interval $\Lambda \subseteq (0, +\infty)$ such that for each parameter $\lambda \in \Lambda$, problem* (3.12) *admits at least one nontrivial solution $u^\lambda \in X$. Moreover, the real function*

$$\lambda \mapsto \frac{(u^\lambda)^t A^\star u^\lambda}{2} - \lambda \sum_{k=1}^{n} \int_0^{u_k^\lambda} f_k(s)\, ds$$

is negative and strictly decreasing on the set Λ.

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