Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 318, pp. 1–22.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

UPPER BOUNDS FOR THE NUMBER OF LIMIT CYCLES OF POLYNOMIAL DIFFERENTIAL SYSTEMS

SELMA ELLAGGOUNE, SABRINA BADI

Abstract. For ε small we consider the number of limit cycles of the polynomial differential system

$$\dot{x} = y - f_1(x,y)y, \quad \dot{y} = -x - g_2(x,y) - f_2(x,y)y,$$

where $f_1(x,y) = \varepsilon f_{11}(x,y) + \varepsilon^2 f_{12}(x,y)$, $g_2(x,y) = \varepsilon g_{21}(x,y) + \varepsilon^2 g_{22}(x,y)$ and $f_2(x,y) = \varepsilon f_{21}(x,y) + \varepsilon^2 f_{22}(x,y)$ where f_{1i}, f_{2i}, g_{2i} have degree l, n, m respectively for each $i = 1, 2$. We provide an accurate upper bound of the maximum number of limit cycles that this class of systems can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of first and second order. We give an example for which this bound is reached.

1. Introduction

One of the main problems in the theory of ordinary differential equations is the study of their limit cycles, their existence, their number and their stability. A limit cycle of a differential equation is a periodic orbit in the set of all isolated periodic orbits of the differential equation. The second part of the 16th Hilbert's problem (see [4]) is related to the least upper bound on the number of limit cycles of polynomial vector fields having a fixed degree. These years many papers have studied the limit cycles of planar polynomial differential systems. In this paper we will try to give a partial answer to this problem for the class of polynomial differential systems given by

$$\dot{x} = y - f_1(x,y)y, \quad \dot{y} = -x - g_2(x,y) - f_2(x,y)y, \tag{1.1}$$

where $f_1(x,y) = \varepsilon f_{11}(x,y) + \varepsilon^2 f_{12}(x,y)$, $g_2(x,y) = \varepsilon g_{21}(x,y) + \varepsilon^2 g_{22}(x,y)$ and $f_2(x,y) = \varepsilon f_{21}(x,y) + \varepsilon^2 f_{22}(x,y)$ wheref_{1i}, f_{2i} and g_{2i} have degree l, n and m respectively for each $i = 1, 2$ and ε is a small parameter. Note that when $f_1(x,y) = 0, g_2(x,y) = 0$ and $f_2(x,y) = f(x)$ these systems coincide with the generalized polynomial Liénard differential systems

$$\dot{x} = y, \quad \dot{y} = -x - f(x)y, \tag{1.2}$$

where $f(x)$ is a polynomial in the variable x of degree n.

In 1977 Lins, de Melo and Pugh [7] studied the classical polynomial Liénard differential system (1.2) and stated the following conjecture:

2010 *Mathematics Subject Classification.* 34C25, 34C29, 37G15.
Key words and phrases. Limit cycle; Liénard differential equation; averaging method.

Submitted October 1, 2016. Published December 14, 2016.

If $f(x)$ has degree $n \geq 1$, then (1.2) has at most $[\frac{n}{2}]$ limit cycles.

Then they proved this conjecture for $n = 1, 2$. The conjecture for $n = 3$ has been proved recently by Chengzi and Llibre [8]. For more information see [11].

Many of the results on the limit cycles of polynomial differential systems have been obtained by considering limit cycles which bifurcate from a single degenerate singular point, that are so called *small amplitude limit cycles*, see for instance [13]. We denote by $\hat{H}(m, n)$ the maximum number of small amplitude limit cycles for systems of the form (1.2). The values of $\hat{H}(m, n)$ give a lower bound for the maximum number $H(m, n)$ (i.e. The Hilbert number) of limit cycles that the differential equation (1.2) with m and n fixed can have. For more information about the Hilbert's 16th problem and related topics see [5].

In [10] the authors ise the averaging theory of first and second order to study the system

$$\begin{aligned} \dot{x} &= y - \varepsilon(g_{11}(x) + f_{11}(x)y) - \varepsilon^2(g_{12}(x) + f_{12}(x)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x) + f_{21}(x)y) - \varepsilon^2(g_{22}(x) + f_{22}(x)y), \end{aligned} \tag{1.3}$$

where $g_{1i}, f_{1i}, g_{2i}, f_{2i}$ have degree l, k, m, n respectively for each $i = 1, 2$, and ε is a small parameter. They provided an accurate upper bound of the maximum number of limit cycles that the above system can have bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$.

In this article, first we consider the more general system

$$\begin{aligned} \dot{x} &= y - \varepsilon(f_{11}(x, y)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x, y) + f_{21}(x, y)y), \end{aligned} \tag{1.4}$$

where f_{11}, g_{21} and f_{21} have degree l, m and n respectively, and ε is a small parameter. We obtain the following result.

Theorem 1.1. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized polynomial differential system* (1.4) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of first order is $[\frac{n}{2}]$.*

The proof of the above theorem is given in section 3.

Secondly we consider the system

$$\begin{aligned} \dot{x} &= y - \varepsilon(f_{11}(x, y)y) - \varepsilon^2(f_{12}(x, y)y), \\ \dot{y} &= -x - \varepsilon(g_{21}(x, y) + f_{21}(x, y)y) - \varepsilon^2(g_{22}(x, y) + f_{22}(x, y)y), \end{aligned} \tag{1.5}$$

where f_{11} andf_{12} have degree l; g_{21} and g_{22} have degree m; and f_{21}, f_{22} have degree n. Furthermore, ε is a small parameter. We obtain the theorem bellow.

Theorem 1.2. *For $|\varepsilon|$ sufficiently small, the maximum number of limit cycles of the generalized polynomial differential system* (1.5) *bifurcating from the periodic orbits of the linear center $\dot{x} = y, \dot{y} = -x$ using the averaging theory of second order is*

$$\lambda = \max\{\lambda_1, \lambda_2 + 1, \lambda_3 + 2\},$$

where

$$\begin{aligned} \lambda_1 = \max \Big\{ &[\frac{O(m) + E(l) - 1}{2}], [\frac{E(m) + O(l) - 1}{2}], [\frac{O(n) + O(l) - 2}{2}], \\ &m - 1, [\frac{E(n)}{2}], [\frac{E(m) + O(n) - 1}{2}] \Big\}, \end{aligned}$$

$$\lambda_2 = \max\Big\{O(n)-1, [\frac{E(m)+O(n)-3}{2}], [\frac{O(m)+E(n)-3}{2}],$$
$$[\frac{O(n)+O(l)-2}{2}], l-1, E(m)-2, [\frac{E(m)+O(n)-3}{2}],$$
$$[\frac{O(m)+E(l)-3}{2}], [\frac{E(n)+O(m)-3}{2}]\Big\},$$
$$\lambda_3 = [\frac{E(n)+E(l)-4}{2}],$$

where $O(i)$ is the largest odd integer less than or equal to i, $E(i)$ is the largest even integer less than or equal to i and $[\cdot]$ denotes the integer part function.

The proof of the above theorem is given in section 4. The results that we shall use from the averaging theory of first and second order for computing limit cycles are presented in section 2.

2. Averaging theory of first and second order

The averaging theory of first and second orders was introduced to study periodic orbits, which is summarized as follows. Consider a differential system

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2.1}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, which are T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that:

(i) $F_1(t,\cdot) \in \mathcal{C}^1(D)$ for all $t \in \mathbb{R}$, F_1, F_2, R, D_xF_1 are locally Lipschitz with respect to x, and R is differentiable with respect to ε. We define

$$F_{10}(z) = \frac{1}{T}\int_0^T F_1(s,z)ds,$$
$$F_{20}(z) = \frac{1}{T}\int_0^T [D_zF_1(s,z)y_1(s,z) + F_2(s,z)]ds,$$

where

$$y_1(s,z) = \int_0^\theta F_1(t,z)dt.$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists an $a_\varepsilon \in V$ such that $F_{10}(a_\varepsilon) + \varepsilon F_{20}(a_\varepsilon) = 0$ and $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$. Then, for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $\varphi(\cdot,\varepsilon)$ of the system (2.1) such that $\varphi(0,\varepsilon) = a_\varepsilon$.

The expression $d_B(F_{10} + \varepsilon F_{20}, V, a_\varepsilon) \neq 0$ means that the Brouwer degree of the function $F_{10} + \varepsilon F_{20} : V \to \mathbb{R}^n$ at the fixed point a_ε is not zero. A sufficient condition for the inequality to be true is that the Jacobian of the function $F_{10} + \varepsilon F_{20}$ at a_ε is not zero.

If F_{10} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly those of F_{10} for ε sufficiently small. In this case the previous result provides the averaging theory of first order.

If F_{10} is identically zero and F_{20} is not identically zero, then the zeros of $F_{10} + \varepsilon F_{20}$ are mainly the zeros of F_{20} for ε sufficiently small. In this case the previous result provides the averaging theory of second order.

For more information about the averaging theory see ([16],[17]).

3. Proof of Theorem 1.1

We need the first order averaging theory, for this we write system (1.4) in polar coordinates (r,θ) where $x = r\cos(\theta)$, $y = r\sin(\theta)$, $r > 0$. In this way system (1.4) is written in the standard form for applying the averaging theory. If we write

$$\begin{gathered} f_{11}(x,y) = \sum_{i+j=0}^{l} a_{ij,1}x^i y^j, \quad f_{21}(x,y) = \sum_{i+j=0}^{n} a_{ij,2}x^i y^j, \\ g_{21}(x,y) = \sum_{i+j=0}^{m} b_{ij,2}x^i y^i, \end{gathered} \tag{3.1}$$

then system (1.4) becomes

$$\begin{aligned} \dot{r} = -\varepsilon\Big(&\sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i+j=0}^{m} b_{ij,2}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\ &+ \sum_{i+j=0}^{l} a_{ij,1}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta)\Big), \\ \dot{\theta} = -1 - \frac{1}{r}\Big[\varepsilon\Big(&\sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) \\ &+ \sum_{i+j=0}^{m} b_{ij,2}r^{i+j}\cos^{i+1}(\theta)\sin^{j}(\theta) \\ &- \sum_{i+j=0}^{l} a_{ij,1}r^{i+j+1}\cos^{i}(\theta)\sin^{j+2}(\theta)\Big)\Big]. \end{aligned} \tag{3.2}$$

Now taking θ as the new independent variable, this system becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + O(\varepsilon^2),$$

where

$$\begin{aligned} F_1(r,\theta) = &\sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i+j=0}^{m} b_{ij,2}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\ &+ \sum_{i+j=0}^{l} a_{ij,1}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta). \end{aligned}$$

Using the notation introduced in section 2 we must calculate

$$F_{10}(r) = \frac{1}{2\pi}\int_0^{2\pi} F_1(r,\theta)d\theta.$$

Since

$$\int_0^{2\pi} \cos^i(\theta)\sin^{j+2}(\theta)d\theta = \begin{cases} 0 & \text{if } i \text{ odd or } j \text{ odd,} \\ \pi\alpha_{ij} & \text{if } i \text{ even, } j \text{ even,} \end{cases}$$

where α_{ij} is a constant, we obtain

$$F_{10}(r) = \frac{1}{2}r\sum_{i+j=0}^{n} a_{ij,2}\alpha_{ij}r^{i+j}, \tag{3.3}$$

where i and j are both even.

Then the polynomial $F_{10}(r)$ has at most $[\frac{n}{2}]$ positive roots, and we can choose the coefficients α_{ij} with i even, j even in such a way that $F_{10}(r)$ has exactly $[\frac{n}{2}]$ simple positive roots, hence Theorem 1.1 is proved.

4. Proof of Theorem 1.2

For the proof we shall use the second order averaging theory as it was stated in section 2. We write f_{11}, f_{21} and g_{21} as in (3.1) and

$$f_{12}(x,y) = \sum_{i+j=0}^{l} C_{ij,1}x^iy^j, \quad f_{22}(x,y) = \sum_{i+j=0}^{n} c_{ij,2}x^iy^j,$$
$$g_{22}(x,y) = \sum_{i+j=0}^{m} d_{ij,2}x^iy^i.$$

Then system (1.5) in polar coordinates becomes

$$\begin{aligned} \dot{r} &= -\varepsilon(A+\varepsilon B), \\ \dot{\theta} &= -1 - \frac{\varepsilon}{r}(A_1+\varepsilon B_1), \end{aligned} \tag{4.1}$$

where

$$\begin{aligned} A &= \sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}\cos^i(\theta)sin^{j+2}(\theta) + \sum_{i+j=0}^{m} b_{ij,2}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\ &\quad + \sum_{i+j=0}^{l} a_{ij,1}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta), \\ B &= \sum_{i+j=0}^{n} c_{ij,2}r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i+j=0}^{m} d_{ij,2}r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta) \\ &\quad + \sum_{i+j=0}^{l} C_{ij,1}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta), \\ A_1 &= \sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i+j=0}^{m} b_{ij,2}r^{i+j}\cos^{i+1}(\theta)\sin^{j}(\theta) \\ &\quad - \sum_{i+j=0}^{l} a_{ij,1}r^{i+j+1}\cos^{i}(\theta)\sin^{j+2}(\theta), \\ B_1 &= \sum_{i+j=0}^{n} c_{ij,2}r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta) + \sum_{i+j=0}^{m} d_{ij,2}r^{i+j}\cos^{i+1}(\theta)\sin^{j}(\theta) \\ &\quad - \sum_{i+j=0}^{l} C_{ij,1}r^{i+j+1}\cos^{i}(\theta)\sin^{j+2}(\theta). \end{aligned}$$

Taking θ as the new independent variable, this system becomes

$$\frac{dr}{d\theta} = \varepsilon F_1(r,\theta) + \varepsilon^2 F_2(r,\theta) + O(\varepsilon^3),$$

where

$$F_1(r,\theta) = A, \quad F_2(r,\theta) = B - \frac{1}{r}AA_1.$$

To compute $F_{20}(r)$, we need that $F_{10}(r)$ be identically zero, which is equivalent to $a_{ij,2} = 0$ for i even, j even.

Now we determine the corresponding function

$$F_{20}(r) = \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{d}{dr}F_1(r,\theta)y_1(r,\theta) + F_2(r,\theta)\Big]d\theta.$$

First, we have

$$\begin{aligned}
\frac{d}{dr}F_1(r,\theta) = &\sum_{\substack{i+j=0\\ i \text{ odd or } j \text{ odd}}}^{n} (i+j+1)a_{ij,2}r^{i+j}\cos^i(\theta)\sin^{j+2}(\theta)\\
&+ \sum_{i+j=0}^{m}(i+j)b_{ij,2}r^{i+j-1}\cos^i(\theta)\sin^{j+1}(\theta)\\
&+ \sum_{i+j=0}^{l}(i+j+1)a_{ij,1}r^{i+j}\cos^{i+1}(\theta)\sin^{j+1}(\theta),
\end{aligned}$$

and we write

$$y_1(r,\theta) = \int_0^\theta F_1(r,t)dt = y_1^1 + y_1^2 + y_1^3,$$

so we obtain

$$\begin{aligned}
y_1^1(r,t) &= \int_0^\theta \sum_{i+j=0}^{n} a_{ij,2}r^{i+j+1}cos^i(t)\sin^{j+2}(t)dt\\
&= a_{10,2}r^2\Big(\alpha_{110}\sin(\theta) + \alpha_{210}\sin(3\theta)\Big) + \dots\\
&\quad + a_{c_1e_1,2}r^{c_1+e_1+1}\Big(\alpha_{1c_1e_1}\sin(\theta) + \alpha_{2c_1e_1}\sin(3\theta) + \dots\\
&\quad + \alpha_{\frac{(c_1+e_1+2)+1}{2}c_1e_1}\sin((c_1+e_1+2)\theta)\Big)\\
&\quad + a_{01,2}r^2\Big(\alpha_{101} + \alpha_{201}\cos(\theta) + \alpha_{301}\cos(3\theta)\Big) + \dots\\
&\quad + a_{p_1q_1,2}r^{p_1+q_1+1}\Big(\alpha_{1p_1q_1} + \alpha_{2p_1q_1}\cos(\theta) + \alpha_{3p_1q_1}\cos(3\theta) + \dots\\
&\quad + \alpha_{\frac{(p_1+q_1+2)+3}{2}p_1q_1}\cos((p_1+q_1+2)\theta)\Big)\\
&\quad + a_{11,2}r^3\Big(\alpha_{111} + \alpha_{211}\cos(2\theta) + \alpha_{311}\cos(4\theta)\Big) + \dots\\
&\quad + a_{c_1q_1,2}r^{c_1+q_1+1}\Big(\alpha_{1c_1q_1} + \alpha_{2c_1q_1}\cos(2\theta) + \alpha_{3c_1q_1}\cos(4\theta) + \dots\\
&\quad + \alpha_{\frac{(c_1+q_1+2)+2}{2}c_1q_1}\cos((c_1+q_1+2)\theta)\Big),
\end{aligned}$$

where c_1 is the greatest odd number and e_1 is the greatest even number so that $c_1 + e_1$ is less than or equal to n. p_1 is the greatest even number and q_1 is the greatest odd number so that $p_1 + q_1$ is less than or equal to n. α_{ijk} are real constants exhibited during the computation of $\int_0^\theta \cos^i(t)\sin^{j+2}(t)dt$ for all i and j.

$$
\begin{aligned}
y_1^2(r,t) &= \int_0^\theta \sum_{i+j=0}^{m} b_{ij,2} r^{i+j} cos^i(t)\sin^{j+1}(t)dt \\
&= b_{00,2}\Big(\tilde\alpha_{100}+\tilde\alpha_{200}\cos(\theta)\Big) + b_{02,2}r^2\Big(\tilde\alpha_{102}+\tilde\alpha_{202}\cos(\theta)+\tilde\alpha_{302}\cos(3\theta)\Big) \\
&\quad + \dots + b_{p_2e_2,2}r^{p_2+e_2}\Big(\tilde\alpha_{1p_2e_2}+\tilde\alpha_{2p_2e_2}\cos(\theta)+\tilde\alpha_{3p_2e_2}\cos(3\theta)+\dots \\
&\quad + \tilde\alpha_{(\frac{p_2+e_2}{2}+2)p_2e_2} cos((p_2+e_2+1)\theta)\Big) + b_{01,2}r\Big(\tilde\alpha_{101}\theta+\tilde\alpha_{201}\sin(2\theta)\Big) \\
&\quad + \dots + b_{p_2q_2,2}r^{p_2+q_2}\Big(\tilde\alpha_{1p_2q_2}\theta+\tilde\alpha_{2p_2q_2}\sin(2\theta)+\tilde\alpha_{3p_2q_2}\sin(4\theta)+\dots \\
&\quad + \tilde\alpha_{(\frac{p_2+q_2+3}{2})p_2q_2}\sin((p_2+q_2+1)\theta)\Big) + b_{10,2}r\Big(\tilde\alpha_{110}+\tilde\alpha_{210}\cos(2\theta)\Big) \\
&\quad + b_{30,2}r^3\Big(\tilde\alpha_{130}+\tilde\alpha_{230}\cos(2\theta)+\tilde\alpha_{330}\cos(4\theta)\Big) + \dots + b_{c_2e_2,2}r^{c_2+e_2} \\
&\quad \times\Big(\tilde\alpha_{1c_2e_2}+\tilde\alpha_{2c_2e_2}\cos(2\theta)+\dots+\tilde\alpha_{(\frac{c_2+e_2+1}{2}+1)c_2e_2}\cos((c_2+e_2+1)\theta)\Big) \\
&\quad + b_{11,2}r^2\Big(\tilde\alpha_{111}\sin(\theta)+\tilde\alpha_{211}\sin(3\theta)\Big) \\
&\quad + b_{13,2}r^4\Big(\tilde\alpha_{113}\sin(\theta)+\tilde\alpha_{213}\sin(3\theta)+\tilde\alpha_{313}\sin(5\theta)\Big) + \dots \\
&\quad + b_{c_2q_2,2}r^{c_2+q_2}\Big(\tilde\alpha_{1c_2q_2\sin(\theta)}+\tilde\alpha_{2c_2q_2}\sin(3\theta)+\dots \\
&\quad + \tilde\alpha_{(\frac{c_2+q_2+2}{2})c_2q_2}\sin((c_2+q_2+1)\theta)\Big),
\end{aligned}
$$

where p_2 is the greatest even number and e_2 is the greatest even number so that p_2+e_2 is less than or equal to m. c_2 is the greatest odd number and q_2 is the greatest odd number so that c_2+q_2 is less than or equal to m. $\tilde\alpha_{ijk}$ are real constants exhibited during the computation of $\int_0^\theta \cos^i(t)\sin^{j+1}(t)dt$ for all i and j.

$$
\begin{aligned}
& y_1^3(r,t) \\
&= \int_0^\theta \sum_{i+j=0}^{l} a_{ij,1} r^{i+j+1} cos^{i+1}(t)\sin^{j+1}(t)dt \\
&= a_{00,1}r\Big(\hat\alpha_{100}+\hat\alpha_{200}\cos(2\theta)\Big) + \dots + a_{p_3e_3,1}r^{p_3+e_3+1}\Big(\hat\alpha_{1p_3e_3}+\hat\alpha_{2p_3e_3}\cos(2\theta) \\
&\quad + \dots + \hat\alpha_{(\frac{p_3+e_3+2}{2}+1)p_3e_3}\cos((p_3+e_3+2)\theta)\Big) \\
&\quad + a_{01,1}r^2\Big(\hat\alpha_{101}\sin(\theta)+\hat\alpha_{201}\sin(3\theta)\Big) + \dots \\
&\quad + a_{p_3q_3,1}r^{p_3+q_3+1}\Big(\hat\alpha_{1p_3q_3}\sin(\theta)+\hat\alpha_{2p_3q_3}\sin(3\theta)+\hat\alpha_{3p_3q_3}\sin(5\theta)+\dots \\
&\quad + \hat\alpha_{(\frac{p_3+q_3+3}{2})p_3q_3}\sin((p_3+q_3+2)\theta)\Big) \\
&\quad + a_{10,1}r^2\Big(\hat\alpha_{110}+\hat\alpha_{210}\cos(\theta)+\hat\alpha_{310}\cos(3\theta)\Big) + \dots \\
&\quad + a_{c_3e_3,1}r^{c_3+e_3+1}\Big(\hat\alpha_{1c_3e_3}+\hat\alpha_{2c_3e_3}\cos(\theta)+\hat\alpha_{3c_3e_3}\cos(3\theta)+\dots \\
&\quad + \hat\alpha_{(\frac{c_3+e_3+1}{2}+2)c_3e_3}\cos((c_3+e_3+2)\theta)\Big) + a_{11,1}r^3\Big(\hat\alpha_{111}\theta+\hat\alpha_{211}\sin(4\theta)\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ a_{13,1}r^5\Big(\hat{\alpha}_{113}\theta + \hat{\alpha}_{213}\sin(2\theta) + \hat{\alpha}_{313}\sin(4\theta) + \hat{\alpha}_{413}\sin(6\theta)\Big) + \dots \\
&+ a_{c_3q_3,1}r^{c_3+q_3+1}\Big(\hat{\alpha}_{1c_3q_3}\theta + \hat{\alpha}_{2c_3q_3}\sin(2\theta) \\
&+ \hat{\alpha}_{3c_3q_3}\sin(4\theta) + \dots + \hat{\alpha}_{(c_3+q_3)c_3q_3}\sin((c_3+q_3+2)\theta)\Big),
\end{aligned}
$$

where p_3 is the greatest even number and e_3 is the greatest even number so that p_3+e_3 is less than or equal to l, c_3 is the greatest odd number and q_3 is the greatest odd number so that c_3+q_3 is less than or equal to l, $\hat{\alpha}_{ijk}$ are real constants exhibited during the computation of $\int_0^\theta \cos^{i+1}(t)\sin^{j+1}(t)dt$ for all i and j.

Finally

$$
\begin{aligned}
&y_1(r,\theta) \\
&= a_{10,2}r^2\Big(\alpha_{110}\sin(\theta) + \alpha_{210}\sin(3\theta)\Big) + \dots + a_{c_1e_1,2}r^{c_1+e_1+1}\Big(\alpha_{1c_1e_1}\sin(\theta) \\
&\quad + \alpha_{2c_1e_1}\sin(3\theta) + \dots + \alpha_{\frac{(c_1+e_1+2)+1}{2}c_1e_1}\sin((c_1+e_1+2)\theta)\Big) \\
&\quad + a_{01,2}r^2\Big(\alpha_{101} + \alpha_{201}\cos(\theta) + \alpha_{301}\cos(3\theta)\Big) + \dots \\
&\quad + a_{p_1q_1,2}r^{p_1+q_1+1}\Big(\alpha_{1p_1q_1} + \alpha_{2p_1q_1}\cos(\theta) + \alpha_{3p_1q_1}\cos(3\theta) + \dots \\
&\quad + \alpha_{\frac{(p_1+q_1+2)+3}{2}p_1q_1}\cos((p_1+q_1+2)\theta)\Big) \\
&\quad + a_{11,2}r^3\Big(\alpha_{111} + \alpha_{211}\cos(2\theta) + \alpha_{311}\cos(4\theta)\Big) + \dots \\
&\quad + a_{c_1q_1,2}r^{c_1+q_1+1}\Big(\alpha_{1c_1q_1} + \alpha_{2c_1q_1}\cos(2\theta) + \alpha_{3c_1q_1}\cos(4\theta) + \dots \\
&\quad + \alpha_{\frac{(c_1+q_1+2)+2}{2}c_1q_1}\cos((c_1+q_1+2)\theta)\Big) \\
&\quad + b_{00,2}\Big(\tilde{\alpha}_{100} + \tilde{\alpha}_{200}\cos(\theta)\Big) + b_{02,2}r^2\Big(\tilde{\alpha}_{102} + \tilde{\alpha}_{202}\cos(\theta) \\
&\quad + \tilde{\alpha}_{302}\cos(3\theta)\Big) + \dots + b_{p_2e_2,2}r^{p_2+e_2}\Big(\tilde{\alpha}_{1p_2e_2} + \tilde{\alpha}_{2p_2e_2}\cos(\theta) + \tilde{\alpha}_{3p_2e_2}\cos(3\theta) + \dots \\
&\quad + \tilde{\alpha}_{(\frac{p_2+e_2}{2}+2)p_2e_2}cos((p_2+e_2+1)\theta)\Big) + b_{01,2}r\Big(\tilde{\alpha}_{101}\theta + \tilde{\alpha}_{201}\sin(2\theta)\Big) + \dots \\
&\quad + b_{p_2q_2,2}r^{p_2+q_2}\Big(\tilde{\alpha}_{1p_2q_2}\theta + \tilde{\alpha}_{2p_2q_2}\sin(2\theta) + \tilde{\alpha}_{3p_2q_2}\sin(4\theta) + \dots \\
&\quad + \tilde{\alpha}_{(\frac{p_2+q_2+3}{2})p_2q_2}\sin((p_2+q_2+1)\theta)\Big) + b_{10,2}r\Big(\tilde{\alpha}_{110} + \tilde{\alpha}_{210}\cos(2\theta)\Big) \\
&\quad + b_{30,2}r^3\Big(\tilde{\alpha}_{130} + \tilde{\alpha}_{230}\cos(2\theta) + \tilde{\alpha}_{330}\cos(4\theta)\Big) + \dots \\
&\quad + b_{c_2e_2,2}r^{c_2+e_2}\Big(\tilde{\alpha}_{1c_2e_2} + \tilde{\alpha}_{2c_2e_2}\cos(2\theta) + \dots \\
&\quad + \tilde{\alpha}_{(\frac{c_2+e_2+1}{2}+1)c_2e_2}\cos((c_2+e_2+1)\theta)\Big) + b_{11,2}r^2\Big(\tilde{\alpha}_{111}\sin(\theta) + \tilde{\alpha}_{211}\sin(3\theta)\Big) \\
&\quad + b_{13,2}r^4\Big(\tilde{\alpha}_{113}\sin(\theta) + \tilde{\alpha}_{213}\sin(3\theta) + \tilde{\alpha}_{313}\sin(5\theta)\Big) + \dots \\
&\quad + b_{c_2q_2,2}r^{c_2+q_2}\Big(\tilde{\alpha}_{1c_2q_2\sin(\theta)} + \tilde{\alpha}_{2c_2q_2}\sin(3\theta) + \dots \\
&\quad + \tilde{\alpha}_{(\frac{c_2+q_2+2}{2})c_2q_2}\sin((c_2+q_2+1)\theta)\Big) + a_{00,1}r\Big(\hat{\alpha}_{100} + \hat{\alpha}_{200}\cos(2\theta)\Big) + \dots \\
&\quad + a_{p_3e_3,1}r^{p_3+e_3+1}\Big(\hat{\alpha}_{1p_3e_3} + \hat{\alpha}_{2p_3e_3}\cos(2\theta) + \dots
\end{aligned}
$$

$$
\begin{aligned}
&+ \alpha_{(\frac{p_3+e_3+2}{2}+1)p_3e_3}\cos((p_3+e_3+2)\theta)\Big) + a_{01,1}r^2\Big(\hat\alpha_{101}\sin(\theta)+\hat\alpha_{201}\sin(3\theta)\Big)\\
&+ \dots + a_{p_3q_3,1}r^{p_3+q_3+1}\Big(\hat\alpha_{1p_3q_3}\sin(\theta)+\hat\alpha_{2p_3q_3}\sin(3\theta)+\hat\alpha_{3p_3q_3}\sin(5\theta)+\dots\\
&+ \hat\alpha_{(\frac{p_3+q_3+3}{2})p_3q_3}\sin((p_3+q_3+2)\theta)\Big) + a_{10,1}r^2\Big(\hat\alpha_{110}+\hat\alpha_{210}\cos(\theta)\\
&+ \hat\alpha_{310}\cos(3\theta)\Big) + \dots + a_{c_3e_3,1}r^{c_3+e_3+1}\Big(\hat\alpha_{1c_3e_3}+\hat\alpha_{2c_3e_3}\cos(\theta)+\hat\alpha_{3c_3e_3}\cos(3\theta)\\
&+ \dots + \hat\alpha_{(\frac{c_3+e_3+1}{2}+2)c_3e_3}\cos((c_3+e_3+2)\theta)\Big) + a_{11,1}r^3\Big(\hat\alpha_{111}\theta+\hat\alpha_{211}\sin(4\theta)\Big)\\
&+ a_{13,1}r^5\Big(\hat\alpha_{113}\theta+\hat\alpha_{213}\sin(2\theta)+\hat\alpha_{313}\sin(4\theta)+\hat\alpha_{413}\sin(6\theta)\Big)+\dots\\
&+ a_{c_3q_3,1}r^{c_3+q_3+1}\Big(\hat\alpha_{1c_3q_3}\theta+\hat\alpha_{2c_3q_3}\sin(2\theta)\\
&+ \hat\alpha_{3c_3q_3}\sin(4\theta)+\dots+\hat\alpha_{(c_3+q_3)c_3q_3}\sin((c_3+q_3+2)\theta)\Big).
\end{aligned}
$$

We know from (3.3) that F_{10} is identically zero if and only if $a_{ij,2}=0$ for all i even, j even.

Now using the integrals given at the appendix, we calculate

$$
\begin{aligned}
&H_1(r)\\
&= \frac{1}{2\pi}\int_0^{2\pi}\Big[\frac{d}{dr}F_1(r,\theta)y_1(r,\theta)\Big]d\theta\\
&= \frac12\Big[\sum_{\substack{i+j=1\\ i\text{ even, }J\text{ odd}}}^{n}(i+j+1)a_{ij,2}r^{i+j}\Big[a_{10,2}r^2\Big(\alpha_{110}A^1_{ij}+\alpha_{210}A^3_{ij}\Big)+\dots\\
&+ a_{c_1e_1,2}r^{c_1+e_1+1}\Big(\alpha_{1c_1e_1}A^1_{ij}+\alpha_{2c_1e_1}A^3_{ij}+\dots+\alpha_{\frac{(c_1+e_1+2)+1}{2}c_1e_1}A^{c_1+e_1+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i\text{ odd, }j\text{ even}}}^{n}(i+j+1)a_{ij,2}r^{i+j}\Big[a_{01,2}r^2\Big(\alpha_{201}B^1_{ij}+\alpha_{301}B^3_{ij}\Big)+\dots\\
&+ a_{p_1q_1,2}r^{p_1+q_1+1}\Big(\alpha_{2p_1q_1}B^1_{ij}+\alpha_{3cp_1q_1}B^3_{ij}+\dots+\alpha_{\frac{(p_1+q_1+2)+3}{2}p_1q_1}B^{p_1+q_1+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i\text{ odd, }j\text{ even}}}^{n}(i+j+1)a_{ij,2}r^{i+j}\Big[b_{00,2}\Big(\tilde\alpha_{200}\tilde B^1_{ij}\Big)+b_{02,2}r^2\Big(\tilde\alpha_{202}\tilde B^1_{ij}+\tilde\alpha_{302}\tilde B^3_{ij}\Big)\\
&+ \dots + b_{p_2e_2,2}r^{p_2+e_2}\Big(\tilde\alpha_{2p_2e_2}\tilde B^1_{ij}+\tilde\alpha_{302}\tilde B^3_{ij}+\dots+\tilde\alpha_{(\frac{p_2+e_2}{2}+2)p_2e_2}\tilde B^{p_2+e_2+1}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=2\\ i\text{ odd, }j\text{ odd}}}^{n}(i+j+1)a_{ij,2}r^{i+j}\Big[b_{01,2}r\Big(\tilde\alpha_{101}\gamma_{ij}+\tilde\alpha_{201}C^2_{ij}\Big)+\dots\\
&+ b_{p_2q_2,2}r^{p_2+q_2}\Big(\tilde\alpha_{1p_2q_2}\gamma_{ij}+\tilde\alpha_{2p_2q_2}C^2_{ij}+\tilde\alpha_{3p_2q_2}C^4_{ij}+\dots\\
&+ \tilde\alpha_{(\frac{p_2+q_2+3}{2})p_2q_2}C^{p_2+q_2+1}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i\text{ even, }j\text{ odd}}}^{n}(i+j+1)a_{ij,2}r^{i+j}\Big[b_{11,2}r^2\Big(\tilde\alpha_{111}\tilde A^1_{ij}+\tilde\alpha_{211}\tilde A^3_{ij}\Big)+\dots
\end{aligned}
$$

$$
\begin{aligned}
&+ b_{c_2q_2,2} r^{c_2+q_2}\Big(\tilde\alpha_{1c_2q_2}\tilde A^1_{ij} + \tilde\alpha_{2c_2q_2}\tilde A^3_{ij} + \dots + \tilde\alpha_{(\frac{c_2+q_2+2}{2})c_2q_2}\tilde A^{c_2+q_2+1}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i \text{ even},\, j \text{ odd}}}^{n} (i+j+1)a_{ij,2} r^{i+j}\Big[a_{01,1}r^2\Big(\hat\alpha_{101}\hat A^1_{ij} + \hat\alpha_{201}\hat A^3_{ij}\Big)\\
&+ \dots + a_{p_3q_3,1} r^{p_3+q_3+1}\Big(\hat\alpha_{1p_3q_3}\hat A^1_{ij} + \hat\alpha_{2p_3q_3}\hat A^3_{ij} + \dots + \hat\alpha_{(\frac{p_3+q_3+3}{2})p_3q_3}\hat A^{p_3+q_3+2}_{ij}\Big)\Big]\\
&\sum_{\substack{i+j=1\\ i \text{ odd},\, j \text{ even}}}^{n} (i+j+1)a_{ij,2} r^{i+j}\Big[a_{10,1}r^2\Big(\hat\alpha_{210}\hat B^1_{ij} + \hat\alpha_{310}\hat B^3_{ij}\Big) + \dots\\
&+ a_{c_3e_3,1} r^{c_3+e_3+1}\Big(\hat\alpha_{2c_3e_3}\hat B^1_{ij} + \hat\alpha_{3c_3e_3}\hat B^3_{ij} + \dots + \hat\alpha_{(\frac{c_3+e_3+1}{2}+2)c_3e_3}\hat B^{c_3+e_3+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{n} (i+j+1)a_{ij,2} r^{i+j}\Big[a_{11,1}r^3\Big(\hat\alpha_{111}\gamma_{ij} + \hat\alpha_{211}\tilde C^4_{ij}\Big) + \dots\\
&+ a_{c_3q_3,1} r^{c_3+q_3+1}\Big(\hat\alpha_{1c_3q_3}\gamma_{ij} + \hat\alpha_{2c_3q_3}\tilde C^2_{ij} + \dots + \hat\alpha_{(c_3+q_3)c_3q_3}\tilde C^{c_3+q_3+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=2\\ i \text{ even},\, j \text{ even}}}^{m} (i+j)b_{ij,2} r^{i+j-1}\Big[a_{10,2}r^2\Big(\alpha_{110}D^1_{ij} + \alpha_{210}D^3_{ij}\Big) + \dots\\
&+ a_{c_1e_1,2} r^{c_1+e_1+1}\Big(\alpha_{1c_1e_1}D^1_{ij} + \alpha_{2c_1e_1}D^3_{ij} + \dots + \alpha_{\frac{(c_1+e_1+2)+1}{2}c_1e_1}D^{c_1+e_1+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{m} (i+j)b_{ij,2} r^{i+j-1}\Big[a_{01,2}r^2\Big(\alpha_{201}E^1_{ij} + \alpha_{301}E^3_{ij}\Big) + \dots\\
&+ a_{p_1q_1,2} r^{p_1+q_1+1}\Big(\alpha_{2p_1q_1}E^1_{ij} + \alpha_{3p_1q_1}E^3_{ij} + \dots\\
&+ \alpha_{\frac{(p_1+q_1+2)+3}{2}p_1q_1}E^{p_1+q_1+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i \text{ even},\, j \text{ odd}}}^{m} (i+j)b_{ij,2} r^{i+j-1}\Big[a_{11,2}r^3\Big(\alpha_{111}\beta_{ij} + \alpha_{211}F^2_{ij} + \alpha_{311}F^4_{ij}\Big) + \dots\\
&+ a_{c_1q_1,2} r^{c_1+q_1+1}\Big(\alpha_{1c_1q_1}\beta_{ij} + \alpha_{2c_1q_1}F^2_{ij} + \alpha_{3c_1q_1}F^4_{ij} + \dots\\
&+ \alpha_{\frac{(c_1+q_1+2)+2}{2}c_1q_1}F^{c_1+q_1+2}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{m} (i+j)b_{ij,2} r^{i+j-1}\Big[b_{00,2}\Big(\tilde\alpha_{200}\tilde E^1_{ij}\Big) + b_{02,2}r^2\Big(\tilde\alpha_{202}\tilde E^1_{ij} + \tilde\alpha_{302}\tilde E^3_{ij}\Big)\\
&+ \dots + b_{p_2e_2,2} r^{p_2+e_2}\Big(\tilde\alpha_{2p_2e_2}\tilde E^1_{ij} + \tilde\alpha_{3p_2e_2}\tilde E^3_{ij} + \dots + \tilde\alpha_{(\frac{p_2+e_2}{2}+2)p_2e_2}\tilde E^{p_2+e_2+1}_{ij}\Big)\Big]\\
&+ \sum_{\substack{i+j=1\\ i \text{ odd},\, j \text{ even}}}^{m} (i+j)b_{ij,2} r^{i+j-1}\Big[b_{01,2}r\Big(\tilde\alpha_{101}\sigma_{ij} + \tilde\alpha_{201}G^2_{ij}\Big) + \dots\\
&+ b_{p_2q_2,2} r^{p_2+q_2}\Big(\tilde\alpha_{1p_2q_21}\sigma_{ij} + \tilde\alpha_{2p_2q_2}G^2_{ij} + \dots
\end{aligned}
$$

$$
\begin{aligned}
&+ \tilde{\alpha}_{(\frac{p_2+q_2+3}{2})p_2q_2} G_{ij}^{p_2+q_2+1}\Big)\Big] \\
&+ \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[b_{10,2} r \Big(\tilde{\alpha}_{110}\beta_{ij} + \tilde{\alpha}_{210}\tilde{F}_{ij}^2 \Big) + \dots \\
&+ b_{c_2e_2,2} r^{c_2+e_2} \Big(\tilde{\alpha}_{1c_2e_2}\beta_{ij} + \tilde{\alpha}_{210}\tilde{F}_{ij}^2 + \dots + \tilde{\alpha}_{(\frac{c_2+e_2+1}{2}+1)c_2e_2}\tilde{F}_{ij}^{c_2+e_2+1}\Big)\Big] \\
&+ \sum_{\substack{i+j=2\\ i \text{ even, } j \text{ even}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[b_{11,2} r^2 \Big(\tilde{\alpha}_{111}\tilde{D}_{ij}^1 + \tilde{\alpha}_{211}\tilde{D}_{ij}^3 \Big) + \dots \\
&+ b_{c_2q_2,2} r^{c_2+q_2} \Big(\tilde{\alpha}_{1c_2q_2}\tilde{D}_{ij}^1 + \tilde{\alpha}_{2c_2q_2}\tilde{D}_{ij}^3 + \dots + \tilde{\alpha}_{(\frac{c_2+q_2+2}{2})c_2q_2}\tilde{D}_{ij}^{c_2+q_2+1}\Big)\Big] \\
&+ \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[a_{00,1} r \Big(\hat{\alpha}_{100}\beta_{ij} + \hat{\alpha}_{200}\hat{F}_{ij}^2 \Big) + \dots \\
&+ a_{p_3e_3,1} r^{p_3+e_3+1} \Big(\hat{\alpha}_{1p_3e_3}\beta_{ij} + \hat{\alpha}_{2p_3e_3}\hat{F}_{ij}^2 + \dots + \hat{\alpha}_{(\frac{p_3+e_3+2}{2}+1)p_3e_3}\hat{F}_{ij}^{p_3+e_3+2}\Big)\Big] \\
&+ \sum_{\substack{i+j=2\\ i \text{ even, } j \text{ even}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[a_{01,1} r^2 \Big(\hat{\alpha}_{101}\hat{D}_{ij}^1 + \hat{\alpha}_{201}\hat{D}_{ij}^3 \Big) + \dots \\
&+ a_{p_3q_3,1} r^{p_3+q_3+1} \Big(\hat{\alpha}_{1p_3q_3}\hat{D}_{ij}^1 + \hat{\alpha}_{2p_3q_3}\hat{D}_{ij}^3 + \dots + \hat{\alpha}_{(\frac{p_3+q_3+3}{2})p_3q_3}\hat{D}_{ij}^{p_3+q_3+2}\Big)\Big] \\
&+ \sum_{\substack{i+j=2\\ i \text{ odd, } j \text{ odd}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[a_{10,1} r^2 \Big(\hat{\alpha}_{210}\hat{E}_{ij}^1 + \hat{\alpha}_{310}\hat{E}_{ij}^3 \Big) + \dots \\
&+ a_{c_3e_3,1} r^{c_3+e_3+1} \Big(\hat{\alpha}_{2c_3e_3}\hat{E}_{ij}^1 + \hat{\alpha}_{3c_3e_3}\hat{E}_{ij}^3 + \dots + \hat{\alpha}_{(\frac{c_3+e_3+1}{2}+2)c_3e_3}\hat{E}_{ij}^{c_3+e_3+2}\Big)\Big] \\
&+ \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{m} (i+j) b_{ij,2} r^{i+j-1} \Big[a_{11,1} r^3 \Big(\hat{\alpha}_{111}\sigma_{ij} + \hat{\alpha}_{211}\tilde{G}_{ij}^4 \Big) + \dots \\
&+ a_{c_3q_3,1} r^{c_3+q_3+1} \Big(\hat{\alpha}_{1c_3q_3}\sigma_{ij} + \hat{\alpha}_{2c_3q_3}\tilde{G}_{ij}^2 + \hat{\alpha}_{3c_3q_3}\tilde{G}_{ij}^4 + \dots \\
&+ \hat{\alpha}_{(c_3+q_3)c_3q_3}\tilde{G}_{ij}^{c_3+q_3+2}\Big)\Big] \\
&+ \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{l} (i+j+1) a_{ij,1} r^{i+j} \Big[a_{10,2} r^2 \Big(\alpha_{110} H_{ij}^1 + \alpha_{210} H_{ij}^3 \Big) + \dots \\
&+ a_{c_1e_1,2} r^{c_1+e_1+1} \Big(\alpha_{1c_1e_1} H_{ij}^1 + \alpha_{2c_1e_1} H_{ij}^3 + \dots + \alpha_{\frac{(c_1+e_1+2)+1}{2}c_1e_1} H_{ij}^{c_1+e_1+2}\Big)\Big] \\
&+ \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{l} (i+j+1) a_{ij,1} r^{i+j} \Big[a_{01,2} r^2 \Big(\alpha_{201} I_{ij}^1 + \alpha_{301} I_{ij}^3 \Big) + \dots \\
&+ a_{p_1q_1,2} r^{p_1+q_1+1} \Big(\alpha_{2p_1q_1} I_{ij}^1 + \alpha_{3p_1q_1} I_{ij}^3 + \dots + \alpha_{\frac{(p_1+q_1+2)+3}{2}p_1q_1} I_{ij}^{p_1+q_1+2}\Big)\Big]
\end{aligned}
$$

$$+\sum_{\substack{i+j=2\\ i\text{ odd},\ j\text{ odd}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[a_{11,2}r^{3}\Big(\alpha_{111}\delta_{ij}+\alpha_{211}K_{ij}^{2}+\alpha_{311}K_{ij}^{4}\Big)+\dots$$

$$+a_{c_1q_1,2}r^{c_1+q_1+1}\Big(\alpha_{1c_1q_1}\delta_{ij}+\alpha_{21c_1q_1}K_{ij}^{2}+\dots+\alpha_{\frac{(c_1+q_1+2)+2}{2}c_1q_1}K_{ij}^{c_1+q_1+2}\Big)\Big]$$

$$+\sum_{\substack{i+j=1\\ i\text{ even},\ j\text{ odd}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[b_{00,2}\Big(\tilde\alpha_{200}\tilde I_{ij}^{1}\Big)+b_{02,2}r^{2}\Big(\tilde\alpha_{202}\tilde I_{ij}^{1}+\tilde\alpha_{302}\tilde I_{ij}^{3}\Big)$$

$$+\dots+b_{p_2e_2,2}r^{p_2+e_2}\Big(\tilde\alpha_{2p_2e_2}\tilde I_{ij}^{1}+\dots+\tilde\alpha_{(\frac{p_2+e_2}{2}+2)p_2e_2}\tilde I_{ij}^{p_2+e_2+1}\Big)\Big]$$

$$+\sum_{\substack{i+j=0\\ i\text{ even},\ j\text{ even}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[b_{01,2}r\Big(\tilde\alpha_{101}\mu_{ij}+\tilde\alpha_{201}L_{ij}^{2}\Big)+\dots$$

$$+b_{p_2q_2,2}r^{p_2+q_2}\Big(\tilde\alpha_{1p_2q_2}\mu_{ij}+\tilde\alpha_{2p_2q_2}L_{ij}^{2}+\dots+\tilde\alpha_{(\frac{p_2+q_2+3}{2})p_2q_2}L_{ij}^{p_2+q_2+1}\Big)\Big]$$

$$+\sum_{\substack{i+j=2\\ i\text{ odd},\ j\text{ odd}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[b_{10,2}r\Big(\tilde\alpha_{110}\delta_{ij}+\tilde\alpha_{210}\tilde K_{ij}^{2}\Big)+\dots$$

$$+b_{c_2e_2,2}r^{c_2+e_2}\Big(\tilde\alpha_{1c_2e_2}\delta_{ij}+\tilde\alpha_{2c_2e_2}\tilde K_{ij}^{2}+\dots+\tilde\alpha_{(\frac{c_2+e_2+1}{2}+1)c_2e_2}\tilde K_{ij}^{c_2+e_2+1}\Big)\Big]$$

$$+\sum_{\substack{i+j=1\\ i\text{ odd},\ j\text{ even}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[b_{11,2}r^{2}\Big(\tilde\alpha_{111}\tilde H_{ij}^{1}+\tilde\alpha_{211}\tilde H_{ij}^{3}\Big)+\dots$$

$$+b_{c_2q_2,2}r^{c_2+q_2}\Big(\tilde\alpha_{1c_2q_2}\tilde H_{ij}^{1}+\dots+\tilde\alpha_{(\frac{c_2+q_2+2}{2})c_2q_2}\tilde H_{ij}^{c_2+q_2+1}\Big)\Big]$$

$$+\sum_{\substack{i+j=2\\ i\text{ odd},\ j\text{ odd}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[a_{00,1}r\Big(\hat\alpha_{100}\delta_{ij}+\hat\alpha_{200}\hat K_{ij}^{2}\Big)+\dots$$

$$+a_{p_3e_3,1}r^{p_3+e_3+1}\Big(\hat\alpha_{1p_3e_3}\delta_{ij}+\hat\alpha_{2p_3e_3}\hat K_{ij}^{2}+\dots+\hat\alpha_{(\frac{p_3+e_3+2}{2}+1)p_3e_3}\hat K_{ij}^{p_3+e_3+2}\Big)\Big]$$

$$+\sum_{\substack{i+j=1\\ i\text{ odd},\ j\text{ even}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[a_{01,1}r^{2}\Big(\hat\alpha_{101}\hat H_{ij}^{1}+\hat\alpha_{201}\hat H_{ij}^{3}\Big)+\dots$$

$$+a_{p_3q_3,1}r^{p_3+q_3+1}\Big(\hat\alpha_{1p_3q_3}\hat H_{ij}^{1}+\hat\alpha_{2p_3q_3}\hat H_{ij}^{3}+\dots+\hat\alpha_{(\frac{p_3+q_3+3}{2})p_3q_3}\hat H_{ij}^{p_3+q_3+2}\Big)\Big]$$

$$+\sum_{\substack{i+j=1\\ i\text{ even},\ j\text{ odd}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[a_{10,1}r^{2}\Big(\hat\alpha_{210}\hat I_{ij}^{1}+\hat\alpha_{310}\hat I_{ij}^{3}\Big)+\dots$$

$$+a_{c_3e_3,1}r^{c_3+e_3+1}\Big(\hat\alpha_{2c_3e_3}\hat I_{ij}^{1}+\dots+\hat\alpha_{(\frac{c_3+e_3+1}{2}+1)c_3e_3}\hat I_{ij}^{c_3+e_3+2}\Big)\Big]$$

$$+\sum_{\substack{i+j=0\\ i\text{ even},\ j\text{ even}}}^{l}(i+j+1)a_{ij,1}r^{i+j}\Big[a_{11,1}r^{3}\Big(\hat\alpha_{111}\mu_{ij}+\hat\alpha_{211}\tilde L_{ij}^{4}\Big)+\dots$$

$$+a_{c_3q_3,1}r^{c_3+q_3+1}\Big(\hat\alpha_{1c_3q_3}\mu_{ij}+\hat\alpha_{2c_3q_3}\tilde L_{ij}^{2}+\dots+\hat\alpha_{(c_3+q_3)c_3q_3}\tilde L_{ij}^{c_3+q_3+2}\Big)\Big]\Big].$$

We also note that

$$\begin{aligned}
&F_2(r,\theta)\\
&= \sum_{i+j=0}^{n} c_{ij,2} r^{i+j+1}\cos^i(\theta)\sin^{j+2}(\theta) + \sum_{i+j=0}^{m} d_{ij,2} r^{i+j}\cos^i(\theta)\sin^{j+1}(\theta)\\
&\quad + \sum_{i+j=0}^{l} C_{ij,1} r^{i+j+1}\cos^{i+1}(\theta)\sin^{j+1}(\theta)\\
&\quad - \sum_{\substack{i+j=0\\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{\substack{k+h=0\\ k \text{ odd or } h \text{ odd}}}^{n} a_{ij,2}a_{kh,2} r^{i+j+k+h+1}\cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)\\
&\quad - \sum_{\substack{i+j=0\\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{k+h=0}^{m} a_{ij,2}b_{kh,2} r^{i+j+k+h}\cos^{i+k+1}(\theta)\sin^{j+h+2}(\theta)\\
&\quad + \sum_{\substack{i+j=0\\ i \text{ odd or } j \text{ odd}}}^{n} \sum_{k+h=0}^{l} a_{ij,2}a_{kh,1} r^{i+j+k+h+1}\cos^{i+k}(\theta)\sin^{j+h+4}(\theta)\\
&\quad - \sum_{i+j=0}^{m} \sum_{\substack{k+h=0\\ k \text{ odd or } h \text{ odd}}}^{n} b_{ij,2}a_{kh,2} r^{i+j+k+h}\cos^{i+k+1}(\theta)\sin^{j+h+2}(\theta)\\
&\quad - \sum_{i+j=0}^{m} \sum_{k+h=0}^{m} b_{ij,2}b_{kh,2} r^{i+j+k+h-1}\cos^{i+k+1}(\theta)\sin^{j+h+1}(\theta)\\
&\quad + \sum_{i+j=0}^{m} \sum_{k+h=0}^{l} b_{ij,2}a_{kh,1} r^{i+j+k+h}\cos^{i+k}(\theta)\sin^{j+h+3}(\theta)\\
&\quad - \sum_{\substack{i+j=0\\ k \text{ odd or } h \text{ odd}}}^{l} \sum_{k+h=0}^{n} a_{ij,1}a_{kh,2} r^{i+j+k+h+1}\cos^{i+k+2}(\theta)\sin^{j+h+2}(\theta)\\
&\quad - \sum_{i+j=0}^{l} \sum_{k+h=0}^{m} a_{ij,1}b_{kh,2} r^{i+j+k+h}\cos^{i+k+2}(\theta)\sin^{j+h+1}(\theta)\\
&\quad + \sum_{i+j=0}^{l} \sum_{k+h=0}^{l} a_{ij,1}a_{kh,1} r^{i+j+k+h+1}\cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta).
\end{aligned}$$

Again, using the integrals given in the appendix, we calculate

$$\begin{aligned}
H_2(r) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(r,\theta)d\theta\\
&= \frac{1}{2}\Big[\sum_{\substack{i+j=0\\ i \text{ even},\ j \text{ even}}}^{n} c_{ij,2} r^{i+j+1}\alpha_{ij} + \sum_{\substack{i+j=1\\ i \text{ even},\ j \text{ odd}}}^{m} d_{ij,2} r^{i+j}\beta_{ij}\\
&\quad + \sum_{\substack{i+j=2\\ i \text{ odd},\ j \text{ odd}}}^{l} C_{ij,1} r^{i+j+1}\delta_{ij}
\end{aligned}$$

$$
\begin{aligned}
&- \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{n} \sum_{\substack{k+h=1\\ k \text{ even, } h \text{ odd}}}^{n} a_{ij,2}a_{kh,2}r^{i+j+k+h+1}M_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{n} \sum_{\substack{k+h=1\\ k \text{ odd, } h \text{ even}}}^{n} a_{ij,2}a_{kh,2}r^{i+j+k+h+1}\tilde{M}_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{n} \sum_{\substack{k+h=0\\ k \text{ even, } h \text{ even}}}^{m} a_{ij,2}b_{kh,2}r^{i+j+k+h}N_{ijkh}\\
&- \sum_{\substack{i+j=2\\ i \text{ odd, } j \text{ odd}}}^{n} \sum_{k+h=1 k \text{ even, } h \text{ odd}}^{m} a_{ij,2}b_{kh,2}r^{i+j+k+h}\tilde{N}_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{n} \sum_{\substack{k+h=2\\ k \text{ odd, } h \text{ odd}}}^{m} a_{ij,2}b_{kh,2}r^{i+j+k+h}\hat{N}_{ijkh}\\
&+ \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{n} \sum_{\substack{k+h=1\\ k \text{ odd, } h \text{ even}}}^{l} a_{ij,2}a_{kh,1}r^{i+j+k+h+1}P_{ijkh}\\
&+ \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{n} \sum_{\substack{k+h=1\\ k \text{ even, } h \text{ odd}}}^{l} a_{ij,2}a_{kh,1}r^{i+j+k+h+1}\tilde{P}_{ijkh}\\
&+ \sum_{\substack{i+j=2\\ i \text{ odd, } j \text{ odd}}}^{n} \sum_{\substack{k+h=2\\ k \text{ odd, } h \text{ odd}}}^{l} a_{ij,2}a_{kh,1}r^{i+j+k+h+1}\hat{P}_{ijkh}\\
&- \sum_{\substack{i+j=2\\ i \text{ odd, } j \text{ odd}}}^{m} \sum_{\substack{k+h=1\\ k \text{ even, } h \text{ odd}}}^{n} b_{ij,2}a_{kh,2}r^{i+j+k+h}\tilde{N}_{ijkh}\\
&- \sum_{\substack{i+j=0\\ i \text{ even, } j \text{ even}}}^{m} \sum_{\substack{k+h=1\\ k \text{ odd, } h \text{ even}}}^{n} b_{ij,2}a_{kh,2}r^{i+j+k+h}N_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ even, } j \text{ odd}}}^{m} \sum_{\substack{k+h=2\\ k \text{ odd, } h \text{ odd}}}^{n} b_{ij,2}a_{kh,2}r^{i+j+k+h}\hat{N}_{ijkh}\\
&- \sum_{\substack{i+j=0\\ i \text{ even, } j \text{ even}}}^{m} \sum_{\substack{k+h=2\\ k \text{ odd, } h \text{ odd}}}^{m} b_{ij,2}b_{kh,2}r^{i+j+k+h-1}Q_{ijkh}\\
&- \sum_{\substack{i+j=2\\ i \text{ odd, } j \text{ odd}}}^{m} \sum_{\substack{k+h=0\\ k \text{ even, } h \text{ even}}}^{m} b_{ij,2}b_{kh,2}r^{i+j+k+h-1}Q_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ odd, } j \text{ even}}}^{m} \sum_{\substack{k+h=1\\ k \text{ even, } h \text{ odd}}}^{m} b_{ij,2}b_{kh,2}r^{i+j+k+h-1}\tilde{Q}_{ijkh}
\end{aligned}
$$

$$
\begin{aligned}
&- \sum_{\substack{i+j=1\\ i \text{ even},\, j \text{ odd}}}^{m} \sum_{\substack{k+h=1\\ k \text{ odd},\, h \text{ even}}}^{m} b_{ij,2} b_{kh,2} r^{i+j+k+h-1} \tilde{Q}_{ijkh}\\
&+ \sum_{\substack{i+j=0\\ i \text{ even},\, j \text{ even}}}^{m} \sum_{\substack{k+h=1\\ k \text{ even},\, h \text{ odd}}}^{l} b_{ij,2} a_{kh,1} r^{i+j+k+h} R_{ijkh}\\
&+ \sum_{\substack{i+j=1\\ i \text{ even},\, j \text{ odd}}}^{m} \sum_{\substack{k+h=0\\ k \text{ even},\, h \text{ even}}}^{l} b_{ij,2} a_{kh,1} r^{i+j+k+h} R_{ijkh}\\
&+ \sum_{\substack{i+j=1\\ i \text{ odd},\, j \text{ even}}}^{m} \sum_{\substack{k+h=2\\ k \text{ odd},\, h \text{ odd}}}^{l} b_{ij,2} a_{kh,1} r^{i+j+k+h} \tilde{R}_{ijkh}\\
&+ \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{m} \sum_{\substack{k+h=1\\ k \text{ odd},\, h \text{ even}}}^{l} b_{ij,2} a_{kh,1} r^{i+j+k+h} \tilde{R}_{ijkh}\\
&- \sum_{i+j=1//i \text{ even},\, j \text{ odd}}^{l} \sum_{\substack{k+h=1\\ k \text{ even},\, h \text{ odd}}}^{n} a_{ij,1} a_{kh,2} r^{i+j+k+h+1} T_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ odd},\, j \text{ even}}}^{l} \sum_{\substack{k+h=1\\ k \text{ odd},\, h \text{ even}}}^{n} a_{ij,1} a_{kh,2} r^{i+j+k+h+1} \tilde{T}_{ijkh}\\
&- \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{l} \sum_{\substack{k+h=2\\ k \text{ odd},\, h \text{ odd}}}^{n} a_{ij,1} a_{kh,2} r^{i+j+k+h+1} \hat{T}_{ijkh}\\
&- \sum_{\substack{i+j=0\\ i \text{ even},\, j \text{ even}}}^{l} \sum_{\substack{k+h=1\\ k \text{ even},\, h \text{ odd}}}^{m} a_{ij,1} b_{kh,2} r^{i+j+k+h} U_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ even},\, j \text{ odd}}}^{l} \sum_{\substack{k+h=0\\ k \text{ even},\, h \text{ even}}}^{m} a_{ij,1} b_{kh,2} r^{i+j+k+h} U_{ijkh}\\
&- \sum_{\substack{i+j=1\\ i \text{ odd},\, j \text{ even}}}^{l} \sum_{\substack{k+h=2\\ k \text{ odd},\, h \text{ odd}}}^{m} a_{ij,1} b_{kh,2} r^{i+j+k+h} \tilde{U}_{ijkh}\\
&- \sum_{\substack{i+j=2\\ i \text{ odd},\, j \text{ odd}}}^{l} \sum_{\substack{k+h=1\\ k \text{ odd},\, h \text{ even}}}^{m} a_{ij,1} b_{kh,2} r^{i+j+k+h} \tilde{U}_{ijkh}\\
&+ \sum_{\substack{i+j=0\\ i \text{ even},\, j \text{ even}}}^{l} \sum_{\substack{k+h=2\\ k \text{ odd},\, h \text{ odd}}}^{l} a_{ij,1} a_{kh,1} r^{i+j+k+h+1} V_{ijkh}
\end{aligned}
$$

$$+ \sum_{\substack{i+j=1 \\ i \text{ even, } j \text{ odd}}}^{l} \sum_{\substack{k+h=1 \\ k \text{ odd, } h \text{ even}}}^{l} a_{ij,1} a_{kh,1} r^{i+j+k+h+1} \tilde{V}_{ijkh}$$
$$+ \sum_{\substack{i+j=1 \\ i \text{ odd, } j \text{ even}}}^{l} \sum_{\substack{k+h=1 \\ k \text{ even, } h \text{ odd}}}^{l} a_{ij,1} a_{kh,1} r^{i+j+k+h+1} \tilde{V}_{ijkh}\Big].$$

Finally, we obtain $H_1(r) + H_2(r)$ is a polynomial in the variable r^2 of the form

$$H_1(r) + H_2(r) = r\Big[P_1(r^2) + r^2 P_2(r^2) + r^4 P_3(r^2)\Big],$$

where $P_1(r^2)$ is a polynomial in the variable r^2 of degree

$$\begin{aligned}\lambda_1 = \max\Big\{&[\frac{O(m)+E(l)-1}{2}], [\frac{E(m)+O(l)-1}{2}], [\frac{O(n)+O(l)-2}{2}],\\ &m-1, [\frac{E(n)}{2}], [\frac{E(m)+O(n)-1}{2}]\Big\},\end{aligned}$$

$P_2(r^2)$ is a polynomial in the variable r^2 of degree

$$\begin{aligned}\lambda_2 = \max\Big\{&O(n)-1, [\frac{E(m)+O(n)-3}{2}], [\frac{O(m)+E(n)-3}{2}], [\frac{O(n)+O(l)-2}{2}],\\ &l-1, E(m)-2, [\frac{E(m)+O(n)-3}{2}], [\frac{O(m)+E(l)-3}{2}], [\frac{E(n)+O(m)-3}{2}]\Big\},\end{aligned}$$

$P_3(r^2)$ is a polynomial in the variable r^2 of degree

$$\lambda_3 = [\frac{E(n)+E(l)-4}{2}],$$

where $O(i)$ is the largest odd integer less than or equal to i, $E(i)$ is the largest even integer less than or equal to i and $[\cdot]$ denotes the integer part function. Then

$$F_{20} = \frac{1}{2\pi} r\Big[P_1(r^2) + r^2 P_2(r^2) + r^4 P_3(r^2)\Big].$$

To find the real positive roots of F_{20} we must find the zeros of a polynomial in r^2 of degree $\lambda = \max\{\lambda_1, \lambda_2 + 1, \lambda_3 + 2\}$. This yields that F_{20} has at most λ real positive roots. Hence, the Theorem 1.2 is proved.

Moreover, we can choose the coefficients $a_{ij,1}, a_{ij,2}, b_{ij,2}, C_{ij,1}, c_{ij,2}, d_{ij,2}$ in such a way that F_{20} has exactly λ real positive roots. In fact, we consider the example

$$\begin{aligned}\dot{x} &= y - \varepsilon(1 + xy - 2x^2 - x)y - \varepsilon^2(-2xy - y^2)y,\\ \dot{y} &= -x - \varepsilon(4xy + (-x^2 - y)y) - \varepsilon^2(x^2 - y + (-xy + y^2 - x)y),\end{aligned} \tag{4.2}$$

in polar coordinates. System (4.2) becomes

$$\begin{aligned}\dot{r} = \varepsilon\Big[&-\cos(\theta)\sin(\theta)r + (\sin(\theta) - 4\sin^2(\theta)\cos(\theta))r^2 + 2\sin(\theta)\cos^3(\theta)r^3\Big]\\ &+ \varepsilon^2\Big[\sin^2(\theta)r + (-\sin(\theta)\cos^2(\theta) + \sin^2(\theta)\cos(\theta))r^2 + (2\cos(\theta)\sin(\theta)\\ &+ 3\cos^2(\theta)\sin^2(\theta) - 2\cos^3 -(\theta)\sin(\theta) - \sin^2(\theta))r^3\Big],\end{aligned}$$

$$\begin{aligned}\dot\theta &= -1+\varepsilon\Big[1-\cos^2(\theta)-4\sin(\theta)\cos^2 r+(\cos(\theta)\sin(\theta)+2\cos^4(\theta)\\&\quad -2\cos^2(\theta))r^2\Big]+\varepsilon^2\Big[\cos(\theta)\sin(\theta)+(-\cos^3(\theta)+\sin(\theta)\cos^2(\theta))r\\&\quad +(3\sin(\theta)\cos^3(\theta)-3\cos(\theta)\sin(\theta)-2\cos^4(\theta)+3\cos^2(\theta)-1)r^2\Big].\end{aligned}$$

Taking θ as the new independent variable, this system becomes

$$\frac{dr}{d\theta}=\varepsilon F_1(r,\theta)+\varepsilon^2F_2(r,\theta)+O(\varepsilon^3),$$

where

$$F_1(r,\theta)=\cos(\theta)r\sin(\theta)-r^2\sin(\theta)-2r^3\sin(\theta)\cos^3(\theta)+4r^2\sin^2(\theta)\cos(\theta),$$

$$\begin{aligned}F_2(r,\theta)&=\cos(s)r\sin(s)-r^2\sin(s)-\cos^3(s)r\sin(s)-4\sin(s)r^5\cos^7(s)\\&\quad+4r^5\sin(s)\cos^5(s)-6r^4\sin(s)\cos^4(s)+6r^4\sin(s)\cos^2(s)\\&\quad-18r^3\sin(s)\cos^3(s)-2\sin(s)r^3\cos(s)+20\cos^5(s)r^3\sin(s)\\&\quad+2r^2\sin(s)\cos^2(s)-r^4\sin^2(s)\cos(s)+16r^4\sin^2(s)\cos^5(s)\\&\quad-2r^5\sin^2(s)\cos^4(s)-8r^2\sin^2(s)\cos^3(s)+2r^3\sin^2(s)\cos^2(s)\\&\quad+r^3\sin^2(s)-r\sin^2(s)+3r^2\sin^2(s)\cos(s)-8r^4\sin^2(s)\cos^3(s).\end{aligned}$$

We have that $F_{10}(r)$ is identically zero, so to look for the limit cycles, we must solve the equation $F_{20}(r)=0$ which is equivalent to

$$r(-\frac{1}{8}r^4-\frac{3}{4}r^3-\frac{1}{2})=0.$$

This equation has exactly the two positive roots

$$r_1=\frac{1}{2}\sqrt{10}+\frac{1}{2}\sqrt{2},\quad r_2=\frac{1}{2}\sqrt{10}-\frac{1}{2}\sqrt{2}.$$

So system (4.2) has exactly two limit cycles bifurcating from the periodic orbits of the linear center $\dot x=y$, $\dot y=-x$.

5. Appendix

In this appendix, we recall some formulae used during this article; for more details see [1]. For $i\ge 0$ and $j\ge 0$, we have

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)d\theta=\begin{cases}\pi\alpha_{ij} & \text{if } i \text{ even, } j \text{ even,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\cos((2h)\theta)d\theta=0\quad\text{if } i \text{ odd or } j \text{ odd, } h=0,1,2,\dots$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+1}(\theta)d\theta=\begin{cases}\pi\beta_{ij} & \text{if } i \text{ even, } j \text{ odd,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\times\theta d\theta=\begin{cases}\pi\gamma_{ij} & \text{if } i \text{ odd, } j \text{ odd,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi}\cos^{i+1}(\theta)\sin^{j+1}(\theta)\times\theta d\theta=\begin{cases}\pi\mu_{ij} & \text{if } i \text{ even, } j \text{ even,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+1}(\theta)\times\theta d\theta=\begin{cases}\pi\sigma_{ij} & \text{if } i \text{ odd, } j \text{ even,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi}\cos^{i+1}(\theta)\sin^{j+1}(\theta)d\theta=\begin{cases}\pi\delta_{ij} & \text{if } i \text{ odd, } j \text{ odd,}\\ 0 & \text{otherwise,}\end{cases}$$

$$\begin{aligned}&\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\cos((2h+1)\theta)d\theta\\ &=\begin{cases}\pi B_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,n\\ \pi\tilde{B}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,m\\ \pi\hat{B}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\sin((2h+1)\theta)d\theta\\ &=\begin{cases}\pi A_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,n\\ \pi\tilde{A}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,m\\ \pi\hat{A}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi}\cos^i(\theta)\sin^{j+2}(\theta)\sin((2h)\theta)d\theta\\ &=\begin{cases}\pi C_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,m\\ \pi\tilde{C}_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi}\cos^i(\theta)\sin^{j+1}(\theta)\sin((2h+1)\theta)d\theta\\ &=\begin{cases}\pi D_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ even, } h=0,1,2,\dots,n\\ \pi\tilde{D}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ even, } h=0,1,2,\dots,m\\ \pi\hat{D}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ even, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

$$\begin{aligned}&\int_0^{2\pi}\cos^i(\theta)\sin^{j+1}(\theta)\cos((2h+1)\theta)d\theta\\ &=\begin{cases}\pi E_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,n\\ \pi\tilde{E}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,m\\ \pi\hat{E}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}\end{aligned}$$

$$\int_0^{2\pi}\cos^i(\theta)\sin^{j+1}(\theta)\cos((2h)\theta)d\theta=\begin{cases}\pi F_{ij}^{2h} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,n\\ \pi\tilde{F}_{ij}^{2h} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,m\\ \pi\hat{F}_{ij}^{2h} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,l\\ 0 & \text{otherwise,}\end{cases}$$

$$\int_0^{2\pi} \cos^i(\theta)\sin^{j+1}(\theta)\sin((2h)\theta)d\theta = \begin{cases} \pi G_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,m \\ \pi \tilde{G}_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,l \\ 0 & \text{otherwise,} \end{cases}$$

$$\begin{aligned} &\int_0^{2\pi} \cos^{i+1}(\theta)\sin^{j+1}(\theta)\sin((2h+1)\theta)d\theta \\ &= \begin{cases} \pi H_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,n \\ \pi \tilde{H}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,m \\ \pi \hat{H}_{ij}^{2h+1} & \text{if } i \text{ odd, } j \text{ even, } h=0,1,2,\dots,l \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} &\int_0^{2\pi} \cos^{i+1}(\theta)\sin^{j+1}(\theta)\cos((2h+1)\theta)d\theta \\ &= \begin{cases} \pi I_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,n \\ \pi \tilde{I}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,m \\ \pi \hat{I}_{ij}^{2h+1} & \text{if } i \text{ even, } j \text{ odd, } h=0,1,2,\dots,l \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\int_0^{2\pi} \cos^{i+1}(\theta)\sin^{j+1}(\theta)\cos((2h)\theta)d\theta = \begin{cases} \pi K_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,n \\ \pi \tilde{K}_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,m \\ \pi \hat{K}_{ij}^{2h} & \text{if } i \text{ odd, } j \text{ odd, } h=0,1,2,\dots,l \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+1}(\theta)\sin^{j+1}(\theta)\sin((2h)\theta)d\theta = \begin{cases} \pi L_{ij}^{2h} & \text{if } i \text{ even, } j \text{ even, } h=0,1,2,\dots,m \\ \pi \tilde{L}_{ij}^{2h} & \text{if } i \text{ even, } j \text{ even, } h=0,1,2,\dots,l \\ 0 & \text{otherwise,} \end{cases}$$

where $\alpha_{ij},\ \beta_{ij},\ \gamma_{ij}, \delta_{ij},\ A_{ij}^{2h+1},\ \tilde{A}_{ij}^{2h+1},\ \hat{A}_{ij}^{2h+1},\ B_{ij}^{2h+1},\ \tilde{B}_{ij}^{2h+1},\ \hat{B}_{ij}^{2h+1},\ C_{ij}^{2h},\ \tilde{C}_{ij}^{2h}$, $D_{ij}^{2h+1},\ \tilde{D}_{ij}^{2h+1},\ \hat{D}_{ij}^{2h+1},\ E_{ij}^{2h+1},\ \tilde{E}_{ij}^{2h+1},\ \hat{E}_{ij}^{2h+1},\ F_{ij}^{2h+1},\ \tilde{F}_{ij}^{2h+1},\ \hat{F}_{ij}^{2h+1},\ G_{ij}^{2h},\ \tilde{G}_{ij}^{2h}$, $H_{ij}^{2h+1},\ \tilde{H}_{ij}^{2h+1},\ \hat{H}_{ij}^{2h+1},\ I_{ij}^{2h+1},\ \tilde{I}_{ij}^{2h+1},\ \hat{I}_{ij}^{2h+1},\ K_{ij}^{2h},\ \tilde{K}_{ij}^{2h},\ \hat{K}_{ij}^{2h},\ L_{ij}^{2h}$ and $\tilde{L}_{ij}^{2h}$ are non-zero constants.

$$\begin{aligned} &\int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)d\theta \\ &= \begin{cases} \pi M_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ even, } h \text{ odd,} \\ \pi \tilde{M}_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ odd, } h \text{ even,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\begin{aligned} &\int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+h+2}(\theta)d\theta \\ &= \begin{cases} \pi N_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ even, } h \text{ even,} \\ & \text{or } i \text{ even, } j \text{ even, } k \text{ odd, } h \text{ even,} \\ \pi \tilde{N}_{ijkh} & \text{if } i \text{ odd, } j \text{ odd, } k \text{ even, } h \text{ is odd,} \\ \pi \hat{N}_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ odd, } h \text{ odd,} \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

$$\int_0^{2\pi} \cos^{i+k}(\theta)\sin^{j+h+4}(\theta)d\theta = \begin{cases} \pi P_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ odd, } h \text{ even,} \\ \pi \tilde{P}_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ even, } h \text{ odd,} \\ \pi \hat{P}_{ijkh} & \text{if } i \text{ odd, } j \text{ odd, } k \text{ odd, } h \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+h+1}(\theta)d\theta = \begin{cases} \pi Q_{ijkh} & \text{if } i \text{ even, } j \text{ even, } k \text{ odd, } h \text{ odd,} \\ & \text{or } i \text{ odd, } j \text{ odd, } k \text{ even, } h \text{ even,} \\ \pi \tilde{Q}_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ odd, } h \text{ even,} \\ & \text{or } i \text{ odd, } j \text{ even, } k \text{ even, } h \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k}(\theta)\sin^{j+h+3}(\theta)d\theta = \begin{cases} \pi R_{ijkh} & \text{if } i \text{ even, } j \text{ even, } k \text{ even, } h \text{ odd,} \\ & \text{or } i \text{ even, } j \text{ odd, } k \text{ even, } h \text{ even,} \\ \pi \tilde{R}_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ odd, } h \text{ odd,} \\ & \text{or } i \text{ odd, } j \text{ odd, } k \text{ odd, } h \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+2}(\theta)\sin^{j+h+2}(\theta)d\theta = \begin{cases} \pi T_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ even, } h \text{ odd,} \\ \pi \tilde{T}_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ is odd, } h \text{ is even,} \\ \pi \hat{T}_{ijkh} & \text{if } i \text{ odd, } j \text{ odd, } k \text{ odd, } h \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+2}(\theta)\sin^{j+h+1}(\theta)d\theta = \begin{cases} \pi U_{ijkh} & \text{if } i \text{ even, } j \text{ even, } k \text{ even, } h \text{ odd,} \\ & \text{or } i \text{ even, } j \text{ odd, } k \text{ even, } h \text{ even,} \\ \pi \tilde{U}_{ijkh} & \text{if } i \text{ odd, } j \text{ even, } k \text{ odd, } h \text{ odd,} \\ & \text{or } i \text{ odd, } j \text{ odd, } k \text{ odd, } h \text{ even,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_0^{2\pi} \cos^{i+k+1}(\theta)\sin^{j+h+3}(\theta)d\theta = \begin{cases} \pi V_{ijkh} & \text{if } i \text{ even, } j \text{ even, } k \text{ odd, } h \text{ odd,} \\ \pi \tilde{V}_{ijkh} & \text{if } i \text{ even, } j \text{ odd, } k \text{ odd, } h \text{ even,} \\ & \text{or } i \text{ odd, } j \text{ even, } k \text{ even, } h \text{ odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where M_{ijkh}, $\tilde{M}_{ijkh}$, N_{ijkh}, $\tilde{N}_{ijkh}$, $\hat{N}_{ijkh}$, Q_{ijkh}, $\tilde{Q}_{ijkh}$, P_{ijkh}, $\tilde{P}_{ijkh}$, $\hat{P}_{ijkh}$, R_{ijkh}, $\tilde{R}ijkh$, T_{ijkh}, $\tilde{T}_{ijkh}$, $\hat{T}_{ijkh}$, U_{ijkh}, $\tilde{U}_{ijkh}$, V_{ijkh}, $\tilde{V}_{ijkh}$ are non-zero constants.

Conclusion. This article concerns the second part of the 16th Hilbert problem in which we study the bifurcation of limit cycles from the periodic orbits of a linear center when we perturb it inside a general class of all polynomial differential systems. We provide an accurate upper bound of the maximum number of limit cycles that this class of systems can have and we give an example which illustrates that this bound can be reached. We would like to stress that although this work ultimately focuses a general class of polynomial differential systems, the "Averaging method" summarized in section 2 can be adapted to the study of other polynomial systems. However, the difficulty lies in the complicated form of the averaged function of second order obtained when the first order one is identically zero. The bifurcation of limit cycles from isochronous center will be the subject of a future work.

Acknowledgements. We want to thank the reviewers all their comments and suggestions which help us to improve the presentation of this article.

References

[1] Abramowitz, M. Stegun, I.; *Handbook of mathematical function with formulas, graphs, and mathematical tables.* National Bureau of Standards Applied Mathematics Series, no. 55. Washington, DC:US Government, Printing Office. 1964
[2] A. Buică, J. Llibre; *Averaging methods for finding periodic orbits via Brouwer degree,* Bull Sci Math 2004; 128: 7-22.
[3] L. Chengzy, J. Llibre; *Uniqucness of limit cycles for Liénard differential equations of degree four,* J. Differ Equ 2012; 252: 3142-62.
[4] D. Hilbert; *Mathematische Problems,* Lecture in: Second Internat. Congr. Math. Paris, 1900, Nachr. Ges. Wiss. Gttingen Math. Phys. Ki 5 (1900), 253-297; English transl. Bull. Amer. Soc. 8 (1902), 437-479.
[5] Y. Ilyashenko; *Centennial history of Hilbert's 16th problem,* Bull. Amer. Math. Soc. 39 (2002), 301-354.
[6] A. Liénard; *Étude des oscillations entrenues,* Revue Gnlectr 1928; 23: 946-54.
[7] A. Lins, W. De Melo, CC. Pugh; *On Liénard's equation,* Lecture notes in mathematics, Vol. 597. Berlin: Springer; 1977. p.335-357.
[8] C. Li, J. Llibre; *Uniqueness of limit cycles for Liénard differential equations of degree four,* Diff Equ 2012; 252: 3142-62.
[9] J. Llibre, A. C. Mereu A, M. A. Teixeira; *Limit cycles of the generalized polynomial Liénard differential equations,* Math Proc Camb phil Soc 2010; 148: 363-83.
[10] J. Llibre, C. Valls; *On the number of limit cycles of a class of polynomial differential systems,* Proc A: R Soc., 2012; 468: 2347-60.
[11] J. Llibre, C. Valls; *Limit cycles for a generalization of polynomial Liénard differential systems,* Chaos, Soltions and Fractals 46 (2013), 65-74.
[12] N. G. Lloyd; *Limit cycles of polynomial systems-some recent developments.* London mathematical society lecture note series, Vol. 127. Cambridge University Press; 1988; p: 192-234.
[13] N. G. Lloyd, S. Lynch; *Small-amplitude limit cycles of certain Liénard systems,* Proc R Soc Lond Ser A 1988; 418: 199-208.
[14] S. Lynch; *limit cycles of generalized Liénard equation,* Appl Math Lett 1995; 8:15-7.
[15] GS. Rychkov; *The maximum number of limit cycle of the system* $\dot{x} = y - a_1x^3 - a_2x^5, \dot{y} = -x$ *is two,* Differ Uravn 1975; 11: 380-91.
[16] J. A. Sanders, F. Verhulst; *Averaging methods in nonlinear dynamical systems,* Applied Mathematical Sci, 59, Springer-Verlag, New york, 1985.
[17] F. Verhulst; *Nonlinear differential equations and dynamical systems,* Universitex, Springer-Verlag, Berlin, 1996.

Selma Ellaggoune
Laboratory of Applied Mathematics and Modeling, University 8 Mai 1945, Guelma, Algeria
E-mail address: sellaggoune@gmail.com

Sabrina Badi
Laboratory of Applied Mathematics and Modeling, University 8 Mai 1945, Guelma, Algeria
E-mail address: badisabrina@yahoo.fr