NONLINEAR PARABOLIC EQUATIONS WITH BLOWING-UP COEFFICIENTS WITH RESPECT TO THE UNKNOWN AND WITH SOFT MEASURE DATA

KHALED ZAKI, HICHAM REDWANE

Abstract. We establish the existence of solutions for the nonlinear parabolic problem with Dirichlet homogeneous boundary conditions,
\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial u}{\partial x_i} \right) = \mu, \quad u(t = 0) = u_0,
\]
in a bounded domain. The coefficients \(d_i(s)\) are continuous on an interval \([-\infty, m]\) of \(\mathbb{R}\) with value in \(\mathbb{R}_+ \cup \{+\infty\}\), \(d_i(s) \geq \alpha > 0\), and such that there exists an index \(p\) such that \(d_p(s)\) blows up at a finite value \(m\) of the unknown \(u\), and \(\mu\) is a diffuse measure.

1. Introduction

In this paper we study the existence of solutions of the problem
\[
\frac{\partial u}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial u}{\partial x_i} \right) = \mu \quad \text{in } Q, \tag{1.1}
\]
\[
u(t = 0) = u_0 \quad \text{in } \Omega, \tag{1.2}
\]
\[
u = 0 \quad \text{on } \partial \Omega \times (0, T), \tag{1.3}
\]
where \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\) \((N \geq 1)\), \(T\) is a positive real number, and we have set \(Q\) the cylinder \(\Omega \times (0, T)\) and \(\partial \Omega \times (0, T)\) its lateral surface. The coefficients \(d_i(s)\) are continuous on an interval \([-\infty, m]\) of \(\mathbb{R}\) (with \(m > 0\)) with value in \(\mathbb{R}^+ \cup \{+\infty\}\), \(d_i(s) \geq \alpha > 0\), and such that there exists an index \(p\) such that \(\lim_{s \to m^-} d_p(s) = +\infty\), and where \(u_0 \in L^1(\Omega), \ u_0 \leq m\) a.e. in \(\Omega\) and \(\mu\) is a measure on \(Q\) with bounded total variation.

When problem (1.1)-(1.3) is studied, the a priori estimates on the above problem do not lead in general to the existence of a weak solution (i.e. in the distributional sense), there are mainly two type of difficulties in studying problem (1.1)-(1.3). One consists to define in a proper way the term \(d_p(u) \frac{\partial u}{\partial x_p}\) on the subset \(\{x, t) \in Q : u(x, t) = m\}\) of \(Q\) on which \(d_p(u) = +\infty\). As an example, one can not set in general \(d_p(u) \frac{\partial u}{\partial x_p} = 0\) on \(\{x, t) \in Q : u(x, t) = m\}\) to obtain the equation in the sense of distributions.

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The second difficulty is represented here by the presence of an \( L^1 \) initial datum and a measure as right-hand side term in (1.1). The measure \( \mu \) is just assumed to be bounded total variation over \( Q \) that do not charge the sets of zero \( p \)-capacity (see Section 2 for the definition), the so called diffuse measures or soft measures, and we will use the symbol \( \mathcal{M}_0(Q) \) to denote them.

To overcome this difficulty we use the framework of renormalized solutions. This notion was introduced by Lions and DiPerna [14] for the study of Boltzmann equation. This notion was then adapted to elliptic version of (1.1)-(1.3) in Boccardo, Diaz, Giachetti, Murat [12], Lions and F. Murat [22], and Murat [22, 23]. At the same the equivalent notion of entropy solutions was developed independently by Bénilan and al. [1] for the study of nonlinear elliptic problems.

The existence of a renormalized solution of (1.1)-(1.3) was proved in [2] in the stationary case where \( \mu \in L^2(\Omega) \). In the stationary and evolution cases of \( u_t - \text{div}(A(x,t,u)\nabla u) = f \) in \( Q \), where the matrix \( A(x,t,s) \) blows up (uniformly with respect to \( (x,t) \)) as \( s \to m^- \) and where \( f \in L^1(Q) \), the existence of renormalized solution was proved by Blanchard, Guibé and Redwane in [3].

The existence and uniqueness of renormalized solution of (1.1)-(1.3) was proved in [4] in the case where
\[
\sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial v}{\partial x_i} \right) = f + F \quad \text{in} \quad Q,
\]

Since \( g \notin L^\infty(Q) \) in general and \( \lim_{s \to m^-} d_p(s) = +\infty \), then the term \( d_p(v+g) \) can not be easily handled. To overcome this difficulty, we use in this paper the following approximation property for the measure \( \mu \) (see Theorem 2.2).

Indeed, every \( \mu \in \mathcal{M}_0(Q) \) can be strongly approximated by measures which admit decomposition (1.4) with \( g \in L^\infty(Q) \) (see [17, Theorem 1.1]).

A large number of papers was then devoted to the study the existence of renormalized solution of parabolic problems with rough data under various assumptions and in different contexts: for a review on classical results, see [5, 6, 8, 9, 18, 19, 20, 24, 25, 26, 30, 32].

We organize this article as follows. In Section 2 we give some preliminaries and, in particular, we provide the definition of parabolic capacity and some basic properties of diffuse measures. Section 3 is devoted to specifying the assumptions on \( d_i, u_0 \) and \( \mu \). We also give the definition of a renormalized solution of (1.1)-(1.3). In Section 4 we establish (Theorem 4.1) the existence of such a solution. In Section 5 (Appendix) we prove Theorem 2.3 that will be a key point in the existence result.
2. Preliminaries on Parabolic Capacity and Diffuse Measures

We recall the notion of parabolic $p$-capacity (with $p = 2$) associated to our problem (for further details see [29, 15]). Let $Q = \Omega \times (0, T)$ for any fixed $T > 0$, and let us recall that $W = \{u \in L^2(0, T; H^1_0(\Omega)) : u_t \in L^2(0, T; H^{-1}(\Omega))\}$, endowed with its natural norm $\|u\|_{L^2(0, T; H^1_0(\Omega))} + \|u_t\|_{L^2(0, T; H^{-1}(\Omega))}$, remark that $W$ is continuously embedded in $C([0, T]; L^2(\Omega))$ and $C_\infty^\infty([0, T] \times \Omega)$ is dense in $W$.

Let $U \subseteq Q$ an open set, we define the parabolic 2-capacity of $U$ as

$$\text{cap}_2(U) = \inf\{\|u\|_W : u \in W, u \geq \chi_U \text{ a.e. in } Q\},$$

where as usual we set $\inf\{0\} = +\infty$. Then for any Borel set $B \subseteq Q$ we define

$$\text{cap}_2(B) = \inf\{\text{cap}_2(U) : U \text{ is open subset of } Q, B \subseteq U\}.$$

We denote by $\mathcal{M}_b(Q)$ the set of all Radon measures with bounded variation on $Q$, while, as we already mentioned, $\mathcal{M}_0(Q)$ denotes the set of all measures with bounded variation over $Q$ that do not charge the sets of zero 2-capacity, that is if $\mu \in \mathcal{M}_0(Q)$, then $\mu(E) = 0$, for all $E \subseteq Q$ such that $\text{cap}_2(E) = 0$.

In [15] the authors proved the following decomposition theorem.

**Theorem 2.1.** Let $\mu$ be a bounded measure on $Q$. If $\mu \in \mathcal{M}_0(Q)$ then there exists $(f, F, g)$ such that $f \in L^1(Q)$, $F \in L^1(0, T; H^{-1}(\Omega))$, $g \in L^1(0, T; H^1_0(\Omega))$ and

$$\int_Q \phi d\mu = \int_Q f \phi dx dt + \int_0^T \langle F, \phi \rangle dt - \int_0^T \langle \phi_t, g \rangle dt \quad \phi \in C_\infty^\infty([0, T] \times \Omega).$$

Such a triplet $(f, F, g)$ will be called a decomposition of $\mu$.

Note that the decomposition of $\mu$ is not uniquely determined. In [17] the authors proved the following approximation of diffuse measures theorem.

**Theorem 2.2.** Let $\mu \in \mathcal{M}_0(Q)$, then, for every $\varepsilon > 0$ there exists $\nu \in \mathcal{M}_0(Q)$ such that

$$\|\mu - \nu\|_{\mathcal{M}(Q)} \leq \varepsilon \quad \text{and} \quad \nu = w_t - \Delta w \text{ in } \mathcal{D}'(Q),$$

where $w \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(Q)$.

The following theorem will be a key point in the existence result given in the next section. The proof follows the arguments in [27, Theorem 1.2].

**Theorem 2.3.** Let $d_i \in C^0(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for every $i \in \{1, \ldots, N\}$, $\mu \in \mathcal{M}_0(Q) \cap L^2(0, T; H^{-1}(\Omega))$ and $u_0 \in L^2(\Omega)$, let $u \in W$ be the (unique) weak solution of

$$\frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i}(d_i(u) \frac{\partial u}{\partial x_i}) = \mu \quad \text{in } Q,$$

$$u = 0 \quad \text{on } (0, T) \times \partial \Omega,$$

$$u(t = 0) = u_0 \quad \text{in } \Omega.$$  \hfill (2.1)

Then

$$\text{cap}_2\{|u| > K\} \leq \frac{C}{\sqrt{K}} \quad \forall K \geq 1,$$

where $C > 0$ is a constant depending on $\|\mu\|_{\mathcal{M}(Q)}$, $\|u_0\|_{L^2(\Omega)}$.

The proof of the above theorem is postponed to the Appendix in Section 5.
Definition 2.4. A sequence of measures \((\mu_n)\) in \(Q\) is equidiffuse if for every \(\eta > 0\) there exists \(\delta > 0\) such that
\[
\text{cap}_2(E) < \delta \iff |\mu_n|(E) < \eta \quad \forall \ n \geq 1.
\]

The following result is proved in [27]:

Lemma 2.5. Let \(\rho_n\) be a sequence of mollifiers on \(Q\). If \(\mu \in M_0(Q)\), then the sequence \((\rho_n * \mu_n)\) is equidiffuse.

Here is some notation we will use throughout the paper. For any nonnegative real number \(K\) we denote by \(T_K(r) = \min(K, \max(r, -K))\) the truncation function at level \(K\). For every \(r \in \mathbb{R}\), let
\[
T_K(z) = \int_0^z T_K(s) \, ds
\]

We consider the following smooth approximation of \(T_K(s)\): for \(m > 0, \eta \in [0, 1]\) and \(\sigma \in [0, 1]\), we define \(S_{K,\sigma}^m : \mathbb{R} \to \mathbb{R}\) by
\[
S_{K,\sigma}^m(s) = \begin{cases} 
1 & \text{if } -K + \eta \leq s \leq m - 2\sigma, \\
0 & \text{if } s \leq -K \text{ and } s \geq m - \sigma, \\
\text{affine} & \text{otherwise},
\end{cases} \quad (2.2)
\]

and let us denote \(T_{K,\sigma}^m(z) = \int_0^z S_{K,\sigma}^m(s) \, ds\) and
\[
T_K^m(s) = \begin{cases} 
s & \text{if } -K \leq s \leq m, \\
-K & \text{if } s \leq -K, \\
m & \text{if } s \geq m.
\end{cases}
\]

By \(\langle \cdot, \cdot \rangle\) we mean the duality between suitable spaces in which function are involved. In particular we will consider both the duality between \(H_0^1(\Omega)\) and \(H^{-1}(\Omega)\) and the duality between \(H_0^1(\Omega) \cap L^\infty(\Omega)\) and \(H^{-1}(\Omega) + L^1(\Omega)\).

3. Main assumptions and definition of renormalized solution

Throughout the paper, we assume that the following assumptions hold: \(\Omega\) is a bounded open set on \(\mathbb{R}^N\) \((N \geq 2)\), \(T > 0\) is given and we set \(Q = \Omega \times (0, T)\).

\(d_i \in C^0(] - \infty, m[; \mathbb{R}^+ \cup \{+\infty\})\) with \(d_i(s) < +\infty \forall s < m, \forall i \in \{1, \ldots, N\};\)
\[
\exists \alpha > 0 \text{ such that } d_i(s) \geq \alpha \forall i \in \{1, \ldots, N\}, \forall s \in ] - \infty, m[; \quad (3.1)
\]

\(\exists \rho \in \{1, \ldots, N\} \text{ such that } \lim_{s \to -m} d_\rho(s) = +\infty \text{ and } \int_0^m d_\rho(s) \, ds < +\infty; \quad (3.2)
\]

\(\mu \in M_0(Q); \quad (3.3)
\]

\(u_0 \in L^1(\Omega)\) such that \(u_0 \leq m\) a.e. in \(\Omega). \quad (3.4)

The definition of a renormalized solution for Problem \([1.1]-[1.3]\) is as follows.

Definition 3.1. Let \(\mu \in M_0(Q)\). A function \(u \in L^1(\Omega)\) is a renormalized solution of Problem \([1.1]-[1.3]\) if
\[
u \leq m \text{ a.e. in } Q, \quad T_K(u) \in L^2(0, T; H_0^1(\Omega)) \quad \forall K > 0; \quad (3.6)
\]

\[
d_i(u) \frac{\partial T_K^m(u)}{\partial x_i} \chi_{\{u < m\}} \in L^2(\Omega) \quad \forall K > 0, \forall i \in \{1, \ldots, N\}, \quad (3.7)
\]
if there exists a sequence of nonnegative measures \((\Lambda_K) \in \mathcal{M}(Q)\) and a nonnegative measure \(\Gamma \in \mathcal{M}(Q)\) such that
\[
\lim_{K \to +\infty} \|\Lambda_K\|_{\mathcal{M}(Q)} = 0,
\]
(3.8)
and if, for every \(K > 0\),
\[
\frac{\partial T^m_K(u)}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial T^m_K(u)}{\partial x_i} \chi_{\{u < m\}} \right) = \mu + \Lambda_K + \Gamma \quad \text{in} \quad D'(Q).
\]
(3.10)

**Remark 3.2.** (1) Note that, in view of (3.6), (3.7) and (3.8) all terms in (3.10) are well defined.

(2) The study of (1.1)-(1.3) under the assumption \(\int m d\rho(s) ds = +\infty\) is easier (see [28] for the elliptic case), because one can then show there exists at least a renormalized solution such that \(u < m\) a.e. in \(Q\).

(3) Let us point out that, in (3.9) the function \(\varphi \in C^1_0([0,T[)\) which does not depend on the variable \(x\), we are not able to prove (3.9) with any function \(\varphi \in L^2(0,T;H^1_0(\Omega)) \cap L^\infty(Q)\) such that \(\nabla \varphi = 0\) a.e. in \(\{ (x,t) : u(x,t) = m \}\) because of a lack of regularity on \(u\) with respect to \(t\) in the parabolic case.

### 4. Existence of solutions

This section is devoted to establish the following existence theorem.

**Theorem 4.1.** Under assumptions (3.1)-(3.7) there exists at least a renormalized solution \(u\) of Problem (1.1)-(1.3).

**Proof.** The proof is divided into 4 steps. In Step 1, we introduce an approximate problem. Step 2 is devoted to establish a few a priori estimates. At last, Step 3 and Step 4 are devoted to prove that \(u\) satisfies (3.7), (3.8), (3.9) and (3.10) of Definition 3.1.

**Step 1.** Let us introduce the following regularization of the data: for \(n \geq 1\) fixed
\[
d_i^n(s) = d_i(T_m - \frac{1}{n} (s^+ - T_n(s^-))) \quad \forall s \in \mathbb{R}, \forall i \in \{1, \ldots, N\},
\]
(4.1)
\[
u_{0n} \in C^\infty_c(\Omega) : u_{0n} \to u_0 \quad \text{strongly in} \quad L^1(\Omega) \quad \text{as} \quad n \to +\infty,
\]
(4.2)
we consider a sequence of mollifiers \((\rho_n)\), and we define the convolution \(\rho_n * \mu\) for every \((t,x) \in Q\) by
\[
\mu^n(t,x) = \rho_n * \mu(t,x) = \int_Q \rho_n(t-s,x-y) d\mu(s,y).
\]
(4.3)
Let us now consider the regularized problem
\[
\frac{\partial u_n}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i^n(u_n) \frac{\partial u_n}{\partial x_i} \right) = \mu^n \quad \text{in} \quad Q,
\]
(4.4)
\[
u_n(t = 0) = u_{0n} \quad \text{in} \quad \Omega,
\]
(4.5)
\[
u_n = 0 \quad \text{on} \quad \partial \Omega \times (0,T).
\]
(4.6)
As a consequence, proving existence of a weak solution \(u_n \in L^2(0,T;H^1_0(\Omega))\) of (4.4)-(4.6) is an easy task (see e.g. [21]).
Step 2. Using $T_K(u_n)$ as a test function in (4.4) leads to
\[
\int_{\Omega} T_K(u^n) \, dx + \sum_{i=1}^{N} \int_{Q} d_i^n(u^n) \left| \frac{\partial T_K(u^n)}{\partial x_i} \right|^2 \, dx \, dt \leq K(\|\mu_n\|_{L^1(Q)} + \|u_0\|_{L^1(\Omega)}) \tag{4.7}
\]
for almost every $t$ in $(0, T)$, and where $T_K(r) = \int_{0}^{r} T_K(s) \, ds$. The properties $T_K(T_K \geq 0, T_K(s) \geq |s| - 1 \forall s \in \mathbb{R})$, and since $\|\mu^n\|_{L^1(Q)}$ and $\|u_{0n}\|_{L^1(\Omega)}$ are bounded, we deduce from (4.7) that
\begin{align*}
&u^n \text{ is bounded in } L^\infty(0, T; L^1(\Omega)), \tag{4.8} \\
&T_K(u^n) \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \tag{4.9} \\
&d_i^n(u^n)\frac{1}{2} \frac{\partial T_K(u^n)}{\partial x_i} \text{ is bounded in } L^2(Q) \tag{4.10}
\end{align*}

independently of $n$ for any $K \geq 0$ and any $i \in \{1, 2, \ldots, N\}$. In view of (3.1)-(3.3), we have that for any $i \in \{1, 2, \ldots, N\}$,
\[
\left| \int_{0}^{m} d_i^n(s) \chi_{[-K,s \leq m]} \, ds \right| \leq \int_{-K}^{m} d_i(s) \, ds \equiv C_K < +\infty,
\]
then we can use $\int_{0}^{m} d_i^n(s) \chi_{[-K,s \leq m]} \, ds$ in $L^2(0, T; H_0^1(\Omega)) \cap L^\infty(Q_T)$ as a test function in (4.4) obtaining
\[
\int_{\Omega} \int_{0}^{m} d_i^n(s) \chi_{[-K,s \leq m]} \, ds \, dx \, dz + \int_{Q} (d_i^n(u^n))^2 \left| \frac{\partial T_K(u^n)}{\partial x_i} \right|^2 \, dx \, dt \\
\leq (\|\mu_n\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)}) \max_{i} \int_{-K}^{m} d_i(s) \, ds
\]
for all $i \in \{1, 2, \ldots, N\}$. Since $\int_{\Omega} \int_{0}^{m} f_i^n(s) \, ds \, dx$ is positive and $\|\mu^n\|_{L^1(Q)}$ and $\|u_{0n}\|_{L^1(\Omega)}$ are bounded, from (4.11) we deduce that
\[
d_i^n(u^n) \nabla T_K^n(u^n) \text{ is bounded in } (L^2(Q))^N. \tag{4.12}
\]
For any $S \in W^{2,\infty}(\mathbb{R})$ such that $S'$ has a compact support ($\text{supp}(S') \subset [-K, m]$), we have
\[
S(u^n) \text{ is bounded in } L^2(0, T; H_0^1(\Omega)), \tag{4.13}
\]
\[
\frac{\partial S(u^n)}{\partial t} \text{ is bounded in } L^1(Q) + L^2(0, T; H^{-1}(\Omega)), \tag{4.14}
\]
indpendently of $n$. In fact, as a consequence of (4.9), by Stampacchia’s Theorem, we obtain (4.13). To show that (4.14) holds true, we multiply the equation (4.4) by $S'(u^n)$ to obtain
\[
\frac{\partial S(u^n)}{\partial t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i^n(u_n) \frac{\partial S(u_n)}{\partial x_i} \right) \\
- \sum_{i=1}^{N} d_i^n(u_n) \left| \frac{\partial u_n}{\partial x_i} \right|^2 S''(u^n) + \mu^n S'(u^n) \text{ in } D'(Q), \tag{4.15}
\]
as a consequence of (4.3), (4.10), (4.12), we obtain (4.14).
Arguing again as in [5, 6, 7, 9] estimates (4.13) and (4.14) imply that, for a subsequence still indexed by \(n\),

\[
u^n \rightharpoonup u \quad \text{almost everywhere in } Q,\tag{4.16}
\]

\[
T_K(u^n) \rightharpoonup T_K(u) \quad \text{weakly in } L^2(0,T; H^1_0(\Omega)),
\tag{4.17}
\]

\[
(d^n(u^n))^{1/2} \nabla T_K(u^n) \rightharpoonup X_K \quad \text{weakly in } (L^2(Q))^N,\tag{4.18}
\]

\[
d^n(u^n) \nabla T^m_K(u^n) \rightharpoonup Y_K \quad \text{weakly in } (L^2(Q))^N,\tag{4.19}
\]

as \(n\) tends to \(+\infty\), for any \(K > 0\).

Using the admissible test function \(T^+_{2m}(u^n) - T^+_m(u^n)\) in (4.4) and the Poincaré inequality, leads to

\[
d_p(m - \frac{1}{n}) \int_Q \left| T^+_{2m}(u^n) - T^+_m(u^n) \right|^2 \, dx \, dt \leq m\left(\|\mu_n\|_{L^1(Q)} + \|u_0n\|_{L^1(\Omega)}\right). \tag{4.20}
\]

In view of (3.3), (4.2) and (4.16) (since \(d_p(m - \frac{1}{n}) \to +\infty\) as \(n\) tends \(+\infty\) passing to the limit in (4.20) as \(n\) tends to \(+\infty\), we deduce that \(T^+_{2m}(u) - T^+_m(u) = 0\) a.e. in \(Q\), hence \(u \leq m\) a.e. in \(Q\).

\[
u^n \rightharpoonup u \quad \text{as } \Omega \text{ tends to } +\infty, \quad \text{for any } K > 0.
\]

Now, in view of (4.18), (4.19) and (4.21) we deduce

\[
X_K = d(u)^{1/2} \nabla T_K(u) \quad \text{and} \quad Y_K = d(u) \nabla T^m_K(u) \quad \text{a.e. in } \{(x,t) \in Q : u(x,t) < m\},
\tag{4.22}
\]

for any \(K > 0\).

For fixed \(K \geq 1, \eta \in ]0,1[\) and \(\sigma \in ]0,1[\), we define the functions, \(h_{K,\eta}\) and \(Z_{\sigma}\) by

\[
h_{K,\eta}(s) = \begin{cases} 
0 & \text{if } -K \leq s \\
-1 & \text{if } s \leq -K - \eta \\
\text{affine otherwise,} & \text{if } s \geq m - \sigma
\end{cases}
\]

\[
Z_{\sigma}(s) = \begin{cases} 
0 & \text{if } s \leq m - 2\sigma \\
1 & \text{if } s \geq m - \sigma
\end{cases}
\]

We remark that \(\max(||h_{K,\eta}||_{L^\infty(\mathbb{R})}, ||Z_{\sigma}||_{L^\infty(\mathbb{R})}) = 1\) for any \(K \geq 1\) any \(0 < \eta < 1\) and any \(0 < \sigma < 1\). Using the admissible test functions \(h_{K,\eta}(u^n)\) and \(Z_{\sigma}(u^n)\) in (4.4) leads to

\[
\int_\Omega h_{K,\eta}(u^n(T)) \, dx + \sum_{i=1}^N \int_Q d^n_i(u^n) \frac{\partial u^n}{\partial x_i} \frac{\partial h_{K,\eta}(u^n)}{\partial x_i} \, dx \, dt 
= \int_Q h_{K,\eta}(u_n) \mu_n \, dx \, dt + \int_\Omega h_{K,\eta}(u_0n) \, dx,
\tag{4.24}
\]

and

\[
\int_\Omega Z_{\sigma}(u^n(T)) \, dx + \sum_{i=1}^N \int_Q d^n_i(u^n) \frac{\partial u^n}{\partial x_i} \frac{\partial Z_{K,\sigma}(u^n)}{\partial x_i} \, dx \, dt 
= \int_Q Z_{K,\sigma}(u_n) \mu_n \, dx \, dt + \int_\Omega Z_{K,\sigma}(u_0n) \, dx,
\tag{4.25}
\]

where

\[
\overline{h_{K,\eta}}(r) = \int_0^r h_{K,\eta}(s) \, ds \geq 0, \quad \overline{Z_{\sigma}}(r) = \int_0^r Z_{\sigma}(s) \, ds \geq 0.
\]
Hence, using (4.21), (4.22) and dropping a nonnegative term,
\[
\sum_{i=1}^{N} \int_{\{K+\eta \leq |u^n| \leq K\}} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \, dx \, dt \\
\leq \int_{\{u^n \leq K\}} |\mu^n| \, dx \, dt + \int_{\{u_{on} \leq K\}} |u_{on}| \, dx \leq C_1,
\] (4.26)
and
\[
\sum_{i=1}^{N} \int_{\{m-2\sigma \leq |u^n| \leq m-\sigma\}} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \, dx \, dt \\
\leq \int_{\{u^n \geq m-2\sigma\}} Z_\sigma(u_n) \mu^n \, dx \, dt + \int_{\{u_{on} \geq m-2\sigma\}} |u_{on}| \, dx \leq C_2.
\] (4.27)

Thus, there exists a bounded Radon measures \( \lambda^{\eta}_K \) and \( \nu_\sigma \) such that, as \( \eta \) tends to zero and \( n \) tends to infinity
\[
\lambda_{K}^{\eta} \equiv \sum_{i=1}^{N} \int_{\{K+\eta \leq |u^n| \leq K\}} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \chi_{\{K+\eta \leq u^n \leq K\}} \rightharpoonup \lambda_K \quad \text{weakly in } M(Q),
\] (4.28)
and
\[
\nu_\sigma \equiv \sum_{i=1}^{N} \int_{\{m-2\sigma \leq |u^n| \leq m-\sigma\}} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \chi_{\{m-2\sigma \leq u^n \leq m-\sigma\}} \rightharpoonup \nu_\sigma \quad \text{weakly in } M(Q). \quad (4.29)
\]

**Step 3.** In this step, \( u \) is shown to satisfy (3.10). For all real numbers \( \eta > 0 \), \( \sigma > 0 \) and \( K > 0 \), let \( \sigma_{K,\sigma}^{m,\eta} \) be the function defined by (2.2), and let us denote \( T^{m,\eta}_{K,\sigma}(z) = \int_{0}^{z} \sigma_{K,\sigma}^{m,\eta}(s) \, ds \). Since \( \text{supp}(\sigma_{K,\sigma}^{m,\eta})' \subset [-K-\eta, -K] \cup [m-2\sigma, m-\sigma] \), the equation (4.15) with \( S = T^{m,\eta}_{K,\sigma} \) gives
\[
\frac{\partial T^{m,\eta}_{K,\sigma}(u^n)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i^n(u^n) \frac{\partial T^{m,\eta}_{K,\sigma}(u^n)}{\partial x_i} \right) \\
= \mu^n + (\sigma_{K,\sigma}^{m,\eta}(u^n) - 1) \mu^n + \frac{1}{\eta} \sum_{i=1}^{N} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \chi_{\{K+\eta \leq u^n \leq K\}} \\
+ \frac{1}{\eta} \sum_{i=1}^{N} d_i^n(u^n) \left| \frac{\partial u^n}{\partial x_i} \right|^2 \chi_{\{m-2\sigma \leq u^n \leq m-\sigma\}}
\] (4.30)
in \( D'(Q) \). Passing to the limit in (4.30) as \( \eta \) tends to zero, and using (4.17), (4.19), (4.21), (4.22), (4.28) and (4.29), we deduce
\[
\frac{\partial T^{m,\eta}_{K,\sigma}(u^n)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d_i^n(u^n) \frac{\partial T^{m,\eta}_{K,\sigma}(u^n)}{\partial x_i} \right) \\
= \mu^n - \mu^n \chi_{\{u^n < -K\}} - \mu^n \chi_{\{u^n > -K\}} + \lambda^{n}_{K} + \nu^{\eta}_{\sigma}
\] (4.31)
in \( D'(Q) \). Now, using the properties of convolution \( \mu_n = \rho_n * \mu \) and in view of (4.26), (4.27), (4.28) and (4.29), we deduce that \( \lambda_{K}^{n} \equiv -\mu^n \chi_{\{u^n < -K\}} + \lambda^{n}_{K} \) and \( \Gamma^{\eta}_{\sigma} \equiv -\mu^n \chi_{\{u^n > -K\}} + \nu^{\eta}_{\sigma} \) are bounded in \( L^1(Q) \). Then there exists a bounded measures \( \Lambda_K \) and \( \Gamma_{\sigma} \) such that \( -\mu^n \chi_{\{u^n < -K\}} + \lambda^{n}_{K} \) converges to \( \Lambda_K \) and \( -\mu^n \chi_{\{u^n > -K\}} + \nu^{n}_{\sigma} \) converges to \( \Gamma_{\sigma} \).
Since $\sigma$ and (4.28) we deduce

$$\frac{\partial T^n_{K,\sigma}(u)}{\partial t} - \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left( d^n_i(u) \frac{\partial T^n_{K,\sigma}(u)}{\partial x_i} \chi_{\{u < m\}} \right) = \mu + \Lambda_K + \Gamma_{\sigma} \text{ in } D'(Q). \tag{4.32}$$

To complete this step, we use

$$\int_Q |\Gamma_{\sigma}| \, dx \, dt \leq \liminf_{n \to +\infty} \int_Q |\Gamma^n_{\sigma}| \, dx \, dt$$

$$= \liminf_{n \to +\infty} \int_Q |\mu^n Z_{\sigma}(u^n) + \nu^n| \, dx \, dt$$

$$\leq 2 \|\mu\|_{\mathcal{M}(Q)} + \|u_0\|_{L^1(\Omega)}$$

then there exists a bounded measure $\Gamma$ such that $\Gamma_{\sigma}$ converges to $\Gamma$ in $\ast$--weakly in $\mathcal{M}(Q)$. Therefore, as $\sigma$ tends to zero in (4.32), it is easy to see that $u$ satisfies (3.10).

**Step 4.** In this step, $\Lambda_K$ and $\Gamma$ are shown to satisfy (3.8) and (3.9). From (4.26) and (4.28) we deduce

$$\|\Lambda^n_K\|_{L^1(Q)} = \|\mu^n \chi_{\{u^n < -K\}} + \Lambda^n_K\|_{L^1(Q)}$$

$$\leq 2 \int_{\{u^n < -K\}} |\mu^n| \, dx \, dt + \int_{\{u_0 < -K\}} |u_{0n}| \, dx. \tag{4.33}$$

Since

$$\|\lambda_K\|_{\mathcal{M}(Q)} \leq \liminf_{n \to +\infty} \|\mu^n \chi_{\{u^n < -K\}} + \Lambda^n_K\|_{\mathcal{M}(Q)},$$

the sequence $(\mu_n)$ is equidiffuse, and the function $u_{0n}$ converges to $u_0$ strongly in $L^1(\Omega)$, we deduce from Theorem 2.3 and (4.33) that $\|\Lambda_K\|_{\mathcal{M}(Q)}$ tends to zero as $K$ tends to infinity, then we obtain (3.8).

On the other hand, for all $\varphi \in C^1_0([0, T])$, we can write

$$\int_Q \varphi \, d\Gamma = \lim_{\sigma \to 0} \int_Q \varphi \, d\Gamma_{\sigma} = \lim_{\sigma \to 0} \lim_{n \to +\infty} \int_Q \varphi \Gamma^n_{\sigma} \, dx \, dt \tag{4.34}$$

where

$$\Gamma^n_{\sigma} = \frac{1}{\sigma} \sum_{i=1}^{N} d^n_i(u_n) \left| \frac{\partial u_n}{\partial x_i} \right|^2 \chi_{\{m-2\sigma < u_n < -m-\sigma\}} - Z_{\sigma}(u_n) \mu^n.$$

Using the admissible function $Z_{\sigma}(u^n) \varphi$ in (4.4), since $\varphi \in C^1_0([0, T])$, it is easy to see that

$$\int_{\Omega} Z_{\sigma}(u_0^n) \varphi(0) \, dx + \int_Q Z_{\sigma}(u^n) \varphi \, dx \, dt$$

$$= \frac{1}{\sigma} \sum_{i=1}^{N} \int_{\{m-2\sigma < u_n < -m-\sigma\}} d^n_i(u_n) \left| \frac{\partial u_n}{\partial x_i} \right|^2 \varphi \, dx \, dt - \int_Q Z_{\sigma}(u^n) \mu^n \varphi \, dx \, dt \tag{4.35}$$

$$= \int_Q \varphi \Gamma^n_{\sigma} \, dx \, dt,$$

where $Z_{\sigma}(r) = \int_0^r Z_{\sigma}(s) \, ds$. Next we pass to the limit in (4.35) as $n$ tends to infinity, and then $\sigma$ tends to zero. Since $Z_{\sigma}(u^n)$ converges to $Z_{\sigma}(u)$ strongly in $L^1(Q)$ and
\( Z_\sigma(u^n) \) converges to \( Z_\sigma(u_0) \) strongly in \( L^1(\Omega) \) as \( n \) tends to infinity, we deduce
\[
\lim_{n \to +\infty} \int_Q Z_\sigma(u^n) \varphi_t \, dx = \int_Q Z_\sigma(u) \varphi_t \, dx \tag{4.36}
\]
Moreover, since \( Z_\sigma(r) \) converges to \((r - m)^+ \) for all \( r \in \mathbb{R} \) and \( u \leq m, u_0 \leq m \) almost everywhere, then it is easy to see that
\[
\lim_{n \to +\infty} \int_\Omega Z_\sigma(u^n) \varphi \, dx = \int_\Omega Z_\sigma(u_0) \varphi \, dx \tag{4.37}
\]
Then, from (4.34), (4.35), (4.37) and (4.38) we deduce (3.9).

5. Appendix: Proof of Theorem 2.3

Sketch of the Proof. For simplicity we assume that \( \mu \geq 0 \) and \( u_0 \geq 0 \). Using the admissible test function \( T_K(u) \) in (2.1) leads to
\[
\int_\Omega T_K(u) \, dx + \sum_{i=1}^n \int_Q \left| d_i(u)^{1/2} \frac{\partial T_K(u)}{\partial x_i} \right|^2 \, dx \, dt \leq K \left[ \|\mu\|_{M(Q)} + \|u_0\|_{L^1(\Omega)} \right] \equiv KM, \tag{5.1}
\]
for almost any \( t \) in \([0, T]\) and where \( T_K(r) = \int_0^r T_K(s) \, ds \). Since \( \frac{1}{2} T_K^2(r) \leq T_K(r) \leq Kr \), from (5.1) we deduce that
\[
\max \left\{ \|T_K(u)\|^2 \sim_{(L^2(\Omega))} \|\nabla T_K(u)\|_{L^2(\Omega)}^2 \right\} \leq KM, \quad \|T_K(u)\|_{L^2(H^1_0(\Omega))}^2 \leq K \frac{M}{\alpha}, \tag{5.2}
\]
Moreover, for \( i \in \{1, \ldots, N\} \) let us choose \( \int_0^{T_K(u)} d_i(r) \, dr \in L^2(0, T; H^1_0(\Omega)) \cap L^\infty(Q) \) as test function in (2.1). Then
\[
\sum_{i=1}^n \int_\Omega \left| d_i(u) \frac{\partial T_K(u)}{\partial x_i} \right|^2 \, dx \, dt \leq K \left[ \|\mu\|_{M(Q)} + \|u_0\|_{L^1(\Omega)} \right] \|d_i\|_{L^\infty(\mathbb{R})}. \tag{5.3}
\]
Let \( v \in W \) be the solution of
\[
- \frac{\partial v}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial v}{\partial x_i} \right) = -2 \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial T_K(u)}{\partial x_i} \right) \quad \text{in } Q, \tag{5.4}
\]
\[
v = 0 \quad \text{on } (0, T) \times \partial \Omega, \quad v(t = T) = T_K(u(t = T)) \quad \text{in } \Omega.
\]
Using the admissible test function \( v \) in (5.4) and integrate between \( \tau \) and \( T \), and by Young’s inequality we obtain
\[
\int_\Omega \frac{|v(\tau)|^2}{2} \, dx + \frac{\alpha}{2} \int_Q |\nabla v|^2 \, dx \, dt \leq C \sum_{i=1}^n \int_Q \left| d_i(u) \frac{\partial T_K(u)}{\partial x_i} \right|^2 \, dx \, dt + \int_\Omega T_K(u(t = T)) \, dx \tag{5.5}
\]
In view of (5.2), (5.3) and (5.5), we deduce that

\[
\max \left\{ \|v\|_{L^\infty(0,T;L^2(\Omega))} : \|\nabla v\|_{L^2(Q)}^2 \right\} \leq CKM. \tag{5.6}
\]

Moreover, by (5.4) we obtain

\[
\|v_t\|_{L^2(0,T;H^{-1}(\Omega))} \leq C \left( \|v\|_{L^2(0,T;H^1_0(\Omega))} + \|T_K(u)\|_{L^2(0,T;H^1_0(\Omega))} \right). \tag{5.7}
\]

Hence, by (5.6) and (5.7) we conclude that

\[
\|v\|_{W} \leq C \sqrt{K}. \tag{5.8}
\]

Since \( \mu \geq 0 \) and \( u_0 \geq 0 \), it follows that

\[
\frac{\partial u}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial u}{\partial x_i} \right) \geq 0
\]

and \( u \geq 0 \) in \( Q \), and by a nonlinear version of Kato's inequality for parabolic equations (see [27]), we deduce that

\[
\frac{\partial T_K(u)}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial T_K(u)}{\partial x_i} \right) \geq 0,
\]

hence by (5.4), we obtain

\[
-\frac{\partial v}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial v}{\partial x_i} \right) \geq -\frac{\partial T_K(u)}{\partial t} - \sum_{i=1}^N \frac{\partial}{\partial x_i} \left( d_i(u) \frac{\partial T_K(u)}{\partial x_i} \right) \quad \text{in } D'(Q).
\]

Now using the standard comparison argument, we easily see that \( v \geq T_K(u) \) a.e. in \( Q \), hence \( v \geq K \) a.e. on \( \{u > K\} \), and by (5.8) we conclude that

\[
\text{cap}_2\{u > K\} \leq \frac{\|\frac{v}{K}\|_{W}}{C} \leq \frac{C}{\sqrt{K}},
\]

the proof is complete. \( \square \)

References


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