REGULARLY VARYING SOLUTIONS WITH INTERMEDIATE GROWTH FOR CYCLIC DIFFERENTIAL SYSTEMS OF SECOND ORDER

JAROSLAV JAROŠ, KUSANO TAKAŠI, TOMOYUKI TANIGAWA

Abstract. In this article, we study the existence and accurate asymptotic behavior as \( t \to \infty \) of positive solutions with intermediate growth for a class of cyclic systems of nonlinear differential equations of the second order

\[
(p_i(t)|x_i'|^{\alpha_i-1}x_i')' + q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1} = 0, \quad i = 1, \ldots, n, \quad x_{n+1} = x_1,
\]

where \( \alpha_i \) and \( \beta_i \), \( i = 1, \ldots, n \), are positive constants such that \( \alpha_1 \ldots \alpha_n > \beta_1 \ldots \beta_n \) and \( p_i, q_i : [a, \infty) \to (0, \infty) \) are continuous regularly varying functions (in the sense of Karamata). It is shown that the situation in which the system possesses regularly varying intermediate solutions can be completely characterized, and moreover that the asymptotic behavior of such solutions is governed by the unique formula describing their order of growth (or decay) precisely. The main results are applied to some classes of partial differential equations with radial symmetry including metaharmonic equations and systems involving \( p \)-Laplace operators on exterior domains.

1. Introduction

In this article we analyze the differential system

\[
(p_i(t)|x_i'|^{\alpha_i-1}x_i')' + q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1} = 0, \quad i = 1, \ldots, n, \quad x_{n+1} = x_1,
\]

where

(a) \( \alpha_i \) and \( \beta_i \), \( i = 1, \ldots, n \), are positive constants such that \( \alpha_1 \ldots \alpha_n > \beta_1 \ldots \beta_n \);
(b) \( p_i \) and \( q_i \) are continuous positive functions on \( [a, \infty) \), \( a > 0 \);
(c) all \( p_i \) simultaneously satisfy either

\[
\int_a^\infty p_i(t)^{-1/\alpha_i} dt = \infty, \quad i = 1, \ldots, n,
\]

or

\[
\int_a^\infty p_i(t)^{-1/\alpha_i} dt < \infty, \quad i = 1, \ldots, n.
\]

By a positive solution of (1.1) we mean a vector function \((x_1, \ldots, x_n)\) consisting of components \( x_i \), \( i = 1, \ldots, n \), which are positive and continuously differentiable together with \( p_i|x_i'|^{\alpha_i-1}x_i' \) on some interval \([T, \infty)\) and satisfy system (1.1) there.
Systems of the form (1.1) with $p_i(t) = t^{N-1}$ and $q_i(t) = t^{N-1}f_i(t)$, $N \geq 2$, $i = 1, \ldots, n$, arise in the study of positive radial solutions in exterior domains in $\mathbb{R}^N$ for the system of $p_i$-Laplacian equations

$$\Delta_{p_i} u_i \equiv \text{div}(|\nabla u_i|^{p_i-2}\nabla u_i) + f_i(|x|)|u_{i+1}|^{\gamma_i-1}u_{i+1} = 0, \quad i = 1, \ldots, n,$$

where $p_i > 1$ and $\gamma_i > 0$, $i = 1, \ldots, n$, are constants, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$ and $f_i$, $i = 1, \ldots, n$, are positive continuous functions on $[a, \infty)$.

Quasilinear elliptic system (1.4) with negative $f_i$, $i = 1, \ldots, n$, and the exponents satisfying the super-homogeneity condition $\gamma_1\gamma_2\ldots\gamma_n > (p_1 - 1)(p_2 - 1)\ldots(p_n - 1)$ was studied by Teramoto [22], while the problem of the existence and precise asymptotic behavior as $|x| \to \infty$ of positive strongly decreasing (resp. strongly increasing) radial solutions of (1.4) in the case $p_1 = \cdots = p_n = p > 1$ under the sub-homogeneity assumption $\gamma_1\gamma_2\ldots\gamma_n < (p - 1)^n$ was investigated in [7] (resp. [8]). (For the special case $p_1 = \cdots = p_n = 2$ see also [23].)

In this article we are concerned with positive solutions $(x_1, \ldots, x_n)$ of (1.1) all components of which have the intermediate growth (or decay) in the sense that they are increasing to infinity as $t \to \infty$ and satisfy

$$\lim_{t \to \infty} p_i(t)|x_i'(t)|^{\alpha_i-1}x_i'(t) = 0, \quad i = 1, \ldots, n, \quad \text{in case (1.2) holds},$$

or decreasing to zero as $t \to \infty$ and satisfy

$$\lim_{t \to \infty} p_i(t)|x_i'(t)|^{\alpha_i-1}x_i'(t) = -\infty, \quad i = 1, \ldots, n, \quad \text{in case (1.3) holds}.$$

Note that this is equivalent to

$$\lim_{t \to \infty} x_i(t) = \infty, \quad \lim_{t \to \infty} \frac{x_i(t)}{P_i(t)} = 0, \quad i = 1, \ldots, n,$$

if (1.2) holds, where $P_i(t) = \int_0^t p_i(s)^{-1/\alpha} \, ds$, or to

$$\lim_{t \to \infty} x_i(t) = 0, \quad \lim_{t \to \infty} \frac{x_i(t)}{\pi_i(t)} = \infty, \quad i = 1, \ldots, n,$$

if (1.3) holds, where $\pi_i(t) = \int_0^t p_i(s)^{-1/\alpha} \, ds$.

In the scalar case, i.e., if (1.1) reduces to a single equation of the form

$$(p(t)|x'|^{\alpha-1}x')' + q(t)|x|^{\beta-1}x = 0,$$

where $\alpha$ and $\beta$ are positive constants such that $\alpha > \beta$ and $p$ and $q$ are positive continuous functions on $[a, \infty)$, necessary and sufficient conditions for the existence of intermediate solutions of (1.9) have been established for the case (1.2) by Naito [20] and for the case (1.3) by Kamo and Usami [11].

It is to be noticed that system (1.1) may possess also positive solutions which have an extreme growth (or decay) in the sense that if (1.2) holds, then each component $x_i$ satisfies either

$$\lim_{t \to \infty} \frac{x_i(t)}{P_i(t)} = \text{const} > 0,$$

or

$$\lim_{t \to \infty} x_i(t) = \text{const} > 0,$$
and if (1.3) holds, then each component $x_i$ satisfies either (1.11) or
\[
\lim_{t \to \infty} \frac{x_i(t)}{\pi_i(t)} = \text{const} > 0.
\] (1.12)

Positive solutions of these types are not considered here.

Once the existence of intermediate solutions of (1.9) (or (1.1)) has been confirmed, a natural question arises as to the possibility of determining their asymptotic behavior at infinity accurately. Partial answers to this question in the scalar case have recently been given in the papers [9, 14] which are concerned exclusively with regularly varying intermediate solutions of equation (1.9) with regularly varying coefficients $p(t)$ and $q(t)$. Restricting our consideration within the framework of regular variation allows us to utilize basic theory of regular variation to acquire thorough and precise information about the existence, the asymptotic behavior and the structure of regularly varying intermediate solutions of equation (1.9). For the definition of regularly varying functions see Section 2.

A prototype of the results we are going to prove says that if $f$ and $g$ are regularly varying functions of indices $\lambda$ and $\mu$, respectively, and $p > N$, then the necessary and sufficient condition for the existence of positive intermediate radial solutions components of which are regularly varying functions with indices in the interval $(0, \frac{p-N}{p-1})$ of the system of two equations
\[
\Delta_p u + f(|x|) v^\alpha = 0, \quad \Delta_p v + g(|x|) u^\beta = 0,
\] (1.13)
where $\alpha \beta < (p-1)^2$, is the satisfaction of the system of inequalities
\[
0 < p + \lambda + \frac{\alpha}{p-1} (p + \mu) < (p - N) \left( 1 - \frac{\alpha \beta}{(p-1)^2} \right),
\]
\[
0 < \frac{\beta}{p-1} (p + \lambda) + p + \mu < (p - N) \left( 1 - \frac{\alpha \beta}{(p-1)^2} \right),
\]
and if $p < N$, then the above two-dimensional system has intermediate RV solutions with indices in $(\frac{p-N}{p-1}, 0)$ if and only if
\[
(p - N) \left( 1 - \frac{\alpha \beta}{(p-1)^2} \right) < p + \lambda + \frac{\alpha}{p-1} (p + \mu) < 0,
\]
and
\[
(p - N) \left( 1 - \frac{\alpha \beta}{(p-1)^2} \right) < \frac{\beta}{p-1} (p + \lambda) + p + \mu < 0.
\]

In both cases the indices $\rho$ and $\sigma$ of regular variation of the components $u$ and $v$, respectively, are given (uniquely) by
\[
\rho = \frac{p-1}{(p-1)^2 - \alpha \beta} \left[ p + \lambda + \frac{\alpha}{p-1} (p + \mu) \right], \quad \sigma = \frac{p-1}{(p-1)^2 - \alpha \beta} \left[ \frac{\beta}{p-1} (p + \lambda) + p + \mu \right],
\]
and any such intermediate solution $(u, v)$ as $|x| \to \infty$ satisfies the asymptotic relation
\[
u(|x|) \sim |x|^\sigma \left[ \frac{\varphi(|x|)}{D(\rho)} \right]^{\frac{1}{p-1}} \left[ \frac{\psi(|x|)}{D(\sigma)} \right]^{\frac{1}{p-1}},
\]
\[
u(|x|) \sim |x|^\sigma \left[ \frac{\varphi(|x|)}{D(\rho)} \right]^{\frac{1}{p-1}} \left[ \frac{\psi(|x|)}{D(\sigma)} \right]^{\frac{1}{p-1}},
\]
where \( \varphi \) and \( \psi \) are the slowly varying parts of \( f \) and \( g \), respectively, and

\[
D(\tau) = (p - N - (p - 1)|\tau|)^\frac{1}{p-1} |\tau|.
\]

The main results of this paper will be presented in Section 4. The existence of intermediate regularly varying solutions of (1.1) is proved by solving the system of integral equations

\[
x_i(t) = c_i + \int_t^\infty \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad i = 1, \ldots, n, \quad \text{if (1.2) holds},
\]

\[
x_i(t) = \int_t^\infty \frac{1}{p_i(s)} \left( \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad i = 1, \ldots, n, \quad \text{if (1.3) holds},
\]

for some constants \( T \geq a \) and \( c_i > 0 \) with the help of fixed point techniques combined with basic theory of regularly varying functions. Furthermore it is shown that the asymptotic behavior of the obtained solutions is governed by the unique explicit law describing their order of growth (in case (1.2) holds) or decay (in case of (1.3) holds) accurately. To this end extensive use is made of the knowledge derived through the analysis of the following systems of asymptotic integral relations associated with (1.14) and (1.15) by means of regular variation:

\[
x_i(t) \sim \int_t^\infty \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1, \ldots, n, \quad (1.16)
\]

\[
x_i(t) \sim \int_t^\infty \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1, \ldots, n. \quad (1.17)
\]

Here and hereafter the notation \( f(t) \sim g(t) \) as \( t \to \infty \) is used to mean

\[
\lim_{t \to \infty} \frac{g(t)}{f(t)} = 1.
\]

The details of the analysis of systems (1.16) and (1.17) in the framework of regular variation is presented in Section 3, which is preceded by Section 2 where the definition and some basic properties of regularly varying functions are summarized for the reader’s convenience. The final Section 5 is designed to explain the effective applicability of our results for (1.1) to some classes of partial differential equations with radial symmetry including metaharmonic equations and systems involving \( p \)-Laplace operators on exterior domains in \( \mathbb{R}^N \).

The systematic study of differential equations in the framework of regular variation was initiated by Marić and Tomić [17, 18, 19]. Since the publication of the monograph of Marić [16] in the year 2000 there has been an increasing interest in the study of asymptotic properties of positive solutions of differential equations by means of regularly varying functions, and it has turned out that theory of regular variation combined with fixed point techniques is so powerful as to cover a wide class of ordinary differential equations including generalized Emden-Fowler and Thomas-Fermi equations, and systems of such equations; see, for example, [2, 3, 5, 6, 9, 10, 12, 13, 14, 15].

2. Regularly varying functions

For the reader’s benefit we recall here the definition and basic properties of regularly varying functions which will be used in this paper.
Definition 2.1. A measurable function \( f : [0, \infty) \to (0, \infty) \) is said to be regularly varying of index \( \rho \in \mathbb{R} \) if
\[
\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.
\]
The set of all regularly varying functions of index \( \rho \) is denoted by RV(\( \rho \)).

Typical example of a function from RV(\( \rho \)) is the power function \( t^\rho \) or, more generally, any function of the form \( ct^\rho(1 + \varepsilon(t)) \) where \( c \) is a positive constant and \( \varepsilon(t) \) a measurable function on \((0, \infty)\) such that \( \varepsilon(t) \to 0 \) as \( t \to \infty \). All these are examples of the so-called trivial regularly varying functions of index \( \rho \), i.e., functions \( f \in \text{RV}(\rho) \) satisfying \( \lim_{t \to \infty} f(t)/t^\rho = \text{const} > 0 \). An example of a nontrivial regularly varying function of index \( \rho \) is \( t^\rho \log(1 + t) \).

We often use the symbol SV for RV(0) and call members of SV slowly varying functions. It is easy to check that any \( f \in \text{RV}(\rho) \) can always be represented in the form \( f(t) = t^\rho g(t) \), where \( g \in \text{SV} \). Thus, the properties of regularly varying functions are easily deduced from the corresponding properties of slowly varying functions.

One of the most important properties of regularly varying functions is the following representation theorem.

Proposition 2.2. \( f \in \text{RV}(\rho) \) if and only if \( f(t) \) is represented in the form
\[
f(t) = c(t) \exp\left\{ \int_{t_0}^t \frac{\delta(s)}{s} ds \right\}, \quad t \geq t_0,
\]
for some \( t_0 > 0 \) and for some measurable functions \( c(t) \) and \( \delta(t) \) such that
\[
\lim_{t \to \infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t \to \infty} \delta(t) = \rho.
\]
If in particular \( c(t) \equiv c_0 \) in (2.1), then \( f(t) \) is referred to as a normalized regularly varying function of index \( \rho \).

Examples of slowly varying functions include all functions tending to some positive constants as \( t \to \infty \), the logarithmic function, its powers \( \log^\gamma t \), \( \gamma \in \mathbb{R} \), iterated logarithms. A more sophisticated example of a member from SV is the function 
\[
\exp\{c \log^\gamma t \cos^\delta (\log t)\},
\]
where \( c > 0 \) and \( \gamma, \delta \geq 0 \) are such that \( \gamma + \delta < 1 \). The following result illustrates operations which preserve slow variation.

Proposition 2.3. Let \( L(t), L_1(t), L_2(t) \) be slowly varying. Then, \( L(t)^\alpha \) for any \( \alpha \in \mathbb{R} \), \( L_1(t) + L_2(t) \), \( L_1(t)L_2(t) \) and \( L_1(L_2(t)) \) (if \( L_2(t) \to \infty \)) are slowly varying.

A slowly varying function may grow to infinity or decay to 0 as \( t \to \infty \). But its order of growth or decay is severely limited as is shown in the following

Proposition 2.4. Let \( f \in \text{SV} \). Then, for any \( \varepsilon > 0 \),
\[
\lim_{t \to \infty} t^\varepsilon f(t) = \infty, \quad \lim_{t \to \infty} t^{-\varepsilon} f(t) = 0.
\]

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition 2.5. A differentiable positive function \( f(t) \) is a normalized regularly varying function of index \( \rho \) if and only if
\[
\lim_{t \to \infty} \frac{f'(t)}{f(t)} = \rho.
\]
The following proposition known as Karamata’s integration theorem will play an important role in this paper.

**Proposition 2.6.** Let \( L(t) \) be a slowly varying function. Then:

(i) if \( \alpha > -1 \),
\[
\int_a^t s^\alpha L(s) ds \sim \frac{1}{\alpha + 1} t^{\alpha + 1} L(t), \quad t \to \infty;
\]

(ii) if \( \alpha < -1 \),
\[
\int_t^\infty s^\alpha L(s) ds \sim -\frac{1}{\alpha + 1} t^{\alpha + 1} L(t), \quad t \to \infty;
\]

(iii) if \( \alpha = -1 \),
\[
l(t) = \int_a^t \frac{L(s)}{s} ds \in \text{SV},
\]
and if, in addition, \( \int_a^\infty s^{-1} L(s) ds < \infty \), then
\[
m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \text{SV}.
\]

**Definition 2.7.** A vector function \( (x_1(t), \ldots, x_n(t)) \) is said to be regularly varying of index \( (\lambda_1, \ldots, \lambda_n) \) if \( x_i(t) \in \text{RV}(\lambda_i) \) for \( i = 1, \ldots, n \). If all \( \lambda_i \) are positive (or negative), then \( (x_1(t), \ldots, x_n(t)) \) is called regularly varying of positive (or negative) index \( (\lambda_1, \ldots, \lambda_n) \). The set of all regularly varying vector functions of index \( (\lambda_1, \ldots, \lambda_n) \) is denoted by \( \text{RV}(\lambda_1, \ldots, \lambda_n) \).

For a complete exposition of theory of regular variation and its applications the reader is referred to the treatise of Bingham, Goldie and Teugels [1]. See also Seneta [21]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [16].

3. Systems of asymptotic relations associated with (1.1)

We assume that \( p_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \) and that they are represented as
\[
p_i(t) = t^{\lambda_i} l_i(t), \quad q_i(t) = t^{\mu_i} m_i(t), \quad l_i, m_i \in \text{SV}, \quad i = 1, \ldots, n.
\]
(3.1)

In addition we require that \( p_i(t) \) satisfy either (1.2) or (1.3). It is easy to see that (1.2) (resp. (1.3)) holds if and only if
\[
\lambda_i < \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} dt = \infty,
\]
resp.
\[
\lambda_i > \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} dt < \infty.
\]

Therefore, in case (1.2) is satisfied, the functions \( P_i(t) = \int_a^t p_i(s)^{-1/\alpha_i} ds, \quad i = 1, \ldots, n \), are given by
\[
P_i(t) = \int_a^t s^{-1} l_i(s)^{-1/\alpha_i} ds \quad \text{if} \quad \lambda_i = \alpha_i,
\]
(3.2)
\[
P_i(t) \sim \frac{\alpha_i - \lambda_i}{\alpha_i} t^{\alpha_i - \lambda_i} l_i(t)^{-1/\alpha_i}, \quad t \to \infty, \quad \text{if} \quad \lambda_i < \alpha_i,
\]
(3.3)
and in case (1.3) holds, the functions \( \pi_i(t) = \int_t^\infty p_i(s)^{-1/\alpha_i} ds, \) \( i = 1, \ldots, n, \) are given by

\[
\pi_i(t) = \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} ds \quad \text{if } \lambda_i = \alpha_i, \tag{3.4}
\]

\[
\pi_i(t) \sim \frac{\lambda_i - \alpha_i}{\alpha_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i}, \quad t \to \infty, \quad \text{if } \lambda_i > \alpha_i. \tag{3.5}
\]

Our task in this section is to solve the following two problems.

Problem (i): Under the condition (1.2) characterize the situation in which the system of asymptotic relations

\[
x_i(t) \sim \int_T^t \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1, \ldots, n, \tag{3.6}
\]

possesses regularly varying solutions \((x_1, \ldots, x_n)\) of positive index \((\rho_1, \ldots, \rho_n)\) satisfying

\[
\lim_{t \to \infty} x_i(t) = \infty, \quad \lim_{t \to \infty} \frac{x_i(t)}{\pi_i(t)} = 0, \quad i = 1, \ldots, n. \tag{3.7}
\]

Problem (ii): Under the condition (1.3) characterize the situation in which the system of asymptotic relations

\[
x_i(t) \sim \int_T^t \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1, \ldots, n, \tag{3.8}
\]

possesses regularly varying solutions \((x_1, \ldots, x_n)\) of negative index \((\rho_1, \ldots, \rho_n)\) satisfying

\[
\lim_{t \to \infty} x_i(t) = 0, \quad \lim_{t \to \infty} \frac{x_i(t)}{\pi_i(t)} = \infty, \quad i = 1, \ldots, n. \tag{3.9}
\]

The positivity or negativity requirement for the regularity indices of solutions excludes the possibility that \(\lambda_i = \alpha_i\) for some or all \(i\) which necessarily reduces the corresponding components \(x_i(t)\) to slowly varying functions \((\rho_i = 0)\) (cf. (3.2) and (3.4)). The presence of slowly varying components in the solutions seems to cause computational difficulty.

We begin with Problem (i). We assume that \(\lambda_i < \alpha_i, \ i = 1, \ldots, n,\) and seek solutions \((x_1, \ldots, x_n)\) of (3.6) belonging to \(RV(\rho_1, \ldots, \rho_n)\) with all \(\rho_i\) positive. In view of (3.3) each \(p_i\) must satisfy \(\rho_i > \frac{\alpha_i - \lambda_i}{\alpha_i} \). Let \((x_1, \ldots, x_n)\) be one such solution on \([T, \infty)\). Suppose that \(x_i\) are expressed in the form

\[
x_i(t) = t^{\mu_i} \xi_i(t), \quad \xi_i \in SV, \quad i = 1, \ldots, n. \tag{3.10}
\]

Using (3.1) and (3.10), we have

\[
\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds = \int_t^\infty s^{\mu_i + \beta_i} m_i(s) \xi_{i+1}(s)^{\beta_i} ds, \tag{3.11}
\]

for \(t \geq T\) and \(i = 1, \ldots, n\). The convergence of (3.11) as \(t \to \infty\) implies that \(\mu_i + \beta_i \rho_{i+1} \leq -1, \ i = 1, \ldots, n,\) but the equality should be ruled out. In fact, if the equality holds for some \(i,\) then since

\[
\left( \frac{1}{p_i(t)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} = t^{-\frac{\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \left( \int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i},
\]

from (3.6) and Karamata’s integration theorem we find that

\[
x_i(t) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \left( \int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} \in RV\left( \frac{\alpha_i - \lambda_i}{\alpha_i} \right)
\]
as $t \to \infty$. This implies that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$, a contradiction. It follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = 1, \ldots, n$, and application of Karamata’s integration theorem to (3.11) gives

\[
\left( \frac{1}{p_i(t)} \int_t^\infty q_i(s)x_{i+1}(s)^{\beta_i} \, ds \right)^{1/\alpha_i} 
\sim \frac{t^{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1} \xi_i(t) - \mu_i + \beta_i \rho_{i+1} + 1}{[-(\mu_i + \beta_i \rho_{i+1} + 1)]^{1/\alpha_i}},
\]

(3.12)
as $t \to \infty$, $i = 1, \ldots, n$.

Because (3.12) is not integrable on $[T, \infty)$ we see that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$, $i = 1, \ldots, n$. We claim that the equality can hold in none of these inequalities. If the equality holds for some $i$, then integrating (3.12) from $T$ to $t$ yields

\[
x_i(t) \sim (\alpha_i - \lambda_i)^{-1/\alpha_i} \int_T^t s^{-1} \xi_i(s)^{-1/\alpha_i} m_i(s)^{-1/\alpha_i} \xi_{i+1}(s)^{\beta_i/\alpha_i} \, ds \in SV, \quad t \to \infty,
\]
an impossibility. It holds therefore that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i > -1$ for all $i$, and hence via application of Karamata’s integration theorem to the integral of (3.12) on $[T, t]$ we conclude that

\[
x_i(t) \sim \frac{t^{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1} \xi_i(t) - \mu_i + \beta_i \rho_{i+1} + 1}{[-(\mu_i + \beta_i \rho_{i+1} + 1)]^{1/\alpha_i} (\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1)},
\]

(3.13)
as $t \to \infty$, $i = 1, \ldots, n$. This implies

\[
\rho_i = \frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1, \quad i = 1, \ldots, n, \quad \rho_{n+1} = \rho_1
\]
or equivalently

\[
\rho_i - \frac{\beta_i}{\alpha_i} \rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = 1, \ldots, n, \quad \rho_{n+1} = \rho_1.
\]

(3.14)
The coefficient matrix

\[
A = A \left( \frac{\beta_1}{\alpha_1}, \ldots, \frac{\beta_n}{\alpha_n} \right) = \begin{pmatrix}
1 & -\frac{\beta_1}{\alpha_1} & 0 & \ldots & 0 & 0 \\
0 & 1 & -\frac{\beta_2}{\alpha_2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1 & -\frac{\beta_{n-1}}{\alpha_{n-1}} \\
-\frac{\beta_n}{\alpha_n} & 0 & 0 & \ldots & 0 & 1
\end{pmatrix}
\]

(3.15)
of the algebraic linear system (3.14) is nonsingular because

\[
\det(A) = 1 - \frac{\beta_1 \beta_2 \ldots \beta_n}{\alpha_1 \alpha_2 \ldots \alpha_n} > 0
\]

(3.16)
because of condition (a). Thus, $A$ is invertible and the explicit calculation gives

$$A^{-1} = \frac{A_n}{A_n - B_n} \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1 \beta_2}{\alpha_1 \alpha_2} & \ldots & \frac{\beta_1 \beta_2 \ldots \beta_{n-1}}{\alpha_1 \alpha_2 \ldots \alpha_{n-1}} \\ \frac{\beta_2}{\alpha_2} & \frac{\beta_2 \beta_3}{\alpha_2 \alpha_3} & \ldots & \frac{\beta_2 \beta_3 \ldots \beta_{n-2}}{\alpha_2 \alpha_3 \ldots \alpha_{n-2}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_{n-1}}{\alpha_{n-1}} & \frac{\beta_{n-1} \beta_n}{\alpha_{n-1} \alpha_n} & \ldots & 1 \end{pmatrix},$$

(3.17)

where $A_n = \alpha_1 \alpha_2 \ldots \alpha_n$, $B_n = \beta_1 \beta_2 \ldots \beta_n$, and the lower triangular elements are omitted for economy of notation. Let $(M_{ij})$ denote the matrix on the right-hand side of (3.17). It is easy to see that the $i$-th row of $(M_{ij})$ is obtained by shifting the vector

$$\left(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i \beta_{i+1}}{\alpha_i \alpha_{i+1}}, \ldots, \frac{\beta_i \beta_{i+1} \ldots \beta_{i+(n-2)}}{\alpha_i \alpha_{i+1} \ldots \alpha_{i+(n-2)}} \right) \alpha_{n+k} = \alpha_k, \ \beta_{n+k} = \beta_k$$

for $k = 1, 2, \ldots, (i - 1)$-times to the right cyclically, so that the lower triangular elements $M_{ij}$ for $j < i$, satisfy the relations

$$M_{ij} M_{ji} = \frac{\beta_1 \beta_2 \ldots \beta_n}{\alpha_1 \alpha_2 \ldots \alpha_n}, \quad i > j, \quad i = 1, 2, \ldots, n.$$  

Then the unique solution $\rho_i$, $i = 1, \ldots, n$, of (3.14) is given explicitly by

$$\rho_i = \frac{A_n}{A_n - B_n} \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = 1, \ldots, n,$$

(3.18)

from which it follows that all $\rho_i$ satisfy $0 < \rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if

$$0 < \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < \frac{\alpha_i - \lambda_i}{\alpha_i} \left(1 - \frac{B_n}{A_n}\right), \quad i = 1, \ldots, n.$$  

(3.19)

We note that (3.13) can be expressed in the form

$$x_i(t) \sim \frac{t^\frac{\alpha_i + 1}{\alpha_i} p_i(t)^{1/\alpha_i} q_i(t)^{1/\alpha_i} x_{i+1}(t)^{\beta_i/\alpha_i}}{D_i}, \quad t \to \infty,$$

(3.20)

where

$$D_i = (\alpha_i - \lambda_i - \alpha_i \rho_i)^{1/\alpha_i},$$

(3.21)

for $i = 1, \ldots, n$. This is a cyclic system of asymptotic relations, from which one can derive without difficulty the following independent explicit asymptotic formula for each $x_i$:

$$x_i(t) \sim \left[ \prod_{j=1}^{n} \frac{t^\frac{\alpha_j + 1}{\alpha_j} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j} x_{j+1}(t)^{\beta_j/\alpha_j}}{D_j} \right] \frac{A_n}{A_n - B_n}, \quad t \to \infty, \quad i = 1, \ldots, n.$$  

(3.22)

This represents the unique law describing precisely the growth order of all possible regularly varying solutions of positive indices of system (3.6) satisfying (3.7). Note that (3.22) is rewritten in the form

$$x_i(t) \sim \rho_i^\alpha \left[ \prod_{j=1}^{n} \frac{t^\frac{\alpha_j + 1}{\alpha_j} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j} x_{j+1}(t)^{\beta_j/\alpha_j}}{D_j} \right] \frac{A_n}{\alpha_n - B_n}, \quad t \to \infty, \quad i = 1, \ldots, n.$$

(3.23)
Now we assume that (3.19) is satisfied and define \( \rho_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i}) \) and \( D_i \) by (3.18) and (3.21), respectively. Let \( X_i \in \text{RV}(\rho_i) \) denote the functions

\[
X_i(t) = \left[ \prod_{j=1}^{n} \left( \frac{t^{-\alpha_j} \rho_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{i,j}} \right]^{\frac{A_{i,n}}{x_{i,n}}} , \quad i = 1, \ldots, n. \tag{3.24}
\]

Then the \( X_i \)'s satisfy the system of asymptotic relations (3.6), i.e.,

\[
\int_{b}^{t} \left( \frac{1}{p_i(s)} \int_{s}^{\infty} q_i(r) X_{i+1}(r)^{\beta_i} \, dr \right)^{1/\alpha_i} \, ds \sim X_i(t), \quad t \to \infty, \ i = 1, \ldots, n, \tag{3.25}
\]

for any \( b \geq a \), where \( X_{n+1}(t) = X_1(t) \). In fact, noting that \( X_i(t) \) are expressed as

\[
X_i(t) = t^{\rho_i} \Xi_i(t), \quad \Xi_i(t) = \left[ \prod_{j=1}^{n} \left( \frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{i,j}} \right]^{\frac{A_{i,n}}{x_{i,n}}},
\]

and using Karamata’s integration theorem, we obtain

\[
\left( \frac{1}{p_i(t)} \int_{b}^{t} q_i(s) X_{i+1}(s)^{\beta_i} \, ds \right)^{1/\alpha_i} \sim \frac{t^{\rho_i-1} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\alpha_i - \lambda_i - \alpha_i \rho_i)^{1/\alpha_i}},
\]

and

\[
\int_{b}^{t} \left( \frac{1}{p_i(s)} \int_{b}^{t} q_i(r) X_{i+1}(r)^{\beta_i} \, dr \right)^{1/\alpha_i} \, ds \sim \frac{t^{\rho_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{D_i}, \quad \text{as } t \to \infty. \tag{3.26}
\]

A simple calculation with the help of the relations

\[
M_{i+1,i} = \frac{B_n}{A_n}, \quad M_{i+1,j} = \frac{B_j}{A_i}, \quad \text{for } j \neq i, \tag{3.27}
\]

(with the convention that \( M_{n+1,i} = M_{1,i} \)) between the \( i \)-th and the \((i + 1)\)-th rows of the matrix \( A \) shows that

\[
\left[ \prod_{j=1}^{n} \left( \frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{i+1,j} \frac{\beta_i}{\alpha_i}} \right]^{\frac{A_{i,n}}{x_{i,n}}} = \Xi_i(t).
\]

From (3.26) we conclude that the \( X_i \)'s satisfy (3.25) as desired.

Summarizing the above observations, we obtain the following result which provides complete information about the existence and asymptotic behavior of regularly varying solutions with positive indices for system (3.6).

**Theorem 3.1.** Let \( p_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \), and suppose that \( \lambda_i < \alpha_i, \ i = 1, \ldots, n \). Then system of asymptotic relations (3.6) has regularly varying solutions \((x_1, \ldots, x_n) \in \text{RV}(p_1, \ldots, p_n) \) with \( p_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i}) \), \( i = 1, \ldots, n \), if and only if (3.19) holds in which case \( p_i \) are uniquely determined by (3.18) and the asymptotic behavior of any such solution is governed by the unique formula (3.22).
Our next task is to study Problem (ii). We assume that \( \lambda_i > \alpha_i, \ i = 1, \ldots, n \), and seek solutions \((x_1, \ldots, x_n)\) in \( \text{RV}(\rho_i) \) with all \( \rho_i \) negative. In view of (3.5) each \( \rho_i \) must satisfy \( \frac{\alpha_i - \lambda_i}{\alpha_i} < \rho_i < 0 \). Our solution to this problem is formulated below with the help of the matrix (3.15) and its inverse (3.17).

**Theorem 3.2.** Let \( \rho_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \) and suppose that \( \lambda_i > \alpha_i, \ i = 1, \ldots, n \). System of asymptotic relations (3.8) has regularly varying solutions \((x_1, \ldots, x_n) \in \text{RV}(\rho_1, \ldots, \rho_n) \) with \( \rho_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0) \), \( i = 1, \ldots, n \), if and only if

\[
\frac{\alpha_i - \lambda_i}{\alpha_i} \left(1 - \frac{B_n}{A_n}\right) < \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0
\]  

(3.28)

in which case \( \rho_i \) are given by (3.18) and the asymptotic behavior of any such solution \((x_1, \ldots, x_n)\) is governed by the formula

\[
x_i(t) \sim \prod_{j=1}^{n} \left( t^{\frac{\alpha_j + 1}{\alpha_j}} p_j(t) - \frac{1}{\alpha_j} q_j(t) t^{1/\alpha_j} \right)^{M_{ij}} \frac{A_n}{\alpha_n}, \quad t \to \infty, \ i = 1, \ldots, n.
\]  

(3.29)

where

\[
\Delta_i = (\lambda_i - \alpha_i + \alpha_i \rho_i)^{1/\alpha_i}(-\rho_i), \quad i = 1, \ldots, n.
\]  

(3.30)

**Proof.** Let \((x_1, \ldots, x_n)\) be one such solution on \([T, \infty)\). Using (3.1) and (3.10) we obtain

\[
\int_T^t q_i(s)x_{i+1}(s)^{\beta_i} ds = \int_T^t s^{\mu_i+\beta_i+1} \mu_i(s)x_{i+1}(s)^{\beta_i} ds,
\]  

(3.31)

for \( t \geq T \) and \( i = 1, \ldots, n \), all of which are required to diverge as \( t \to \infty \). Therefore \( \mu_i + \beta_i + 1 \geq -1 \) for all \( i \). If the equality holds for some \( i \), then noting that

\[
\left( \frac{1}{p_i(t)} \right) \int_T^t q_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} = t^{-\frac{\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \left( \int_T^t s^{-1} m_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i},
\]  

and integrating this from \( t \) to \( \infty \), we obtain via Karamata’s integration theorem,

\[
x_i(t) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} t^{-\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \left( \int_T^t s^{-1} m_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} \in \text{RV} \left( \frac{\alpha_i - \lambda_i}{\alpha_i} \right),
\]  

which is a contradiction. It follows that \( \mu_i + \beta_i + 1 > -1 \) for all \( i \). Applying Karamata’s integration theorem to (3.31), we have

\[
\left( \frac{1}{p_i(t)} \right) \int_T^t q_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} \sim \frac{t^{-\frac{\lambda_i + \mu_i + \beta_i + 1}{\alpha_i}}}{(\mu_i + \beta_i + 1)^{1/\alpha_i}}, \quad t \to \infty.
\]  

(3.32)

Since (3.32) is integrable over \([T, \infty)\), we see that \( -\frac{\lambda_i + \mu_i + \beta_i + 1}{\alpha_i} \leq -1 \) for all \( i \). Note that all of these inequalities should be strict, because if the equality holds for some \( i \), then integrating on \([t, \infty)\), we have

\[
x_i(t) \sim (\lambda_i - \alpha_i)^{-1/\alpha_i} \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} m_i(s)^{1/\alpha_i} x_{i+1}(s)^{\beta_i/\alpha_i} ds \in \text{SV},
\]
a contradiction. It follows that \( \frac{\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} < -1 \) for all \( i \), in which case integration of (3.32) on \( [t, \infty) \) yields

\[
x_i(t) \sim \frac{\rho_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{1/\alpha_i}}^\alpha_i \xi_{i+1}(t),
\]

as \( t \to \infty \) and \( i = 1, \ldots, n \). This implies

\[
\rho_i = \frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}, \quad i = 1, \ldots, n,
\]

which is equivalent to the linear algebraic system \( (3.14) \) in \( \rho_i \). From this point on one can proceed exactly as in the proof of the “only if” part of Theorem 3.1, asserting that system (3.8) may have regularly varying solutions of negative indices \( \rho_i \in (\frac{\alpha - \lambda_i}{\alpha_i}, 0) \) only if \( (3.28) \) is fulfilled.

Now we assume that \( (3.28) \) holds. Define \( \rho_i \in (0, \frac{\alpha - \lambda_i}{\alpha_i}) \) by \( (3.18) \) and let \( X_i \in RV(\rho_i) \) denote the functions

\[
X_i(t) = \left[ \prod_{j=1}^n \left( \frac{\rho_{j+1}^{1/\alpha_j} p_{j+1}(t)^{-1/\alpha_j} q_{j+1}(t)^{1/\alpha_j}}{M_{ij}^{1/\alpha_j}} \right)^{A_{i+1}^n_{ji}} \right]^{1/\alpha_i}, \quad i = 1, \ldots, n.
\]

Then the \( X_i \)'s satisfy the system of asymptotic relations (3.8) (with \( T = b \)), i.e.,

\[
\int_t^\infty \left( \frac{1}{p_i(s)} \int_b^s q_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds \sim X_i(t), \quad t \to \infty, \quad i = 1, \ldots, n,
\]

for any \( b \geq a \), where \( X_{n+1}(t) = X_1(t) \). In fact, using the following expression for \( X_i(t) \),

\[
X_i(t) = t^{\rho_i} \Xi_i(t), \quad \Xi_i(t) = \left[ \prod_{j=1}^n \left( \frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j} \xi_{j+1}(t)^{\beta_j/\alpha_j}}{\Delta_j} \right)^{A_{i+1}^n_{ji}} \right]^{1/\alpha_i},
\]

we obtain

\[
\left( \frac{1}{p_i(t)} \int_b^t q_i(s) X_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} \sim \frac{t^{\rho_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\alpha_i - \lambda_i + \alpha_i \rho_i)^{1/\alpha_i}},
\]

and

\[
\int_t^\infty \left( \frac{1}{p_i(s)} \int_b^s q_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds \sim \frac{t^{\rho_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{\Delta_i},
\]

as \( t \to \infty \). Since it can be verified with the help of (3.27) that

\[
\frac{l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}}{\Delta_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i} = \Xi_i(t),
\]

the desired relation (3.35) immediately follows from (3.36). This completes the proof. \( \Box \)
4. REGULARLY VARYING INTERMEDIATE SOLUTIONS OF \( (1.1) \)

We are now in a position to state and prove our main results on the existence and the precise asymptotic behavior of regularly varying intermediate solutions of system \( (1.1) \) with regularly varying coefficients \( p_i \) and \( q_i \). Use is made of the notation and properties of the matrix \( (3.15) \) and \( (3.17) \).

**Theorem 4.1.** Let \( p_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \), \( i = 1, \ldots, n \). Suppose that \( \lambda_i < \alpha_i \) for \( i = 1, \ldots, n \). System \( (1.1) \) possesses intermediate solutions \( \text{RV}(p_1, \ldots, p_n) \) with \( p_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i}) \), \( i = 1, \ldots, n \), if and only if \( (3.19) \) holds, in which case \( p_i \) are given by \( (3.18) \) and the asymptotic behavior of any such solution \( (x_1, \ldots, x_n) \) is governed by the unique formula \( (3.22) \).

**Theorem 4.2.** Let \( p_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \), \( i = 1, \ldots, n \). Suppose that \( \lambda_i > \alpha_i \) for \( i = 1, \ldots, n \). System \( (1.1) \) possesses intermediate solutions \( \text{RV}(p_1, \ldots, p_n) \) with \( p_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0) \), \( i = 1, \ldots, n \), if and only if \( (3.28) \) holds, in which case \( p_i \) are given by \( (3.18) \) and the asymptotic behavior of any such solution \( (x_1, \ldots, x_n) \) is governed by the unique formula \( (3.29) \).

We remark that the “only if” parts of these theorems follow immediately from the corresponding parts of Theorems 3.1 and 3.2 because any solution \( (x_1, \ldots, x_n) \) of \( (1.1) \) with the indicated property satisfies the systems of asymptotic relations \( (3.6) \) plus \( (3.7) \) or \( (3.8) \) plus \( (3.9) \). The “if” parts are proved by way of the following existence theorems for intermediate solutions for system \( (1.1) \) with nearly regularly varying coefficients \( p_i(t) \) and \( q_i(t) \) in the sense defined below.

**Definition 4.3.** Let \( f(t) \) be a regularly varying function of index \( \sigma \) and suppose that \( g(t) \) satisfies \( Kf(t) \leq g(t) \leq kf(t) \) for some positive constants \( k \) and \( K \) and for all large \( t \). Then \( g(t) \) is said to be a nearly regularly varying function of index \( \sigma \). Such a relation between \( f(t) \) and \( g(t) \) is denoted by \( g(t) \asymp f(t) \) as \( t \to \infty \).

**Theorem 4.4.** Let \( p_i \) and \( q_i \) be nearly regularly varying of indices \( \lambda_i \) and \( \mu_i \), respectively; that is, there exist \( \bar{p}_i \in \text{RV}(\lambda_i) \) and \( \bar{q}_i \in \text{RV}(\mu_i) \) such that

\[
p_i(t) \asymp \bar{p}_i(t), \quad q_i(t) \asymp \bar{q}_i(t), \quad t \to \infty, \quad i = 1, \ldots, n.
\]

Suppose in addition that \( \lambda_i < \alpha_i \), \( i = 1, \ldots, n \), and that \( (3.19) \) holds. Then, system \( (1.1) \) possesses intermediate solutions \( (x_1, \ldots, x_n) \) which are nearly regularly varying of positive index \( (p_1, \ldots, p_n) \) with \( p_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i}) \) in the sense that

\[
x_i(t) \asymp \prod_{j=1}^{n} \left( \frac{t^{\frac{\alpha_j+1}{\alpha_j}} \bar{p}_j(t)^{-1/\alpha_j} \bar{q}_j(t)^{1/\alpha_j}}{D_j} \right)^{M_{ij}} \rho_i^{\Delta_i},
\]

for \( t \to \infty \) and \( i = 1, \ldots, n \), where \( \rho_i \) and \( D_i \) are defined by \( (3.18) \) and \( (3.21) \), respectively.

**Theorem 4.5.** Let \( p_i \) and \( q_i \) be nearly regularly varying of indices \( \lambda_i \) and \( \mu_i \), respectively; that is, there exist \( \bar{p}_i \in \text{RV}(\lambda_i) \) and \( \bar{q}_i \in \text{RV}(\mu_i) \) satisfying \( (4.1) \). Suppose that \( \lambda_i > \alpha_i \), \( i = 1, \ldots, n \), and that \( (3.28) \) holds. Then, system \( (1.1) \) possesses intermediate solutions solutions \( (x_1, \ldots, x_n) \) which are nearly regularly varying of negative index \( (p_1, \ldots, p_n) \) with \( p_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0) \), \( i = 1, \ldots, n \), and satisfy

\[
x_i(t) \asymp \prod_{j=1}^{n} \left( \frac{t^{\frac{\alpha_j+1}{\alpha_j}} \bar{p}_j(t)^{-1/\alpha_j} \bar{q}_j(t)^{1/\alpha_j}}{\Delta_j} \right)^{M_{ij}} \rho_i^{\Delta_i}, \quad t \to \infty, \quad i = 1, \ldots, n,
\]

as \( t \to \infty \).
where \( \rho_i \) and \( \Delta_i \) are defined by (3.18) and (3.30).

**Proof of Theorem 4.4.** We assume that the regularly varying functions \( \tilde{p}_i(t) \) and \( \tilde{q}_i(t) \) are expressed in the form
\[
\tilde{p}_i(t) = t^{\alpha_i} \tilde{l}_i(t), \quad \tilde{q}_i(t) = t^{\alpha_i} \tilde{m}_i(t), \quad \tilde{l}_i, \tilde{m}_i \in SV, \quad i = 1, \ldots, n.
\]
(4.4)

By (4.1) there exist positive constants \( h_i, H_i, k_i \) and \( K_i \) such that
\[
h_i \tilde{p}_i(t) \leq p_i(t) \leq H_i \tilde{p}_i(t), \quad k_i \tilde{q}_i(t) \leq q_i(t) \leq K_i \tilde{q}_i(t),
\]
(4.5)

for \( t \geq a \) and \( i = 1, \ldots, n \). Define the functions \( X_i \in RV(\rho_i) \) by
\[
X_i(t) = t^\rho_i \left[ \prod_{j=1}^{n} \left( \frac{\tilde{m}_j(t)^{1/\alpha_j}}{D_j} \right)^{\alpha_n/\alpha_n} \right]^{\frac{1}{\alpha_i}}, \quad t \geq a, \ i = 1, \ldots, n.
\]
(4.6)

It is known that
\[
\int_b^t \left( \frac{1}{\tilde{p}_i(s)} \right) \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \frac{1}{\alpha_i} ds \sim X_i(t), \quad t \to \infty, \ i = 1, \ldots, n,
\]
(4.7)

for any \( b \geq a \), from which it follows that there exists \( T > b \) such that
\[
\int_T^t \left( \frac{1}{\tilde{p}_i(s)} \right) \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \frac{1}{\alpha_i} ds \leq 2X_i(t), \quad t \geq T, \ i = 1, \ldots, n.
\]
(4.8)

Without loss of generality we may assume that each \( X_i(t) \) is increasing on \([T, \infty)\) because it is known that any regularly varying function of positive index is asymptotically equivalent to an increasing RV function of the same index (cf. [II, Theorem 1.5.3]). Since (4.7) holds for \( b = T \) it is possible to choose \( T_1 > T \) so large that
\[
\int_T^{t_1} \left( \frac{1}{\tilde{p}_i(s)} \right) \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr \frac{1}{\alpha_i} ds \geq \frac{1}{2} X_i(t)
\]
(4.9)

for \( t \geq T_1, \ i = 1, \ldots, n \). We define the positive constants \( l_i \) and \( L_i \) \( (l_i \leq L_i) \) by
\[
l_i = \prod_{j=1}^{n} \left\{ \frac{1}{2} \left( \frac{k_j}{H_j} \right)^{1/\alpha_j} M_j \right\}^{\alpha_n/\alpha_n}, \quad L_i = \prod_{j=1}^{n} \left\{ 4 \left( \frac{K_j}{h_j} \right)^{1/\alpha_j} M_j \right\}^{\alpha_n/\alpha_n}, \quad i = 1, \ldots, n.
\]
(4.10)

As is easily verified, \( l_i \) and \( L_i \) satisfy the cyclic systems of equalities
\[
l_i = \frac{1}{2} \left( \frac{K_i}{H_i} \right)^{1/\alpha_i} l_{i+1}, \quad L_i = 4 \left( \frac{K_i}{h_i} \right)^{1/\alpha_i} L_{i+1}, \quad i = 1, \ldots, n,
\]
\[
L_{n+1} = L_1, \quad l_{n+1} = l_1.
\]

Since
\[
\frac{L_i}{l_i} = \prod_{j=1}^{n} \left\{ 8 \left( \frac{H_j K_j}{H_j K_j} \right)^{1/\alpha_j} M_j \right\}^{\alpha_n/\alpha_n},
\]
the constants \( h_i, H_i, k_i \) and \( K_i \) can be chosen so that \( L_i/l_i \geq 2X_i(T_1)/X_i(T) \); that is,
\[
2l_i X_i(T_1) \leq L_i X_i(T), \quad i = 1, \ldots, n,
\]
(4.11)

because these constants are independent of \( X_i(t) \) and the choice of \( T \) and \( T_1 \).

Let \( \mathcal{X} \) denote the set consisting of continuous vector functions \((x_1, \ldots, x_n)\) on \([T, \infty)\) satisfying
\[
l_i X_i(t) \leq x_i(t) \leq L_i X_i(t), \quad t \geq T, \ i = 1, \ldots, n.
\]
(4.12)
It is clear that \( \mathcal{X} \) is a closed convex subset of the locally convex space \( C[T, \infty)^n \). We consider the integral operators \( \mathcal{F}_i \) given by

\[
\mathcal{F}_i x(t) = c_i + \int_T^t \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x(r)^{\beta_i} dr \right)^{1/\alpha_i} ds,
\]

where \( c_i \) are positive constants such that

\[
l_iX_i(T_1) \leq c_i \leq \frac{1}{2}L_iX_i(T), \quad i = 1, \ldots, n, \tag{4.14}
\]

and define the mapping \( \Phi : \mathcal{X} \to C[T, \infty)^n \) by

\[
\Phi(x_1, x_2, \ldots, x_n)(t) = (\mathcal{F}_1x_2(t), \mathcal{F}_2x_3(t), \ldots, \mathcal{F}_nx_{n+1}(t)),
\]

\[
t \geq T, \quad (x_{n+1}(t) = x_1(t)). \tag{4.15}
\]

We will show that the Schauder-Tychonoff fixed point theorem is applicable to \( \Phi \).

(i) \( \Phi \) maps \( \mathcal{X} \) into itself. Let \( (x_1, \ldots, x_n) \in \mathcal{X} \). Then, using (4.8–4.15), we see that

\[
\mathcal{F}_i x_{i+1}(t) \leq \frac{1}{2}L_iX_i(t) + \left( \frac{K_iL_i^{\beta_i}}{h_i} \right)^{1/\alpha_i} \int_T^t \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds
\]

\[
\leq \frac{1}{2}L_iX_i(t) + \left( \frac{K_iL_i^{\beta_i}}{h_i} \right)^{1/\alpha_i} X_i(t)
\]

\[
\leq \frac{1}{2}L_iX_i(t) + \frac{1}{2}L_iX_i(t) = L_iX_i(t) \quad \text{for } t \geq T,
\]

and

\[
\mathcal{F}_i x_{i+1}(t) \geq c_i \geq l_iX_i(T_1) \geq l_iX_i(t) \quad \text{for } T \leq t \leq T_1,
\]

\[
\mathcal{F}_i x_{i+1}(t) \geq \left( \frac{k_iL_i^{\beta_i}}{H_i} \right)^{1/\alpha_i} \int_T^t \left( \frac{1}{p_i(s)} \int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds
\]

\[
\geq \frac{1}{2} \left( \frac{k_iL_i^{\beta_i}}{H_i} \right)^{1/\alpha_i} X_i(t) = l_iX_i(t) \quad t \geq T_1.
\]

This shows that \( \Phi(x_1, \ldots, x_n) \in \mathcal{X} \); that is, \( \Phi \) is a self-map on \( \mathcal{X} \).

(ii) \( \Phi(\mathcal{X}) \) is relatively compact. From the inclusion \( \Phi(\mathcal{X}) \subset \mathcal{X} \) proven above it follows that \( \Phi(\mathcal{X}) \) is locally uniformly bounded on \([T, \infty)\). From the inequalities

\[
0 \leq (\mathcal{F}_i x_{i+1})'(t) \leq L_i^{\beta_i/\alpha_i} \left( \frac{1}{p_i(t)} \int_t^\infty q_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i}, \quad t \geq T, \quad i = 1, \ldots, n,
\]

holding for all \( (x_1, \ldots, x_n) \in \mathcal{X} \) we see that \( \Phi(\mathcal{X}) \) is locally equicontinuous on \([T, \infty)\). The relative compactness of \( \Phi(\mathcal{X}) \) is an immediate consequence of the Arzela-Ascoli lemma.

(iii) \( \Phi \) is continuous. Let \( \{(x_1^\nu(t), \ldots, x_n^\nu(t))\} \) be a sequence in \( \mathcal{X} \) converging as \( \nu \to \infty \) to \( (x_1(t), \ldots, x_n(t)) \in \mathcal{X} \) uniformly on compact subintervals of \([T, \infty)\). Using (4.13) we obtain

\[
|\mathcal{F}_i x_{i+1}^\nu(t) - \mathcal{F}_i x_{i+1}(t)| \leq \int_T^t p_i(s)^{-1/\alpha_i} F_i^\nu(s) ds, \quad t \geq T, \tag{4.16}
\]

where

\[
F_i^\nu(t) = \left| \left( \int_t^\infty q_i(s)x_{i+1}^\nu(s)^{\beta_i} ds \right)^{1/\alpha_i} - \left( \int_t^\infty q_i(s)x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i} \right|.
\]
It is easy to see that
\[
F_i^\nu(t) \leq \left( \int_t^\infty q_i(s) \left| x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i} \right| ds \right)^{1/\alpha_i}, \quad (4.17)
\]
if \( \alpha_i \geq 1 \) and
\[
F_i^\nu(t) \leq \frac{1}{\alpha_i} \left( \frac{L_{i+1}}{\rho_i(s)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds \right)^{1/\alpha_i - 1} \times \int_t^\infty q_i(s) \left| x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i} \right| ds, \quad (4.18)
\]
if \( \alpha_i < 1 \). Combine (4.16) with (4.17) or (4.18) and apply the Lebesgue dominated convergence theorem. Then we conclude that
\[
\lim_{\nu \to \infty} F_i x_{i+1}^\nu = F_i x_{i+1}(t)
\]
uniformly on any compact subset of \( [T, \infty) \), \( i = 1, \ldots, n \), proving the continuity of \( \Phi \).

Therefore, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and \( \Phi \) has a fixed point \( (x_1, \ldots, x_n) \in \mathcal{X} \), which satisfies
\[
x_i(t) = F_i x_{i+1}(t)
\]
\[
= c_i + \int_T^t \left( \frac{1}{\rho_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds, \quad t \geq T, \ i = 1, \ldots, n. \quad (4.19)
\]
This shows that \( (x_1, \ldots, x_n) \) is a solution of system (1.1) on \( [T, \infty) \). Since the obtained solution is a member of \( \mathcal{X} \), it is nearly regularly varying of positive index \( (\rho_1, \ldots, \rho_n) \) and hence is an intermediate solution of (1.1). This completes the proof. \( \square \)

**Proof of Theorem 4.5.** Assume that \( \tilde{p}_i(t) \) and \( \tilde{q}_i(t) \) are expressed in the form (4.4) and satisfy (4.5) for positive constants \( h_i, H_i, k_i \) and \( K_i \). Suppose that (3.28) holds. Define \( \rho_i \) and \( \Delta_i \) by (3.18) and (3.30), respectively, and consider the regularly varying functions of indices \( \rho_i \)
\[
Y_i(t) = t^{\rho_i} \left[ \prod_{j=1}^n \left( \frac{\tilde{L}_j(t)^{-1/\alpha_j} \tilde{m}_j(t)^{1/\alpha_j}}{M_j} \right)^{\Delta_j} \right]^{1/\alpha_i}, \quad i = 1, \ldots, n. \quad (4.20)
\]
Since \( Y_i(t) \) satisfy the asymptotic relations
\[
\int_t^\infty \left( \frac{1}{\tilde{p}_i(s)} \int_b^s \tilde{q}_i(r) Y_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds \sim Y_i(t), \quad t \to \infty, \ i = 1, \ldots, n, \quad (4.21)
\]
one can choose \( T > a \) so that
\[
\frac{1}{2} Y_i(t) \leq \int_t^\infty \left( \frac{1}{\tilde{p}_i(s)} \int_b^s \tilde{q}_i(r) Y_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds \leq 2 Y_i(t), \quad (4.22)
\]
for \( t \geq T \) and \( i = 1, \ldots, n \). All of \( Y_i(t) \) may be assumed to be decreasing on \( [T, \infty) \) because it is known \( \square \) Theorem 1.5.3] that a regularly varying function of negative index is asymptotic to a decreasing RV function of the same index. Denote by \( \mathcal{Y} \) the set consisting of continuous vector functions \( (x_1, \ldots, x_n) \) such that
\[
l_i Y_i(t) \leq x_i(t) \leq L_i Y_i(t), \quad t \geq T, \ i = 1, \ldots, n, \quad (4.23)
\]
where
\[
l_i = \left( \prod_{j=1}^{n} \left( \frac{1}{2} \left( \frac{K_j}{h_j} \right)^{1/\alpha_j} \right) ^{M_{i,j}} \right)^{\frac{A_{i,0}}{m_i}} , \quad L_i = \left( \prod_{j=1}^{n} \left( 2 \left( \frac{K_j}{h_j} \right)^{1/\alpha_j} \right) ^{M_{i,j}} \right)^{\frac{A_{i,0}}{m_i}},
\]
for \( i = 1, \ldots, n \), which satisfy the cyclic systems of equalities
\[
l_i = \frac{1}{2} \left( \frac{K_i}{h_i} \right)^{1/\alpha_i} l_{i+1}^{\beta_{i}/\alpha_i} , \quad L_i = 2 \left( \frac{K_i}{h_i} \right)^{1/\alpha_i} L_{i+1}^{\beta_{i}/\alpha_i} , \quad l_{n+1} = l_1, \quad L_{n+1} = L_1. \tag{4.25}
\]

We now consider the mapping \( \Psi : Y \to C(T, \infty)^n \) defined by
\[
\Psi(x_1, \ldots, x_n)(t) = (\mathcal{G}_1 x_2(t), \mathcal{G}_2 x_3(t), \ldots, \mathcal{G}_n x_{n+1}(t)), \tag{4.26}
\]
for \( t \geq T \) and \( x_{n+1}(t) = x_1(t) \), where \( \mathcal{G}_i \) denotes the integral operator
\[
\mathcal{G}_i x(t) = \int_t^\infty \left( \frac{1}{\tilde{p}_i(s)} \int_s^t q_i(s) x(r)^{\beta_i} dr \right)^{1/\alpha_i} ds , \quad t \geq T, \quad i = 1, \ldots, n. \tag{4.27}
\]

It is a matter of straightforward calculation to verify that \( \Psi \) is a self-map on \( Y \) and sends \( Y \) into a relatively compact subset of \( C(T, \infty)^n \). The details may be omitted. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of a fixed point \( (x_1, \ldots, x_n) \in Y \) of \( \Psi \). This fixed point gives rise to an intermediate solutions of (1.1) which is nearly regularly varying of negative index \((\rho_1, \ldots, \rho_n)\). This completes the proof. \( \square \)

To complete the proof of the “if” parts of Theorems 4.1 and 4.2 it suffices to show that if \( p_i(t) \) and \( q_i(t) \) are assumed to be regularly varying, then the nearly regularly varying solutions obtained in Theorems 4.4 and 4.5 actually become regularly varying of the same specified indices. For this purpose use is made of the following generalized L’Hospital’s rule. See, for example, Haupt and Aumann [4].

**Lemma 4.6.** Let \( f(t), g(t) \in C^1[T, \infty) \) and suppose that
\[
\lim_{t \to \infty} f(t) = \lim_{t \to \infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0 \quad \text{for all large } t,
\]
or
\[
\lim_{t \to \infty} f(t) = \lim_{t \to \infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0 \quad \text{for all large } t.
\]
Then
\[
\lim_{t \to \infty} \inf \frac{f'(t)}{g'(t)} \leq \lim_{t \to \infty} \inf \frac{f(t)}{g(t)} , \quad \lim_{t \to \infty} \sup \frac{f(t)}{g(t)} \leq \lim_{t \to \infty} \sup \frac{f'(t)}{g'(t)}.
\]

Proof of the “if” parts of Theorem 4.4 Suppose that \( p_i \in \text{RV}(\lambda_i) \) and \( q_i \in \text{RV}(\mu_i) \). Suppose in addition that \( \lambda_i < \alpha_i \). Define the positive constants \( \rho_i \) and \( D_i \) by (3.18) and (3.21), respectively, and let \( X_i \in \text{RV}(\rho_i) \) denote the functions on the right-hand side of (4.2) with \( \tilde{p}_i(t) \) and \( \tilde{q}_i(t) \) replaced with \( p_i(t) \) and \( q_i(t) \), respectively. Then, by Theorem 4.4 system (1.1) has a nearly regularly varying solution \((x_1, \ldots, x_n)\) such that \( x_i(t) \asymp X_i(t) \) as \( t \to \infty, \ i = 1, \ldots, n \). Notice that \( x_i(t) \) satisfy the system of integral equations (4.19).

It remains to verify that \( x_i \) are regularly varying functions of indices \( \rho_i, \ i = 1, \ldots, n \), respectively. We define
\[
u_i(t) = \int_T^t \left( \frac{1}{p_i(s)} \int_s^t q_i(r) X_{i+1}(r)^{\beta_i} dr \right)^{1/\alpha_i} ds , \quad i = 1, \ldots, n, \tag{4.28}
\]
and put
\[
\omega_i = \liminf_{t \to \infty} \frac{x_i(t)}{\nu_i(t)} , \quad \Omega_i = \limsup_{t \to \infty} \frac{x_i(t)}{\nu_i(t)} .
\]
Since \( x_i(t) \asymp X_i(t) \) and
\[
u_i(t) \sim X_i(t), \quad t \to \infty, \quad i = 1, \ldots, n, \tag{4.29}\]
it follows that \( 0 < \omega_i \leq \Omega_i < \infty, \quad i = 1, \ldots, n \). Using Lemma 4.6 we obtain
\[
\omega_i \geq \liminf_{t \to \infty} \frac{x_i'(t)}{u_i'(t)} = \liminf_{t \to \infty} \frac{\beta_i}{\alpha_i} \left( \frac{1}{\nu_i(t)} \int_t^\infty q_i(s) X_i(s)^{\beta_i} ds \right)^{1/\alpha_i} \geq \left( \liminf_{t \to \infty} \frac{q_i(t) x_i(t)^{\beta_i}}{q_i(t) X_i(t)^{\beta_i}} \right)^{1/\alpha_i} = \liminf_{t \to \infty} \frac{x_{i+1}(t)^{\beta_i/\alpha_i}}{X_{i+1}(t)^{\beta_i/\alpha_i}} = \omega_{i+1}^{\beta_i/\alpha_i},
\]
where (4.29) has been used in the last step. Thus, \( \omega_i \) satisfy the cyclic system of inequalities
\[
\omega_i \geq \omega_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \ldots, n, \quad \omega_{n+1} = \omega_1. \tag{4.30}
\]
Likewise, by taking the upper limits instead of the lower limits we are led to the cyclic inequalities
\[
\Omega_i \leq \Omega_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \ldots, n, \quad \Omega_{n+1} = \Omega_1. \tag{4.31}
\]
From (4.30) and (4.31) we easily see that
\[
\omega_i \geq \omega_{i+1}^{\beta_i/\alpha_i}, \quad \Omega_i \leq \Omega_{i+1}^{\beta_i/\alpha_i},
\]
whence, because of the hypothesis \( \beta_1 \ldots \beta_n/\alpha_1 \ldots \alpha_n < 1 \), we find that \( \omega_i \geq 1 \) and \( \Omega_i \leq 1 \). It follows therefore that \( \omega_i = \Omega_i = 1 \) or \( \lim_{t \to \infty} x_i(t)/u_i(t) = 1 \) for \( i = 1, \ldots, n \). This combined with (4.29) implies that \( x_i(t) \sim u_i(t) \sim X_i(t) \) as \( t \to \infty \), which shows that each \( x_i(t) \) is a regularly varying function of index \( \rho_i \).

Thus the proof of the “if” part of Theorem 4.1 is complete.

In essentially the same way one can complete the proof of the “if” part of Theorem 4.1.

\[\square\]

5. Applications to partial differential equations

The purpose of the final section is to demonstrate that our results on cyclic systems of ordinary differential equations (1.1) can be applied to some classes of partial differential equations to provide new information about the existence and asymptotic behavior of their radial positive solutions. Throughout this section \( x = (x_1, \ldots, x_N) \) represents the space variable in \( \mathbb{R}^N, \quad N \geq 2 \), and \( |x| \) denotes the Euclidean length of \( x \). All partial differential equations will be considered in an exterior domain \( \Omega_R = \{ x \in \mathbb{R}^N : |x| \geq R \}, \quad R > 0 \).

5.1. Systems of \( p \)-Laplacian equations. We are concerned with the system of nonlinear \( p \)-Laplacian equations
\[
\div (|\nabla u_i|^{p-2} \nabla u_i) + f_i(|x|) u_i^{\gamma_i-1} u_{i+1} = 0, \quad i = 1, \ldots, n, \quad (u_{n+1} = u_1) \tag{5.1}
\]
where \( p > 1 \) and \( \gamma_i > 0 \) are constants, and \( f_i(t) \) are positive continuous functions on \([a, \infty)\) which are regularly varying of indices \( \nu_i, i = 1, \ldots, n \). Our attention will be focused on radial solutions \((u_1(|x|), \ldots, u_n(|x|))\) of (5.1) defined in \( \Omega_R, R > a \).

A radial vector function \((u_1(|x|), \ldots, u_n(|x|))\) is a solution of (5.1) in \( \Omega_R \) if and only if \((u_1(t), \ldots, u_n(t))\) is a solution of the system of ordinary differential equations

\[
\begin{align*}
(t^{N-1}|u'|^{p-2}u_t)' + t^{N-1}f_i(t)|u_i|^{\gamma_i-1}u_{i+1} &= 0, & t \geq a, & i = 1, \ldots, n, \\
u_{n+1} &= u_1
\end{align*}
\]

which is a special case of system (4.1) with

\[
\begin{align*}
\alpha_1 = \cdots = \alpha_n &= p - 1, & \beta_i &= \gamma_i, & i = 1, \ldots, n; \\
\lambda_1 = \cdots = \lambda_n &= N - 1, & \mu_i &= N - 1 + \nu_i, & i = 1, \ldots, n.
\end{align*}
\]

It is always assumed that

\[
\gamma_1 \cdots \gamma_n < (p - 1)^n.
\]

We need the matrix \( A(p; \gamma_1, \ldots, \gamma_n) \) associated with (5.2) and its inverse (cf. (3.15) and (3.17)). We define

\[
(M_{ij}) = \frac{(p - 1)^n - \gamma_1 \cdots \gamma_n}{(p - 1)^n} A\left(\frac{\gamma_1}{p - 1}, \ldots, \frac{\gamma_n}{p - 1}\right)^{-1}.
\]

To analyze (5.2) it is necessary to distinguish the two cases \( p > N \) and \( p < N \).

(i) Suppose that \( p > N \). In this case applying Theorem 4.1 to (5.2), we conclude that system (5.1) possesses decreasing radial solutions \((u_1(|x|), \ldots, u_n(|x|))\) such that \( u_i \in \text{RV}(\rho_i), 0 < \rho_i < \frac{p-N}{p-1}, i = 1, \ldots, n, \) if and only if

\[
0 < \sum_{j=1}^{n} M_{ij}(p + \nu_j) < (p - N) \left(1 - \frac{\gamma_1 \cdots \gamma_n}{(p - 1)^n}\right), & i = 1, \ldots, n. \tag{5.5}
\]

In this case the \( \rho_i \)'s are uniquely determined by

\[
\rho_i = \frac{(p - 1)^{n-1}}{(p - 1)^n - \gamma_1 \cdots \gamma_n} \sum_{j=1}^{n} M_{ij}(p + \nu_j), & i = 1, \ldots, n, \tag{5.6}
\]

and moreover the asymptotic behavior of any such solution as \( |x| \to \infty \) is governed by the unique growth law

\[
u_i(|x|) \sim |x|^\rho_i \prod_{j=1}^{n} \left(\varphi_j(|x|) \frac{\varphi_j(|x|)}{(p - N - (p - 1)\rho_j)\rho_j^{p_j-1}}\right)^{M_{ij}} \left(\frac{\rho_i^{-1}}{(p - 1)^n - \gamma_1 \cdots \gamma_n}\right)^{M_{ij}}, & |x| \to \infty, \tag{5.7}
\]

for \( i = 1, \ldots, n, \) where \( \varphi_i \in \text{SV} \) is the slowly varying part of \( f_i; \) that is, \( f_i(t) = t^{\nu_i\varphi_i(t)} \).

(ii) Suppose that \( p < N \). In this case from Theorem 4.2 applied to (5.2) it follows that system (5.1) possesses decreasing radial solutions \((u_1(|x|), \ldots, u_n(|x|))\) such that \( u_i \in \text{RV}(\rho_i), \frac{p-N}{p-1} < \rho_i < 0, i = 1, \ldots, n, \) if and only if

\[
(p - N) \left(1 - \frac{\gamma_1 \cdots \gamma_n}{(p - 1)^n}\right) < \sum_{j=1}^{n} M_{ij}(p + \nu_j) < 0, & i = 1, \ldots, n. \tag{5.8}
\]
In this case, \( \rho_i \) are uniquely determined by (5.6) and the asymptotic behavior of any such solution as \(|x| \to \infty\) is governed by the unique decay law

\[
u_i (|x|) \sim |x|^{p_i} \prod_{j=1}^{n} \left( \frac{\varphi_j(|x|)}{(N - p + (p - 1)\rho_j)(-\rho_j)^{p-1}} \right)^{M_{ij}} |x|^{(p-1)n-\gamma_i},
\]

as \(|x| \to \infty\), for \(i = 1, \ldots, n\), where \(\varphi_j\) is the slowly varying part of \(f_i\).

Consider the particular case of (5.1) in which \(f_i(t) \equiv c_i > 0\), i.e.,

\[
\text{div}\left(|\nabla u_i|^{p-2}\nabla u_i\right) + c_i |u_{i+1}|^{\gamma-1} u_{i+1} = 0, \quad i = 1, \ldots, n, \quad u_{n+1} = u_1. \quad (5.9)
\]

In this case, \(\nu_i = 0\) for all \(i\), and so (5.5) and (5.8) are always violated. Therefore, system (5.9) cannot admit intermediate radial solutions \((u_1(|x|), \ldots, u_N(|x|))\) such that \(u_i \in \text{RV}(\rho_i)\), where \(\rho_i\) satisfy

\[
0 < \rho_i < \frac{p-N}{p-1}, \quad i = 1, \ldots, n, \quad \text{if } p > N;
\]

\[
\frac{p-N}{p-1} < \rho_i < 0, \quad i = 1, \ldots, n, \quad \text{if } p < N.
\]

5.2. **Nonlinear metaharmonic equations.** Now we consider the nonlinear metaharmonic equation

\[
\Delta^m u = (-1)^m g(|x|)|u|^\gamma-1 u, \quad x \in \Omega_R, \quad (5.10)
\]

where \(N \geq 3, \ m \geq 2\) and \(\gamma > 0\) are constants, and \(g(t)\) is a positive continuous function on \([a, \infty)\) which is regularly varying of index \(\nu\). We are interested in radial positive solutions \(u\) of (5.10) such that \(u\) and \((-1)^i \Delta u, i = 1, \ldots, m-1\), are regularly varying of negative indices. It is clear that seeking such solutions of (5.10) is equivalent to seeking radial regularly varying solutions of negative indices of the system

\[
\Delta u_i + u_{i+1} = 0, \quad i = 1, \ldots, m-1,
\]

\[
\Delta u_m + g(|x|)|u_{m+1}|^{\gamma-1} u_{m+1} = 0, \quad x \in \Omega_R, \quad (5.11)
\]

where \(u_{m+1} = u_1\). This system is equivalent to the system of ordinary differential equations

\[
(t^{N-1} u_i^\prime) + t^{N-1} u_{i+1} = 0, \quad i = 1, \ldots, m-1,
\]

\[
(t^{N-1} u_m^\prime) + t^{N-1} g(t)|u_{m+1}|^{\gamma-1} u_{m+1} = 0, \quad t \geq R, \quad (5.12)
\]

which is a special case of (1.1) with

\[
\alpha_1 = \ldots = \alpha_m = 1, \quad \beta_i = \ldots = \beta_{m-1} = 1, \quad \beta_m = \gamma,
\]

\[
\lambda_1 = \ldots = \lambda_m = N - 1, \quad \mu_1 = \ldots = \mu_{m-1} = N - 1, \quad \mu_m = N - 1 + \nu.
\]

We assume that \(\gamma < 1\). The \(m \times m\)-matrix (3.15) associated with (5.12) reads \(A(1, \ldots, 1, \gamma)\). Define the matrix \((M_{ij})\) by

\[
(M_{ij}) = (1 - \gamma) A(1, \ldots, 1, \gamma)^{-1}, \quad (5.13)
\]

As is easily checked, \(M_{ij} = 1\) for \(1 \leq i \leq j \leq m\) and \(M_{ij} = \gamma\) for \(1 \leq j < i \leq m\).

Since \(\lambda_1 = N - 1 > 1 = \alpha_i\) for all \(i\), Theorem 4.1 can be utilized to determine the structure of decreasing regularly varying solutions \((u_1, \ldots, u_m) \in \text{RV}(\rho_1, \ldots, \rho_m), \)
2 – N < ρ_i < 0, of the cyclic system (5.12). The regularity indices ρ_i should be given by (3.18) which in the present situation reduce to
\[ ρ_i = \frac{2m + ν}{1 − γ} − 2(i − 1), \quad i = 1, \ldots, m, \]  
(5.14)

from which we see that all ρ_i are admissible if and only if
\[ 2m − N < \frac{2m + ν}{1 − γ} < 0 \iff −2m + (2m − N)(1 − γ) < ν < −2m. \]  
(5.15)

Clearly, (5.15) makes sense only if N > 2m, in which case it is concluded that equation (5.10) possesses radial positive solutions u(|x|) ∈ RV(ρ_1), where
\[ ρ_1 = \frac{2m + ν}{1 − γ} ∈ (2 − N, 0), \]  
(5.16)
such that (−1)^i Δ^i u(|x|) ∈ RV(ρ_{i+1}), 2 − N < ρ_{i+1} < 0, for i = 1, \ldots, m − 1.

Furthermore, the asymptotic behavior of any such solution u(|x|) is governed by the formula
\[ u(|x|) ∼ |x|^{ρ_1} \left[ \frac{ψ(|x|)}{\prod_{j=1}^{m−1}(N−2 + ρ_j)(−ρ_j)} \right]^{1/(1−γ)} \]  
(5.17)
as |x| → ∞, where ψ(t) denotes the slowly varying function such that g(t) = t^ν ψ(t).

We remark that the particular case of (5.10),
\[ Δ^m u = (−1)^m c|u|^−1 u, \quad x ∈ Ω_R, \]
where c > 0 is a constant, can by no means possess radial solutions u(|x|) ∈ RV(ρ_1) with 2 − N < ρ_1 < 0.

Acknowledgments. The authors would like to express their sincere thanks to the anonymous referees for their valuable comments and suggestions.

The first author was supported by the grant No.1/0071/14 of the Slovak Grant Agency VEGA.

References


[17] V. Marić, M. Tomić; Asymptotic properties of solutions of the equation $y'' = f(x)\phi(y)$, Math. Z., 149 (1976), 261–266.


Jaroslav Jaroš  
Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, 842 48 Bratislava, Slovakia  
E-mail address: jaro@fmph.uniba.sk

Kusano Takaši  
Department of Mathematics, Faculty of Science, Hiroshima University, Higashi Hiroshima 739-8526, Japan  
E-mail address: kusanot@zj8.so-net.ne.jp

Tomoyuki Tanigawa  
Department of Mathematics, Faculty of Education, Kumamoto University, Kumamoto 860-8555, Japan  
E-mail address: tanigava@educ.kumamoto-u.ac.jp