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REGULARLY VARYING SOLUTIONS WITH INTERMEDIATE GROWTH FOR CYCLIC DIFFERENTIAL SYSTEMS OF SECOND ORDER

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Abstract. In this article, we study the existence and accurate asymptotic behavior as $t \to \infty$ of positive solutions with intermediate growth for a class of cyclic systems of nonlinear differential equations of the second order

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' + q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1} = 0, \quad i = 1,\dots,n,\ x_{n+1} = x_1,$$

where α_i and β_i, $i = 1,\dots,n$, are positive constants such that $\alpha_1 \dots \alpha_n > \beta_1 \dots \beta_n$ and $p_i, q_i : [a,\infty) \to (0,\infty)$ are continuous regularly varying functions (in the sense of Karamata). It is shown that the situation in which the system possesses regularly varying intermediate solutions can be completely characterized, and moreover that the asymptotic behavior of such solutions is governed by the unique formula describing their order of growth (or decay) precisely. The main results are applied to some classes of partial differential equations with radial symmetry including metaharmonic equations and systems involving p-Laplace operators on exterior domains.

1. Introduction

In this article we analyze the differential system

$$(p_i(t)|x_i'|^{\alpha_i-1}x_i')' + q_i(t)|x_{i+1}|^{\beta_i-1}x_{i+1} = 0, \quad i = 1,\dots,n,\ x_{n+1} = x_1, \tag{1.1}$$

where

(a) α_i and β_i, $i = 1,\dots,n$, are positive constants such that $\alpha_1 \dots \alpha_n > \beta_1 \dots \beta_n$;
(b) p_i and q_i are continuous positive functions on $[a,\infty)$, $a > 0$;
(c) all p_i simultaneously satisfy either

$$\int_a^\infty p_i(t)^{-1/\alpha_i} dt = \infty, \quad i = 1,\dots,n, \tag{1.2}$$

or

$$\int_a^\infty p_i(t)^{-1/\alpha_i} dt < \infty, \quad i = 1,\dots,n. \tag{1.3}$$

By a positive solution of (1.1) we mean a vector function $(x_1,\dots,x_n)$ consisting of components x_i, $i = 1,\dots,n$, which are positive and continuously differentiable together with $p_i|x_i'|^{\alpha-1}x_i'$ on some interval $[T,\infty)$ and satisfy system (1.1) there.

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Systems of the form (1.1) with $p_i(t) = t^{N-1}$ and $q_i(t) = t^{N-1} f_i(t)$, $N \ge 2$, $i = 1, \dots, n$, arise in the study of positive radial solutions in exterior domains in $\mathbb{R}^N$ for the system of p_i-Laplacian equations

$$\Delta_{p_i} u_i \equiv \operatorname{div}\big(|\nabla u_i|^{p_i-2}\nabla u_i\big) + f_i(|x|)|u_{i+1}|^{\gamma_i-1}u_{i+1} = 0, i = 1, \dots, n, \\ u_{n+1} = u_1, \tag{1.4}$$

where $p_i > 1$ and $\gamma_i > 0, i = 1, \dots, n$, are constants, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}^N$ and f_i, $i = 1, \dots, n$, are positive continuous functions on $[a, \infty)$.

Quasilinear elliptic system (1.4) with negative $f_i, i = 1, \dots, n$, and the exponents satisfying the super-homogeneity condition $\gamma_1\gamma_2 \dots \gamma_n > (p_1 - 1)(p_2 - 1) \dots (p_n - 1)$ was studied by Teramoto [22], while the problem of the existence and precise asymptotic behavior as $|x| \to \infty$ of positive strongly decreasing (resp. strongly increasing) radial solutions of (1.4) in the case $p_1 = \dots = p_n = p > 1$ under the sub-homogeneity assumption $\gamma_1\gamma_2 \dots \gamma_n < (p - 1)^n$ was investigated in [7] (resp. [8]). (For the special case $p_1 = \dots = p_n = 2$ see also [23].)

In this article we are concerned with positive solutions $(x_1, \dots, x_n)$ of (1.1) all components of which have the intermediate growth (or decay) in the sense that they are increasing to infinity as $t \to \infty$ and satisfy

$$\lim_{t\to\infty} p_i(t)|x_i'(t)|^{\alpha_i-1}x_i'(t) = 0, \quad i = 1, \dots, n, \quad \text{in case (1.2) holds}, \tag{1.5}$$

or decreasing to zero as $t \to \infty$ and satisfy

$$\lim_{t\to\infty} p_i(t)|x_i'(t)|^{\alpha_i-1}x_i'(t) = -\infty, \quad i = 1, \dots, n, \quad \text{in case (1.3) holds}. \tag{1.6}$$

Note that this is equivalent to

$$\lim_{t\to\infty} x_i(t) = \infty, \quad \lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = 0, \quad i = 1, \dots, n, \tag{1.7}$$

if (1.2) holds, where $P_i(t) = \int_a^t p_i(s)^{-1/\alpha_i} ds$, or to

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = \infty, \quad i = 1, \dots, n, \tag{1.8}$$

if (1.3) holds, where $\pi_i(t) = \int_t^\infty p_i(s)^{-1/\alpha_i} ds$.

In the scalar case, i.e., if (1.1) reduces to a single equation of the form

$$(p(t)|x'|^{\alpha-1}x')' + q(t)|x|^{\beta-1}x = 0, \tag{1.9}$$

where α and β are positive constants such that $\alpha > \beta$ and p and q are positive continuous functions on $[a, \infty)$, necessary and sufficient conditions for the existence of intermediate solutions of (1.9) have been established for the case (1.2) by Naito [20] and for the case (1.3) by Kamo and Usami [11].

It is to be noticed that system (1.1) may possess also positive solutions which have an extreme growth (or decay) in the sense that if (1.2) holds, then each component x_i satisfies either

$$\lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = \text{const} > 0, \tag{1.10}$$

or

$$\lim_{t\to\infty} x_i(t) = \text{const} > 0, \tag{1.11}$$

and if (1.3) holds, then each component x_i satisfies either (1.11) or

$$\lim_{t\to\infty}\frac{x_i(t)}{\pi_i(t)} = \text{const} > 0. \tag{1.12}$$

Positive solutions of these types are not considered here.

Once the existence of intermediate solutions of (1.9) (or (1.1)) has been confirmed, a natural question arises as to the possibility of determining their asymptotic behavior at infinity accurately. Partial answers to this question in the scalar case have recently been given in the papers [9, 14] which are concerned exclusively with regularly varying intermediate solutions of equation (1.9) with regularly varying coefficients $p(t)$ and $q(t)$. Restricting our consideration within the framework of regular variation allows us to utilize basic theory of regular variation to acquire thorough and precise information about the existence, the asymptotic behavior and the structure of regularly varying intermediate solutions of equation (1.9). For the definition of regularly varying functions see Section 2.

A prototype of the results we are going to prove says that if f and g are regularly varying functions of indices λ and μ, respectively, and $p > N$, then the necessary and sufficient condition for the existence of positive intermediate radial solutions components of which are regularly varying functions with indices in the interval $(0, \frac{p-N}{p-1})$ of the system of two equations

$$\Delta_p u + f(|x|)v^{\alpha} = 0, \quad \Delta_p v + g(|x|)u^{\beta} = 0, \tag{1.13}$$

where $\alpha\beta < (p-1)^2$, is the satisfaction of the system of inequalities

$$\begin{gathered} 0 < p+\lambda+\frac{\alpha}{p-1}(p+\mu) < (p-N)\Big(1-\frac{\alpha\beta}{(p-1)^2}\Big), \\ 0 < \frac{\beta}{p-1}(p+\lambda)+p+\mu < (p-N)\Big(1-\frac{\alpha\beta}{(p-1)^2}\Big), \end{gathered}$$

and if $p < N$, then the above two-dimensional system has intermediate RV solutions with indices in $(\frac{p-N}{p-1}, 0)$ if and only if

$$(p-N)\Big(1-\frac{\alpha\beta}{(p-1)^2}\Big) < p+\lambda+\frac{\alpha}{p-1}(p+\mu) < 0,$$

and

$$(p-N)\Big(1-\frac{\alpha\beta}{(p-1)^2}\Big) < \frac{\beta}{p-1}(p+\lambda)+p+\mu < 0.$$

In both cases the indices ρ and σ of regular variation of the components u and v, respectively, are given (uniquely) by

$$\rho = \frac{p-1}{(p-1)^2-\alpha\beta}\Big[p+\lambda+\frac{\alpha}{p-1}(p+\mu)\Big], \quad \sigma = \frac{p-1}{(p-1)^2-\alpha\beta}\Big[\frac{\beta}{p-1}(p+\lambda)+p+\mu\Big],$$

and any such intermediate solution (u, v) as $|x| \to \infty$ satisfies the asymptotic relation

$$\begin{gathered} u(|x|) \sim |x|^{\rho}\Big[\frac{\varphi(|x|)^{\frac{1}{p-1}}}{D(\rho)}\Big(\frac{\psi(|x|)^{\frac{1}{p-1}}}{D(\sigma)}\Big)^{\frac{\alpha}{p-1}}\Big]^{\frac{(p-1)^2}{(p-1)^2-\alpha\beta}}, \\ v(|x|) \sim |x|^{\sigma}\Big[\Big(\frac{\varphi(|x|)^{\frac{1}{p-1}}}{D(\rho)}\Big)^{\frac{\beta}{p-1}}\frac{\psi(|x|)^{\frac{1}{p-1}}}{D(\sigma)}\Big]^{\frac{(p-1)^2}{(p-1)^2-\alpha\beta}}, \end{gathered}$$

where φ and ψ are the slowly varying parts of f and g, respectively, and $D(\tau) = \big(p - N - (p-1)|\tau|\big)^{\frac{1}{p-1}}|\tau|$.

The main results of this paper will be presented in Section 4. The existence of intermediate regularly varying solutions of (1.1) is proved by solving the system of integral equations

$$x_i(t) = c_i + \int_T^t \Big(\frac{1}{p_i(s)}\int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds, \quad i = 1,\dots,n, \quad \text{if (1.2) holds,} \tag{1.14}$$

$$x_i(t) = \int_t^\infty \frac{1}{p_i(s)}\Big(\int_T^s q_i(r)x_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds, \quad i = 1,\dots,n, \quad \text{if (1.3) holds,} \tag{1.15}$$

for some constants $T \ge a$ and $c_i > 0$ with the help of fixed point techniques combined with basic theory of regularly varying functions. Furthermore it is shown that the asymptotic behavior of the obtained solutions is governed by the unique explicit law describing their order of growth (in case (1.2) holds) or decay (in case of (1.3) holds) accurately. To this end extensive use is made of the knowledge derived through the analysis of the following systems of asymptotic integral relations associated with (1.14) and (1.15) by means of regular variation:

$$x_i(t) \sim \int_T^t \Big(\frac{1}{p_i(s)}\int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds, \quad t\to\infty, \quad i = 1,\dots,n, \tag{1.16}$$

$$x_i(t) \sim \int_t^\infty \Big(\frac{1}{p_i(s)}\int_T^s q_i(r)x_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds, \quad t\to\infty, \quad i = 1,\dots,n. \tag{1.17}$$

Here and hereafter the notation $f(t) \sim g(t)$ as $t \to \infty$ is used to mean

$$\lim_{t\to\infty}\frac{g(t)}{f(t)} = 1.$$

The details of the analysis of systems (1.16) and (1.17) in the framework of regular variation is presented in Section 3, which is preceded by Section 2 where the definition and some basic properties of regularly varying functions are summarized for the reader's convenience. The final Section 5 is designed to explain the effective applicability of our results for (1.1) to some classes of partial differential equations with radial symmetry including metaharmonic equations and systems involving p-Laplace operators on exterior domains in $\mathbb{R}^N$.

The systematic study of differential equations in the framework of regular variation was initiated by Marić and Tomić [17, 18, 19]. Since the publication of the monograph of Marić [16] in the year 2000 there has been an increasing interest in the study of asymptotic properties of positive solutions of differential equations by means of regularly varying functions, and it has turned out that theory of regular variation combined with fixed point techniques is so powerful as to cover a wide class of ordinary differential equations including generalized Emden-Fowler and Thomas-Fermi equations, and systems of such equations; see, for example, [2, 3, 5, 6, 9, 10, 12, 13, 14, 15].

2. Regularly varying functions

For the reader's benefit we recall here the definition and basic properties of regularly varying functions which will be used in this paper.

Definition 2.1. A measurable function $f : [0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for all } \lambda > 0.$$

The set of all regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$.

Typical example of a function from $\mathrm{RV}(\rho)$ is the power function t^{ρ} or, more generally, any function of the form $ct^{\rho}(1 + \varepsilon(t))$ where c is a positive constant and $\varepsilon(t)$ a measurable function on $(0, \infty)$ such that $\varepsilon(t) \to 0$ as $t \to \infty$. All these are examples of the so-called *trivial* regularly varying functions of index ρ, i.e., functions $f \in \mathrm{RV}(\rho)$ satisfying $\lim_{t\to\infty} f(t)/t^{\rho} = \text{const} > 0$. An example of a *nontrivial* regularly varying function of index ρ is $t^{\rho}\log(1 + t)$.

We often use the symbol SV for $\mathrm{RV}(0)$ and call members of SV *slowly varying functions*. It is easy to check that any $f \in \mathrm{RV}(\rho)$ can always be represented in the form $f(t) = t^{\rho}g(t)$, where $g \in \mathrm{SV}$. Thus, the properties of regularly varying functions are easily deduced from the corresponding properties of slowly varying functions.

One of the most important properties of regularly varying functions is the following *representation theorem*.

Proposition 2.2. *$f \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is represented in the form*

$$f(t) = c(t)\exp\Big\{\int_{t_0}^{t} \frac{\delta(s)}{s} ds\Big\}, \quad t \ge t_0, \tag{2.1}$$

for some $t_0 > 0$ and for some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0, \infty) \quad \textit{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

If in particular $c(t) \equiv c_0$ in (2.1), then $f(t)$ is referred to as a normalized *regularly varying function of index ρ.*

Examples of slowly varying functions include all functions tending to some positive constants as $t \to \infty$, the logarithmic function, its powers $\log^{\gamma} t, \gamma \in R$, iterated logarithms. A more sophisticated example of a member from SV is the function $\exp\{c\log^{\gamma} t\cos^{\delta}(\log t)\}$, where $c > 0$ and $\gamma, \delta \ge 0$ are such that $\gamma + \delta < 1$. The following result illustrates operations which preserve slow variation.

Proposition 2.3. *Let $L(t)$, $L_1(t)$, $L_2(t)$ be slowly varying. Then, $L(t)^{\alpha}$ for any $\alpha \in \mathbb{R}$, $L_1(t) + L_2(t)$, $L_1(t)L_2(t)$ and $L_1(L_2(t))$ (if $L_2(t) \to \infty$) are slowly varying.*

A slowly varying function may grow to infinity or decay to 0 as $t \to \infty$. But its order of growth or decay is severely limited as is shown in the following

Proposition 2.4. *Let $f \in \mathrm{SV}$. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} f(t) = \infty, \quad \lim_{t\to\infty} t^{-\varepsilon} f(t) = 0.$$

A simple criterion for determining the regularity of differentiable positive functions follows.

Proposition 2.5. *A differentiable positive function $f(t)$ is a normalized regularly varying function of index ρ if and only if*

$$\lim_{t\to\infty} t\frac{f'(t)}{f(t)} = \rho.$$

The following proposition known as Karamata's integration theorem will play an important role in this paper.

Proposition 2.6. *Let $L(t)$ be a slowly varying function. Then:*

(i) *if* $\alpha > -1$,
$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$
(ii) *if* $\alpha < -1$,
$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$
(iii) *if* $\alpha = -1$,
$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV},$$
and if, in addition, $\int_a^\infty s^{-1}L(s)ds < \infty$, *then*
$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV}.$$

Definition 2.7. A vector function $(x_1(t), \dots, x_n(t))$ is said to be regularly varying of index $(\rho_1, \dots, \rho_n)$ if $x_i \in \mathrm{RV}(\rho_i)$ for $i = 1, \dots, n$. If all ρ_i are positive (or negative), then $(x_1(t), \dots, x_n(t))$ is called regularly varying of positive (or negative) index $(\rho_1, \dots, \rho_n)$. The set of all regularly varying vector functions of index $(\rho_1, \dots, \rho_n)$ is denoted by $\mathrm{RV}(\rho_1, \dots, \rho_n)$.

For a complete exposition of theory of regular variation and its applications the reader is referred to the treatise of Bingham, Goldie and Teugels [1]. See also Seneta [21]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [16].

3. Systems of asymptotic relations associated with (1.1)

We assume that $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ and that they are represented as
$$p_i(t) = t^{\lambda_i} l_i(t), \quad q_i(t) = t^{\mu_i} m_i(t), \quad l_i,\ m_i \in \mathrm{SV}, \quad i = 1, \dots, n. \tag{3.1}$$
In addition we require that $p_i(t)$ satisfy either (1.2) or (1.3). It is easy to see that (1.2) (resp. (1.3)) holds if and only if
$$\lambda_i < \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} dt = \infty,$$
resp.
$$\lambda_i > \alpha_i, \quad \text{or} \quad \lambda_i = \alpha_i \quad \text{and} \quad \int_a^\infty t^{-1} l_i(t)^{-1/\alpha_i} dt < \infty.$$
Therefore, in case (1.2) is satisfied, the functions $P_i(t) = \int_a^t p_i(s)^{-1/\alpha_i} ds$, $i = 1, \dots, n$, are given by
$$P_i(t) = \int_a^t s^{-1} l_i(s)^{-1/\alpha_i} ds \quad \text{if } \lambda_i = \alpha_i, \tag{3.2}$$
$$P_i(t) \sim \frac{\alpha_i - \lambda_i}{\alpha_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i}, \quad t \to \infty, \quad \text{if} \quad \lambda_i < \alpha_i, \tag{3.3}$$

and in case (1.3) holds, the functions $\pi_i(t) = \int_t^\infty p_i(s)^{-1/\alpha_i} ds$, $i = 1,\dots,n$, are given by

$$\pi_i(t) = \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} ds \quad \text{if } \lambda_i = \alpha_i, \tag{3.4}$$

$$\pi_i(t) \sim \frac{\lambda_i - \alpha_i}{\alpha_i} t^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i}, \quad t \to \infty, \quad \text{if } \lambda_i > \alpha_i. \tag{3.5}$$

Our task in this section is to solve the following two problems.

Problem (i): Under the condition (1.2) characterize the situation in which the system of asymptotic relations

$$x_i(t) \sim \int_T^t \Big(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1,\dots,n, \tag{3.6}$$

possesses regularly varying solutions $(x_1,\dots,x_n)$ of positive index $(\rho_1,\dots,\rho_n)$ satisfying

$$\lim_{t\to\infty} x_i(t) = \infty, \quad \lim_{t\to\infty} \frac{x_i(t)}{P_i(t)} = 0, \quad i = 1,\dots,n. \tag{3.7}$$

Problem (ii): Under the condition (1.3) characterize the situation in which the system of asymptotic relations

$$x_i(t) \sim \int_t^\infty \Big(\frac{1}{p_i(s)} \int_T^s q_i(r) x_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds, \quad t \to \infty, \quad i = 1,\dots,n, \tag{3.8}$$

possesses regularly varying solutions $(x_1,\dots,x_n)$ of negative index $(\rho_1,\dots,\rho_n)$ satisfying

$$\lim_{t\to\infty} x_i(t) = 0, \quad \lim_{t\to\infty} \frac{x_i(t)}{\pi_i(t)} = \infty, \quad i = 1,\dots,n. \tag{3.9}$$

The positivity or negativity requirement for the regularity indices of solutions excludes the possibility that $\lambda_i = \alpha_i$ for some or all i which necessarily reduces the corresponding components $x_i(t)$ to slowly varying functions ($\rho_i = 0$) (cf. (3.2) and (3.4)). The presence of slowly varying components in the solutions seems to cause computational difficulty.

We begin with Problem (i). We assume that $\lambda_i < \alpha_i$, $i = 1,\dots,n$, and seek solutions $(x_1,\dots,x_n)$ of (3.6) belonging to $\mathrm{RV}(\rho_1,\dots,\rho_n)$ with all ρ_i positive. In view of (3.3) each ρ_i must satisfy $\rho_i > \frac{\alpha_i-\lambda_i}{\alpha_i}$. Let $(x_1,\dots,x_n)$ be one such solution on $[T,\infty)$. Suppose that x_i are expressed in the form

$$x_i(t) = t^{\rho_i}\xi_i(t), \quad \xi_i \in \mathrm{SV}, \quad i = 1,\dots,n. \tag{3.10}$$

Using (3.1) and (3.10), we have

$$\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds = \int_t^\infty s^{\mu_i+\beta_i\rho_{i+1}} m_i(s) \xi_{i+1}(s)^{\beta_i} ds, \tag{3.11}$$

for $t \ge T$ and $i = 1,\dots,n$. The convergence of (3.11) as $t \to \infty$ implies that $\mu_i + \beta_i\rho_{i+1} \le -1$, $i = 1,\dots,n$, but the equality should be ruled out. In fact, if the equality holds for some i, then since

$$\Big(\frac{1}{p_i(t)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} = t^{-\frac{\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \Big(\int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i},$$

from (3.6) and Karamata's integration theorem we find that

$$x_i(t) \sim \frac{\alpha_i}{\alpha_i - \lambda_i} t^{\frac{\alpha_i-\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \Big(\int_t^\infty s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} \in \mathrm{RV}\Big(\frac{\alpha_i - \lambda_i}{\alpha_i}\Big)$$

as $t \to \infty$. This implies that $\rho_i = \frac{\alpha_i - \lambda_i}{\alpha_i}$, a contradiction. It follows that $\mu_i + \beta_i \rho_{i+1} < -1$ for $i = 1, \dots, n$, and application of Karamata's integration theorem to (3.11) gives

$$\begin{aligned} &\Big(\frac{1}{p_i(t)} \int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} \\ &\sim \frac{t^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{[-(\mu_i + \beta_i \rho_{i+1} + 1)]^{1/\alpha_i}}, \end{aligned} \tag{3.12}$$

as $t \to \infty$, $i = 1, \dots, n$.

Because (3.12) is not integrable on $[T, \infty)$ we see that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i \geq -1$, $i = 1, \dots, n$. We claim that the equality can hold in none of these inequalities. If the equality holds for some i, then integrating (3.12) from T to t yields

$$x_i(t) \sim (\alpha_i - \lambda_i)^{-1/\alpha_i} \int_T^t s^{-1} l_i(s)^{-1/\alpha_i} m_i(s)^{-1/\alpha_i} \xi_{i+1}(s)^{\beta_i/\alpha_i} ds \in \mathrm{SV}, \quad t \to \infty,$$

an impossibility. It holds therefore that $(-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1)/\alpha_i > -1$ for all i, and hence via application of Karamata's integration theorem to the integral of (3.12) on $[T, t]$ we conclude that

$$x_i(t) \sim \frac{t^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1} l_i(t)^{-1/\alpha_i} m_i(t)^{-1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{[-(\mu_i + \beta_i \rho_{i+1} + 1)]^{1/\alpha_i} \big(\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1\big)}, \tag{3.13}$$

as $t \to \infty$, $i = 1, \dots, n$. This implies

$$\rho_i = \frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} + 1, \quad i = 1, \dots, n, \quad \rho_{n+1} = \rho_1$$

or equivalently

$$\rho_i - \frac{\beta_i}{\alpha_i} \rho_{i+1} = \frac{\alpha_i - \lambda_i + \mu_i + 1}{\alpha_i}, \quad i = 1, \dots, n, \quad \rho_{n+1} = \rho_1 \,. \tag{3.14}$$

The coefficient matrix

$$A = A\Big(\frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_n}{\alpha_n}\Big) = \begin{pmatrix} 1 & -\frac{\beta_1}{\alpha_1} & 0 & \dots & 0 & 0 \\ 0 & 1 & -\frac{\beta_2}{\alpha_2} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -\frac{\beta_{n-1}}{\alpha_{n-1}} \\ -\frac{\beta_n}{\alpha_n} & 0 & 0 & \dots & 0 & 1 \end{pmatrix} \tag{3.15}$$

of the algebraic linear system (3.14) is nonsingular because

$$\det(A) = 1 - \frac{\beta_1 \beta_2 \dots \beta_n}{\alpha_1 \alpha_2 \dots \alpha_n} > 0 \tag{3.16}$$

because of condition (a). Thus, A is invertible and the explicit calculation gives

$$A^{-1} = \frac{A_n}{A_n - B_n} \begin{pmatrix} 1 & \frac{\beta_1}{\alpha_1} & \frac{\beta_1\beta_2}{\alpha_1\alpha_2} & \dots & \dots & \frac{\beta_1\beta_2\dots\beta_{n-1}}{\alpha_1\alpha_2\dots\alpha_{n-1}} \\ & 1 & \frac{\beta_2}{\alpha_2} & \frac{\beta_2\beta_3}{\alpha_2\alpha_3} & \dots & \frac{\beta_2\beta_3\dots\beta_{n-1}}{\alpha_2\alpha_3\dots\alpha_{n-1}} \\ & & 1 & \frac{\beta_3}{\alpha_3} & \dots & \frac{\beta_3\dots\beta_{n-1}}{\alpha_3\dots\alpha_{n-1}} \\ & & & \ddots & \ddots & \vdots \\ & & & & 1 & \frac{\beta_{n-1}}{\alpha_{n-1}} \\ * & & & & & 1 \end{pmatrix}, \tag{3.17}$$

where $A_n = \alpha_1\alpha_2\dots\alpha_n$, $B_n = \beta_1\beta_2\dots\beta_n$, and the lower triangular elements are omitted for economy of notation. Let (M_{ij}) denote the matrix on the right-hand side of (3.17). It is easy to see that the i-th row of (M_{ij}) is obtained by shifting the vector

$$\Big(1, \frac{\beta_i}{\alpha_i}, \frac{\beta_i\beta_{i+1}}{\alpha_i\alpha_{i+1}}, \dots, \frac{\beta_i\beta_{i+1}\dots\beta_{i+(n-2)}}{\alpha_i\alpha_{i+1}\dots\alpha_{i+(n-2)}}\Big) \quad \alpha_{n+k} = \alpha_k,\ \beta_{n+k} = \beta_k \text{ for } k = 1, 2, \dots$$

$(i-1)$-times to the right cyclically, so that the lower triangular elements M_{ij} for $j < i$, satisfy the relations

$$M_{ij}M_{ji} = \frac{\beta_1\beta_2\dots\beta_n}{\alpha_1\alpha_2\dots\alpha_n}, \quad i > j, \quad i = 1, 2, \dots, n.$$

Then the unique solution ρ_i, $i = 1, \dots, n$, of (3.14) is given explicitly by

$$\rho_i = \frac{A_n}{A_n - B_n} \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j}, \quad i = 1, \dots, n, \tag{3.18}$$

from which it follows that all ρ_i satisfy $0 < \rho_i < \frac{\alpha_i - \lambda_i}{\alpha_i}$ if and only if

$$0 < \sum_{j=1}^{n} M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < \frac{\alpha_i - \lambda_i}{\alpha_i}\Big(1 - \frac{B_n}{A_n}\Big), \quad i = 1, \dots, n. \tag{3.19}$$

We note that (3.13) can be expressed in the form

$$x_i(t) \sim \frac{t^{\frac{\alpha_i+1}{\alpha_i}} p_i(t)^{-1/\alpha_i} q_i(t)^{1/\alpha_i} x_{i+1}(t)^{\beta_i/\alpha_i}}{D_i}, \quad t \to \infty, \tag{3.20}$$

where

$$D_i = (\alpha_i - \lambda_i - \alpha_i\rho_i)^{1/\alpha_i} \rho_i, \tag{3.21}$$

for $i = 1, \dots, n$. This is a cyclic system of asymptotic relations, from which one can derive without difficulty the following independent explicit asymptotic formula for each x_i:

$$x_i(t) \sim \Big[\prod_{j=1}^{n} \Big(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \quad t \to \infty, \quad i = 1, \dots, n. \tag{3.22}$$

This represents the unique law describing precisely the growth order of all possible regularly varying solutions of positive indices of system (3.6) satisfying (3.7). Note that (3.22) is rewritten in the form

$$x_i(t) \sim t^{\rho_i} \Big[\prod_{j=1}^{n} \Big(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \quad t \to \infty, \quad i = 1, \dots, n. \tag{3.23}$$

Now we assume that (3.19) is satisfied and define $\rho_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i})$ and D_i by (3.18) and (3.21), respectively. Let $X_i \in \mathrm{RV}(\rho_i)$ denote the functions

$$X_i(t) = \Big[\prod_{j=1}^{n}\Big(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}}, \quad i = 1, \dots, n. \tag{3.24}$$

Then the X_i's satisfy the system of asymptotic relations (3.6), i.e.,

$$\int_b^t \Big(\frac{1}{p_i(s)} \int_s^\infty q_i(r) X_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds \sim X_i(t), \quad t \to \infty,\ i = 1,,,,n, \tag{3.25}$$

for any $b \geq a$, where $X_{n+1}(t) = X_1(t)$. In fact, noting that $X_i(t)$ are expressed as

$$X_i(t) = t^{\rho_i}\Xi_i(t), \quad \Xi_i(t) = \Big[\prod_{j=1}^{n}\Big(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}},$$

and using Karamata's integration theorem, we obtain

$$\Big(\frac{1}{p_i(t)}\int_b^t q_i(s) X_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} \sim \frac{t^{\rho_i - 1} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\alpha_i - \lambda_i - \alpha_i\rho_i)^{1/\alpha_i}},$$

and

$$\begin{aligned} &\int_b^t \Big(\frac{1}{p_i(s)}\int_b^t q_i(r) X_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds \\ &\sim \frac{t^{\rho_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \Xi_{i+1}(t)^{\beta_i/\alpha_i}}{D_i}, \quad \text{as } t \to \infty. \end{aligned} \tag{3.26}$$

A simple calculation with the help of the relations

$$M_{i+1,i}\frac{\beta_i}{\alpha_i} = \frac{B_n}{A_n}, \quad M_{i+1,j}\frac{\beta_i}{\alpha_i} = M_{ij}, \quad \text{for } j \neq i, \tag{3.27}$$

(with the convection that $M_{n+1,j} = M_{1,j}$) between the i-th and the $(i+1)$-th rows of the matrix A shows that

$$\begin{aligned} &\frac{l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}}{D_i}\Xi_{i+1}(t)^{\beta_i/\alpha_i} \\ &= \frac{l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}}{D_i}\Big[\prod_{j=1}^{n}\Big(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{i+1,j}\frac{\beta_i}{\alpha_i}}\Big]^{\frac{A_n}{A_n-B_n}} \\ &= \Big[\prod_{j=1}^{n}\Big(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}} = \Xi_i(t). \end{aligned}$$

From (3.26) we conclude that the X_i's satisfy (3.25) as desired.

Summarizing the above observations, we obtain the following result which provides complete information about the existence and asymptotic behavior of regularly varying solutions with positive indices for system (3.6).

Theorem 3.1. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$, and suppose that $\lambda_i < \alpha_i$, $i = 1, \dots, n$. Then system of asymptotic relations* (3.6) *has regularly varying solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i \in (0, \frac{\alpha_i-\lambda_i}{\alpha_i})$, $i = 1, \dots, n$, if and only if* (3.19) *holds in which case ρ_i are uniquely determined by* (3.18) *and the asymptotic behavior of any such solution is governed by the unique formula* (3.22).

Our next task is to study Problem (ii). We assume that $\lambda_i > \alpha_i$, $i = 1, \dots, n$, and seek solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_i)$ with all ρ_i negative. In view of (3.5) each ρ_i must satisfy $\frac{\alpha_i - \lambda_i}{\alpha_i} < \rho_i < 0$. Our solution to this problem is formulated below with the help of the matrix (3.15) and its inverse (3.17).

Theorem 3.2. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$ and suppose that $\lambda_i > \alpha_i$, $i = 1, \dots, n$. System of asymptotic relations* (3.8) *has regularly varying solutions $(x_1, \dots, x_n) \in \mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0)$, $i = 1, \dots, n$, if and only if*

$$\frac{\alpha_i - \lambda_i}{\alpha_i}\Big(1 - \frac{B_n}{A_n}\Big) < \sum_{j=1}^n M_{ij} \frac{\alpha_j - \lambda_j + \mu_j + 1}{\alpha_j} < 0 \tag{3.28}$$

in which case ρ_i are given by (3.18) *and the asymptotic behavior of any such solution $(x_1, \dots, x_n)$ is governed by the unique formula*

$$x_i(t) \sim \Big[\prod_{j=1}^n \Big(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{\Delta_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \quad t \to \infty, \ i = 1, \dots, n. \tag{3.29}$$

where

$$\Delta_i = (\lambda_i - \alpha_i + \alpha_i \rho_i)^{1/\alpha_i}(-\rho_i), \quad i = 1, \dots, n. \tag{3.30}$$

Proof. Let $(x_1, \dots, x_n)$ be one such solution on $[T, \infty)$. Using (3.1) and (3.10) we obtain

$$\int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds = \int_T^t s^{\mu_i + \beta_i \rho_{i+1}} \mu_i(s) \xi_{i+1}(s)^{\beta_i} ds, \tag{3.31}$$

for $t \ge T$ and $i = 1, \dots, n$, all of which are required to diverge as $t \to \infty$. Therefore $\mu_i + \beta_i \rho_{i+1} \ge -1$ for all i. If the equality holds for some i, then noting that

$$\Big(\frac{1}{p_i(t)} \int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} = t^{-\frac{\lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \Big(\int_T^t s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i},$$

and integrating this from t to ∞, we obtain via Karamata's integration theorem,

$$x_i(t) \sim \frac{\alpha_i}{\lambda_i - \alpha_i} t^{\frac{\alpha_i - \lambda_i}{\alpha_i}} l_i(t)^{-1/\alpha_i} \Big(\int_T^t s^{-1} m_i(s) \xi_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} \in \mathrm{RV}\Big(\frac{\alpha_i - \lambda_i}{\alpha_i}\Big),$$

which is a contradiction. It follows that $\mu_i + \beta_i \rho_{i+1} > -1$ for all i. Applying Karamata's integration theorem to (3.31), we have

$$\begin{aligned} &\Big(\frac{1}{p_i(t)} \int_T^t q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i} \\ &\sim \frac{t^{\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i}} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\mu_i + \beta_i \rho_{i+1} + 1)^{1/\alpha_i}}, \quad t \to \infty. \end{aligned} \tag{3.32}$$

Since (3.32) is integrable over $[T, \infty)$, we see that $\frac{-\lambda_i + \mu_i + \beta_i \rho_{i+1} + 1}{\alpha_i} \le -1$ for all i. Note that all of these inequalities should be strict, because if the equality holds for some i, then integrating on $[t, \infty)$, we have

$$x_i(t) \sim (\lambda_i - \alpha_i)^{-1/\alpha_i} \int_t^\infty s^{-1} l_i(s)^{-1/\alpha_i} m_i(s)^{1/\alpha_i} \xi_{i+1}(s)^{\beta_i/\alpha_i} ds \in \mathrm{SV},$$

a contradiction. It follows that $\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i} < -1$ for all i, in which case integration of (3.32) on $[t,\infty)$ yields

$$x_i(t) \sim \frac{t^{\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i} \xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\mu_i+\beta_i\rho_{i+1}+1)^{1/\alpha_i}\Big[-\Big(\frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1\Big)\Big]}, \tag{3.33}$$

as $t \to \infty$ and $i = 1,\dots,n$. This implies

$$\rho_i = \frac{-\lambda_i+\mu_i+\beta_i\rho_{i+1}+1}{\alpha_i}+1, \quad i = 1,\dots,n,$$

which is equivalent to the linear algebraic system (3.14) in ρ_i. From this point on one can proceed exactly as in the proof of the "only if" part of Theorem 3.1, asserting that system (3.8) may have regularly varying solutions of negative indices $\rho_i \in (\frac{\alpha_i-\lambda_i}{\alpha_i}, 0)$ only if (3.28) is fulfilled.

Now we assume that (3.28) holds. Define $\rho_i \in (0, \frac{\alpha_i-\lambda_i}{\alpha_i})$ by (3.18) and let $X_i \in \mathrm{RV}(\rho_i)$ denote the functions

$$X_i(t) = \Big[\prod_{j=1}^{n}\Big(\frac{t^{\frac{\alpha_j+1}{\alpha_j}} p_j(t)^{-1/\alpha_j} q_j(t)^{1/\alpha_j}}{\Delta_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}}, \quad i = 1,\dots,n. \tag{3.34}$$

Then the X_i's satisfy the system of asymptotic relations (3.8) (with $T = b$), i.e.,

$$\int_t^\infty \Big(\frac{1}{p_i(s)}\int_b^s q_i(r)X_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i} ds \sim X_i(t), \quad t \to \infty,\ i = 1,,,,n, \tag{3.35}$$

for any $b \ge a$, where $X_{n+1}(t) = X_1(t)$. In fact, using the following expression for $X_i(t)$,

$$X_i(t) = t^{\rho_i}\Xi_i(t), \quad \Xi_i(t) = \Big[\prod_{j=1}^{n}\Big(\frac{l_j(t)^{-1/\alpha_j} m_j(t)^{1/\alpha_j}}{\Delta_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}},$$

we obtain

$$\Big(\frac{1}{p_i(t)}\int_b^t q_i(s)X_{i+1}(s)^{\beta_i}ds\Big)^{1/\alpha_i} \sim \frac{t^{\rho_i-1} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}\Xi_{i+1}(t)^{\beta_i/\alpha_i}}{(\alpha_i-\lambda_i+\alpha_i\rho_i)^{1/\alpha_i}},$$

and

$$\int_t^\infty \Big(\frac{1}{p_i(s)}\int_b^s q_i(r)X_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i} ds \sim \frac{t^{\rho_i} l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}\Xi_{i+1}(t)^{\beta_i/\alpha_i}}{\Delta_i}, \tag{3.36}$$

as $t \to \infty$. Since it can be verified with the help of (3.27) that

$$\frac{l_i(t)^{-1/\alpha_i} m_i(t)^{1/\alpha_i}}{\Delta_i}\Xi_{i+1}(t)^{\beta_i/\alpha_i} = \Xi_i(t),$$

the desired relation (3.35) immediately follows from (3.36). This completes the proof. $\square$

4. Regularly varying intermediate solutions of (1.1)

We are now in a position to state and prove our main results on the existence and the precise asymptotic behavior of regularly varying intermediate solutions of system (1.1) with regularly varying coefficients p_i and q_i. Use is made of the notation and properties of the matrix (3.15) and (3.17).

Theorem 4.1. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. Suppose that $\lambda_i < \alpha_i$ for $i = 1, \dots, n$. System (1.1) possesses intermediate solutions in $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i})$, $i = 1, \dots, n$, if and only if (3.19) holds, in which case ρ_i are given by (3.18) and the asymptotic behavior of any such solution $(x_1, \dots, x_n)$ is governed by the unique formula (3.22).*

Theorem 4.2. *Let $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$, $i = 1, \dots, n$. Suppose that $\lambda_i > \alpha_i$ for $i = 1, \dots, n$. System (1.1) possesses intermediate solutions in $\mathrm{RV}(\rho_1, \dots, \rho_n)$ with $\rho_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0)$, $i = 1, \dots, n$, if and only if (3.28) holds, in which case ρ_i are given by (3.18) and the asymptotic behavior of any such solution $(x_1, \dots, x_n)$ is governed by the unique formula (3.29).*

We remark that the "only if" parts of these theorems follow immediately from the corresponding parts of Theorems 3.1 and 3.2 because any solution $(x_1, \dots, x_n)$ of (1.1) with the indicated property satisfies the systems of asymptotic relations (3.6) plus (3.7) or (3.8) plus (3.9). The "if" parts are proved by way of the following existence theorems for intermediate solutions for system (1.1) with nearly regularly varying coefficients $p_i(t)$ and $q_i(t)$ in the sense defined below.

Definition 4.3. Let $f(t)$ be a regularly varying function of index σ and suppose that $g(t)$ satisfies $kf(t) \le g(t) \le Kf(t)$ for some positive constants k and K and for all large t. Then $g(t)$ is said to be a *nearly regularly varying function of index σ*. Such a relation between $f(t)$ and $g(t)$ is denoted by $g(t) \asymp f(t)$ as $t \to \infty$.

Theorem 4.4. *Let p_i and q_i be nearly regularly varying of indices λ_i and μ_i, respectively; that is, there exist $\tilde{p}_i \in \mathrm{RV}(\lambda_i)$ and $\tilde{q}_i \in \mathrm{RV}(\mu_i)$ such that*

$$p_i(t) \asymp \tilde{p}_i(t), \quad q_i(t) \asymp \tilde{q}_i(t), \quad t \to \infty,\ i = 1, \dots, n. \tag{4.1}$$

Suppose in addition that $\lambda_i < \alpha_i$, $i = 1, \dots, n$, and that (3.19) holds. Then, system (1.1) possesses intermediate solutions $(x_1, \dots, x_n)$ which are nearly regularly varying of positive index $(\rho_1, \dots, \rho_n)$ with $\rho_i \in (0, \frac{\alpha_i - \lambda_i}{\alpha_i})$ in the sense that

$$x_i(t) \asymp \Big[\prod_{j=1}^{n} \Big(\frac{t^{\frac{\alpha_j + 1}{\alpha_j}} \tilde{p}_j(t)^{-1/\alpha_j} \tilde{q}_j(t)^{1/\alpha_j}}{D_j} \Big)^{M_{ij}} \Big]^{\frac{A_n}{A_n - B_n}}, \tag{4.2}$$

for $t \to \infty$ and $i = 1, \dots, n$, where ρ_i and D_i are defined by (3.18) and (3.21), respectively.

Theorem 4.5. *Let p_i and q_i be nearly regularly varying of indices λ_i and μ_i, respectively; that is, there exist $\tilde{p}_i \in \mathrm{RV}(\lambda_i)$ and $\tilde{q}_i \in \mathrm{RV}(\mu_i)$ satisfying (4.1). Suppose that $\lambda_i > \alpha_i$, $i = 1, \dots, n$, and that (3.28) holds. Then, system (1.1) possesses intermediate solutions solutions $(x_1, \dots, x_n)$ which are nearly regularly varying of negative index $(\rho_1, \dots, \rho_n)$ with $\rho_i \in (\frac{\alpha_i - \lambda_i}{\alpha_i}, 0)$, $i = 1, \dots, n$, and satisfy*

$$x_i(t) \asymp \Big[\prod_{j=1}^{n} \Big(\frac{t^{\frac{\alpha_j + 1}{\alpha_j}} \tilde{p}_j(t)^{-1/\alpha_j} \tilde{q}_j(t)^{1/\alpha_j}}{\Delta_j} \Big)^{M_{ij}} \Big]^{\frac{A_n}{A_n - B_n}}, \quad t \to \infty,\ i = 1, \dots, n, \tag{4.3}$$

where ρ_i *and* Δ_i *are defined by* (3.18) *and* (3.30).

Proof of Theorem 4.4. We assume that the regularly varying functions $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ are expressed in the form

$$\tilde{p}_i(t) = t^{\lambda_i}\tilde{l}_i(t), \quad \tilde{q}_i(t) = t^{\mu_i}\tilde{m}_i(t), \quad \tilde{l}_i, \tilde{m}_i \in \mathrm{SV}, \quad i = 1, \dots, n. \tag{4.4}$$

By (4.1) there exist positive constants h_i, H_i, k_i and K_i such that

$$h_i\tilde{p}_i(t) \le p_i(t) \le H_i\tilde{p}_i(t), \quad k_i\tilde{q}_i(t) \le q_i(t) \le K_i\tilde{q}_i(t), \tag{4.5}$$

for $t \ge a$ and $i = 1, \dots, n$. Define the functions $X_i \in \mathrm{RV}(\rho_i)$ by

$$X_i(t) = t^{\rho_i}\Big[\prod_{j=1}^{n}\Big(\frac{\tilde{l}_j(t)^{-1/\alpha_j}\tilde{m}_j(t)^{1/\alpha_j}}{D_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \quad t \ge a, \ i = 1, \dots, n. \tag{4.6}$$

It is known that

$$\int_b^t\Big(\frac{1}{\tilde{p}_i(s)}\int_s^\infty \tilde{q}_i(r)X_{i+1}(s)^{\beta_i}dr\Big)^{1/\alpha_i}ds \sim X_i(t), \quad t \to \infty, \ i = 1, \dots, n, \tag{4.7}$$

for any $b \ge a$, from which it follows that there exists $T > b$ such that

$$\int_T^t\Big(\frac{1}{\tilde{p}_i(s)}\int_s^\infty \tilde{q}_i(r)X_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds \le 2X_i(t), \quad t \ge T, \ i = 1, \dots, n. \tag{4.8}$$

Without loss of generality we may assume that each $X_i(t)$ is increasing on $[T, \infty)$ because it is known that any regularly varying function of positive index is asymptotically equivalent to an increasing RV function of the same index (cf. [1, Theorem 1.5.3]). Since (4.7) holds for $b = T$ it is possible to choose $T_1 > T$ so large that

$$\int_T^t\Big(\frac{1}{\tilde{p}_i(s)}\int_s^\infty \tilde{q}_i(r)X_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds \ge \frac{1}{2}X_i(t) \tag{4.9}$$

for $t \ge T_1$, $i = 1, \dots, n$. We define the positive constants l_i and L_i ($l_i \le L_i$) by

$$l_i = \Big[\prod_{j=1}^{n}\Big\{\frac{1}{2}\Big(\frac{k_j}{H_j}\Big)^{1/\alpha_j}\Big\}^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \quad L_i = \Big[\prod_{j=1}^{n}\Big\{4\Big(\frac{K_j}{h_j}\Big)^{1/\alpha_j}\Big\}^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}}, \tag{4.10}$$

$i = 1, \dots, n$. As is easily verified, l_i and L_i satisfy the cyclic systems of equalities

$$l_i = \frac{1}{2}\Big(\frac{k_i}{H_i}\Big)^{1/\alpha_i}l_{i+1}^{\beta_i/\alpha_i}, \quad L_i = 4\Big(\frac{K_i}{h_i}\Big)^{1/\alpha_i}L_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \dots, n,$$
$$L_{n+1} = L_1, \quad l_{n+1} = l_1.$$

Since

$$\frac{L_i}{l_i} = \Big[\prod_{j=1}^{n}\Big\{8\Big(\frac{H_jK_j}{h_jk_j}\Big)^{1/\alpha_j}\Big\}^{M_{ij}}\Big]^{\frac{A_n}{A_n - B_n}},$$

the constants h_i, H_i, k_i and K_i can be chosen so that $L_i/l_i \ge 2X_i(T_1)/X_i(T)$; that is,

$$2l_iX_i(T_1) \le L_iX_i(T), \quad i = 1, \dots, n, \tag{4.11}$$

because these constants are independent of $X_i(t)$ and the choice of T and T_1.

Let $\mathcal{X}$ denote the set consisting of continuous vector functions $(x_1, \dots, x_n)$ on $[T, \infty)$ satisfying

$$l_iX_i(t) \le x_i(t) \le L_iX_i(t), \quad t \ge T, \ i = 1, \dots, n. \tag{4.12}$$

It is clear that $\mathcal{X}$ is a closed convex subset of the locally convex space $C[T,\infty)^n$. We consider the integral operators $\mathcal{F}_i$ given by

$$\mathcal{F}_i x(t) = c_i + \int_T^t \Big(\frac{1}{p_i(s)} \int_s^\infty q_i(r) x(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds, \quad t \geq T,\ i = 1, \dots, n, \tag{4.13}$$

where c_i are positive constants such that

$$l_i X_i(T_1) \leq c_i \leq \frac{1}{2} L_i X_i(T), \quad i = 1, \dots, n, \tag{4.14}$$

and define the mapping $\Phi : \mathcal{X} \to C[T,\infty)^n$ by

$$\begin{gathered} \Phi(x_1, x_2, \dots, x_n)(t) = (\mathcal{F}_1 x_2(t), \mathcal{F}_2 x_3(t), \dots, \mathcal{F}_n x_{n+1}(t)), \\ t \geq T, \quad (x_{n+1}(t) = x_1(t)). \end{gathered} \tag{4.15}$$

We will show that the Schauder-Tychonoff fixed point theorem is applicable to Φ.

(i) Φ maps $\mathcal{X}$ into itself. Let $(x_1, \dots, x_n) \in \mathcal{X}$. Then, using (4.8)–(4.15), we see that

$$\begin{aligned} \mathcal{F}_i x_{i+1}(t) &\leq \frac{1}{2} L_i X_i(T) + \Big(\frac{K_i L_{i+1}^{\beta_i}}{h_i}\Big)^{1/\alpha_i} \int_T^t \Big(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds \\ &\leq \frac{1}{2} L_i X_i(T) + 2\Big(\frac{K_i L_{i+1}^{\beta_i}}{h_i}\Big)^{1/\alpha_i} X_i(t) \\ &\leq \frac{1}{2} L_i X_i(t) + \frac{1}{2} L_i X_i(t) = L_i X_i(t) \quad \text{for } t \geq T, \end{aligned}$$

and

$$\begin{gathered} \mathcal{F}_i x_{i+1}(t) \geq c_i \geq l_i X_i(T_1) \geq l_i X_i(t) \quad \text{for } T \leq t \leq T_1, \\ \begin{aligned} \mathcal{F}_i x_{i+1}(t) &\geq \Big(\frac{k_i l_{i+1}^{\beta_i}}{H_i}\Big)^{1/\alpha_i} \int_T^t \Big(\frac{1}{\tilde{p}_i(s)} \int_s^\infty \tilde{q}_i(r) X_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds \\ &\geq \frac{1}{2}\Big(\frac{k_i l_{i+1}^{\beta_i}}{H_i}\Big)^{1/\alpha_i} X_i(t) = l_i X_i(t) \quad t \geq T_1. \end{aligned} \end{gathered}$$

This shows that $\Phi(x_1, \dots, x_n) \in \mathcal{X}$; that is, Φ is a self-map on $\mathcal{X}$.

(ii) $\Phi(\mathcal{X})$ is relatively compact. From the inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ proven above it follows that $\Phi(\mathcal{X})$ is locally uniformly bounded on $[T,\infty)$. From the inequalities

$$0 \leq (\mathcal{F}_i x_{i+1})'(t) \leq L_i^{\beta_i/\alpha_i} \Big(\frac{1}{p_i(t)} \int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i}, \quad t \geq T,\ i = 1, \dots, n,$$

holding for all $(x_1, \dots, x_n) \in \mathcal{X}$ we see that $\Phi(\mathcal{X})$ is locally equicontinuous on $[T,\infty)$. The relative compactness of $\Phi(\mathcal{X})$ is an immediate consequence of the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_1^\nu(t), \dots, x_n^\nu(t))\}$ be a sequence in $\mathcal{X}$ converging as $\nu \to \infty$ to $(x_1(t), \dots, x_n(t)) \in \mathcal{X}$ uniformly on compact subintervals of $[T,\infty)$. Using (4.13) we obtain

$$|\mathcal{F}_i x_{i+1}^\nu(t) - \mathcal{F}_i x_{i+1}(t)| \leq \int_T^t p_i(s)^{-1/\alpha_i} F_i^\nu(s) ds, \quad t \geq T, \tag{4.16}$$

where

$$F_i^\nu(t) = \Big|\Big(\int_t^\infty q_i(s) x_{i+1}^\nu(s)^{\beta_i} ds\Big)^{1/\alpha_i} - \Big(\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i}\Big|.$$

It is easy to see that

$$F_i^\nu(t) \le \Big(\int_t^\infty q_i(s)\Big|x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i}\Big|ds\Big)^{1/\alpha_i}, \tag{4.17}$$

if $\alpha_i \ge 1$ and

$$\begin{aligned} F_i^\nu(t) \le \frac{1}{\alpha_i}\Big(L_{i+1}^{\beta_i}\int_t^\infty q_i(s)X_{i+1}(s)^{\beta_i}ds\Big)^{\frac{1}{\alpha_i}-1} \\ \times \int_t^\infty q_i(s)\Big|x_{i+1}^\nu(s)^{\beta_i} - x_{i+1}(s)^{\beta_i}\Big|ds, \end{aligned} \tag{4.18}$$

if $\alpha_i < 1$. Combine (4.16) with (4.17) or (4.18) and apply the Lebesgue dominated convergence theorem. Then we conclude that

$$\lim_{\nu\to\infty} \mathcal{F}_i x_{i+1}^\nu(t) = \mathcal{F}_i x_{i+1}(t)$$

uniformly on any compact subset of $[T,\infty)$, $i = 1,\dots,n$, proving the continuity of Φ.

Therefore, all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled and Φ has a fixed point $(x_1,\dots,x_n) \in \mathcal{X}$, which satisfies

$$\begin{aligned} x_i(t) &= \mathcal{F}_i x_{i+1}(t) \\ &= c_i + \int_T^t \Big(\frac{1}{p_i(s)}\int_s^\infty q_i(r)x_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_i}ds, \quad t \ge T,\ i = 1,\dots,n. \end{aligned} \tag{4.19}$$

This shows that $(x_1,\dots,x_n)$ is a solution of system (1.1) on $[T,\infty)$. Since the obtained solution is a member of $\mathcal{X}$, it is nearly regularly varying of positive index $(\rho_1,\dots,\rho_n)$ and hence is an intermediate solution of (1.1). This completes the proof. □

Proof of Theorem 4.5. Assume that $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ are expressed in the form (4.4) and satisfy (4.5) for positive constants h_i, H_i, k_i and K_i. Suppose that (3.28) holds. Define ρ_i and Δ_i by (3.18) and (3.30), respectively, and consider the regularly varying functions of indices ρ_i

$$Y_i(t) = t^{\rho_i}\Big[\prod_{j=1}^n\Big(\frac{\tilde{l}_j(t)^{-1/\alpha_j}\tilde{m}_j(t)^{1/\alpha_j}}{\Delta_j}\Big)^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}}, \quad i = 1,\dots,n. \tag{4.20}$$

Since $Y_i(t)$ satisfy the asymptotic relations

$$\int_t^\infty\Big(\frac{1}{\tilde{p}_i(s)}\int_b^s \tilde{q}_i(r)Y_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_j}ds \sim Y_i(t), \quad t\to\infty,\ i = 1,\dots,n, \tag{4.21}$$

one can choose $T > a$ so that

$$\frac{1}{2}Y_i(t) \le \int_t^\infty\Big(\frac{1}{\tilde{p}_i(s))}\int_b^s \tilde{q}_i(r)Y_{i+1}(r)^{\beta_i}dr\Big)^{1/\alpha_j}ds \le 2Y_i(t), \tag{4.22}$$

for $t \ge T$ and $i = 1,\dots,n$. All of $Y_i(t)$ may be assumed to be decreasing on $[T,\infty)$ because it is known [1, Theorem 1.5.3] that a regularly varying function of negative index is asymptotic to a decreasing RV function of the same index. Denote by $\mathcal{Y}$ the set consisting of continuous vector functions $(x_1,\dots,x_n)$ such that

$$l_iY_i(t) \le x_i(t) \le L_iY_i(t), \quad t \ge T,\ i = 1,\dots,n, \tag{4.23}$$

where

$$l_i = \Big[\prod_{j=1}^{n}\Big\{\frac{1}{2}\Big(\frac{k_j}{H_j}\Big)^{1/\alpha_j}\Big\}^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}}, \quad L_i = \Big[\prod_{j=1}^{n}\Big\{2\Big(\frac{K_j}{h_j}\Big)^{1/\alpha_j}\Big\}^{M_{ij}}\Big]^{\frac{A_n}{A_n-B_n}}, \tag{4.24}$$

for $i = 1, \dots, n$, which satisfy the cyclic systems of equalities

$$l_i = \frac{1}{2}\Big(\frac{k_i}{H_i}\Big)^{1/\alpha_i} l_{i+1}^{\beta_i/\alpha_i}, \quad L_i = 2\Big(\frac{K_i}{h_i}\Big)^{1/\alpha_i} L_{i+1}^{\beta_i/\alpha_i}, \quad l_{n+1} = l_1,\ L_{n+1} = L_1. \tag{4.25}$$

We now consider the mapping $\Psi : \mathcal{Y} \to C[T,\infty)^n$ defined by

$$\Psi(x_1, \dots, x_n)(t) = (\mathcal{G}_1 x_2(t), \mathcal{G}_2 x_3(t), \dots, \mathcal{G}_n x_{n+1}(t)), \tag{4.26}$$

for $t \ge T$ and $x_{n+1}(t) = x_1(t)$, where $\mathcal{G}_i$ denotes the integral operator

$$\mathcal{G}_i x(t) = \int_t^\infty \Big(\frac{1}{\tilde{p}_i(s)}\int_T^s \tilde{q}_i(r) x(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds, \quad t \ge T, \quad i = 1, \dots, n. \tag{4.27}$$

It is a matter of straightforward calculation to verify that Ψ is a self-map on $\mathcal{Y}$ and sends $\mathcal{Y}$ into a relatively compact subset of $C[T,\infty)^n$. The details may be omitted. Therefore, the Schauder-Tychonoff fixed point theorem ensures the existence of a fixed point $(x_1, \dots, x_n) \in \mathcal{Y}$ of Ψ. This fixed point gives rise to an intermediate solutions of (1.1) which is nearly regularly varying of negative index $(\rho_1, \dots, \rho_n)$. This completes the proof. $\square$

To complete the proof of the "if" parts of Theorems 4.1 and 4.2 it suffices to show that if $p_i(t)$ and $q_i(t)$ are assumed to be regularly varying, then the nearly regularly varying solutions obtained in Theorems 4.4 and 4.5 actually become regularly varying of the same specified indices. For this purpose use is made of the following generalized L'Hôpital's rule. See, for example, Haupt and Aumann [4].

Lemma 4.6. *Let $f(t), g(t) \in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = \infty \quad \text{and} \quad g'(t) > 0 \quad \text{for all large } t,$$

or

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} g(t) = 0 \quad \text{and} \quad g'(t) < 0 \quad \text{for all large } t.$$

Then

$$\liminf_{t\to\infty} \frac{f'(t)}{g'(t)} \le \liminf_{t\to\infty} \frac{f(t)}{g(t)}, \quad \limsup_{t\to\infty} \frac{f(t)}{g(t)} \le \limsup_{t\to\infty} \frac{f'(t)}{g'(t)}.$$

Proof of the "if" parts of Theorem 4.1. Suppose that $p_i \in \mathrm{RV}(\lambda_i)$ and $q_i \in \mathrm{RV}(\mu_i)$. Suppose in addition that $\lambda_i < \alpha_i$. Define the positive constants ρ_i and D_i by (3.18) and (3.21), respectively, and let $X_i \in \mathrm{RV}(\rho_i)$ denote the functions on the right-hand side of (4.2) with $\tilde{p}_i(t)$ and $\tilde{q}_i(t)$ replaced with $p_i(t)$ and $q_i(t)$, respectively. Then, by Theorem 4.4 system (1.1) has a nearly regularly varying solution $(x_1, \dots, x_n)$ such that $x_i(t) \asymp X_i(t)$ as $t \to \infty$, $i = 1, \dots, n$. Notice that $x_i(t)$ satisfy the system of integral equations (4.19).

It remains to verify that x_i are regularly varying functions of indices ρ_i, $i = 1, \dots, n$, respectively. We define

$$u_i(t) = \int_T^t \Big(\frac{1}{p_i(s)}\int_s^\infty q_i(r) X_{i+1}(r)^{\beta_i} dr\Big)^{1/\alpha_i} ds, \quad i = 1, \dots, n, \tag{4.28}$$

and put

$$\omega_i = \liminf_{t\to\infty} \frac{x_i(t)}{u_i(t)}, \quad \Omega_i = \limsup_{t\to\infty} \frac{x_i(t)}{u_i(t)}.$$

Since $x_i(t) \asymp X_i(t)$ and

$$u_i(t) \sim X_i(t), \quad t \to \infty,\ i = 1, \dots, n, \tag{4.29}$$

it follows that $0 < \omega_i \le \Omega_i < \infty$, $i = 1, \dots, n$. Using Lemma 4.6 we obtain

$$\begin{aligned}
\omega_i &\ge \liminf_{t\to\infty} \frac{x_i'(t)}{u_i'(t)} = \liminf_{t\to\infty} \frac{\Big(\frac{1}{p_i(t)}\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i}}{\Big(\frac{1}{p_i(t)}\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} ds\Big)^{1/\alpha_i}} \\
&= \liminf_{t\to\infty} \Big(\frac{\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds}{\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} ds}\Big)^{1/\alpha_i} \\
&= \Big(\liminf_{t\to\infty} \frac{\int_t^\infty q_i(s) x_{i+1}(s)^{\beta_i} ds}{\int_t^\infty q_i(s) X_{i+1}(s)^{\beta_i} ds}\Big)^{1/\alpha_i} \\
&\ge \Big(\liminf_{t\to\infty} \frac{q_i(t) x_{i+1}(t)^{\beta_i}}{q_i(t) X_{i+1}(t)^{\beta_i}}\Big)^{1/\alpha_i} = \liminf_{t\to\infty} \Big(\frac{x_{i+1}(t)}{X_{i+1}(t)}\Big)^{\beta_i/\alpha_i} = \omega_{i+1}^{\beta_i/\alpha_i},
\end{aligned}$$

where (4.29) has been used in the last step. Thus, ω_i satisfy the cyclic system of inequalities

$$\omega_i \ge \omega_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \dots, n, \quad \omega_{n+1} = \omega_1. \tag{4.30}$$

Likewise, by taking the upper limits instead of the lower limits we are led to the cyclic inequalities

$$\Omega_i \le \Omega_{i+1}^{\beta_i/\alpha_i}, \quad i = 1, \dots, n, \quad \Omega_{n+1} = \Omega_1. \tag{4.31}$$

From (4.30) and (4.31) we easily see that

$$\omega_i \ge \omega_i^{\frac{\beta_1\cdots\beta_n}{\alpha_1\cdots\alpha_n}}, \quad \Omega_i \le \Omega_i^{\frac{\beta_1\cdots\beta_n}{\alpha_1\cdots\alpha_n}},$$

whence, because of the hypothesis $\beta_1 \dots \beta_n / \alpha_1 \dots \alpha_n < 1$, we find that $\omega_i \ge 1$ and $\Omega_i \le 1$. It follows therefore that $\omega_i = \Omega_i = 1$ or $\lim_{t\to\infty} x_i(t)/u_i(t) = 1$ for $i = 1, \dots, n$. This combined with (4.29) implies that $x_i(t) \sim u_i(t) \sim X_i(t)$ as $t \to \infty$, which shows that each $x_i(t)$ is a regularly varying function of index ρ_i. Thus the proof of the "if" part of Theorem 4.1 is complete.

In essentially the same way one can complete the proof of the "if" part of Theorem 4.2. □

5. Applications to partial differential equations

The purpose of the final section is to demonstrate that our results on cyclic systems of ordinary differential equations (1.1) can be applied to some classes of partial differential equations to provide new information about the existence and asymptotic behavior of their radial positive solutions. Throughout this section $x = (x_1, \dots, x_N)$ represents the space variable in $\mathbb{R}^N$, $N \ge 2$, and $|x|$ denotes the Euclidean length of x. All partial differential equations will be considered in an exterior domain $\Omega_R = \{x \in \mathbb{R}^N : |x| \ge R\}$, $R > 0$.

5.1. **Systems of p-Laplacian equations.** We are concerned with the system of nonlinear p-Laplacian equations

$$\operatorname{div}\Big(|\nabla u_i|^{p-2}\nabla u_i\Big) + f_i(|x|)|u_{i+1}|^{\gamma_i - 1}u_{i+1} = 0, \quad i = 1, \dots, n, \quad (u_{n+1} = u_1) \tag{5.1}$$

where $p > 1$ and $\gamma_i > 0$ are constants, and $f_i(t)$ are positive continuous functions on $[a, \infty)$ which are regularly varying of indices ν_i, $i = 1, \dots, n$. Our attention will be focused on radial solutions $(u_1(|x|), \dots, u_n(|x|))$ of (5.1) defined in Ω_R, $R > a$. A radial vector function $(u_1(|x|), \dots, u_n(|x|))$ is a solution of (5.1) in Ω_R if and only if $(u_1(t), \dots, u_n(t))$ is a solution of the system of ordinary differential equations

$$\begin{gathered} (t^{N-1}|u_i'|^{p-2}u_i')' + t^{N-1}f_i(t)|u_{i+1}|^{\gamma_i - 1}u_{i+1} = 0, \quad t \ge a, \quad i = 1, \dots, n, \\ u_{n+1} = u_1 \end{gathered} \tag{5.2}$$

which is a special case of system (1.1) with

$$\begin{gathered} \alpha_1 = \dots = \alpha_n = p - 1, \quad \beta_i = \gamma_i, \quad i = 1, \dots, n; \\ \lambda_1 = \ldots = \lambda_n = N - 1, \quad \mu_i = N - 1 + \nu_i, \quad i = 1, \dots, n. \end{gathered}$$

It is always assumed that

$$\gamma_1 \dots \gamma_n < (p-1)^n. \tag{5.3}$$

We need the matrix $A\Big(\frac{\gamma_1}{p-1}, \dots, \frac{\gamma_n}{p-1}\Big)$ associated with (5.2) and its inverse (cf. (3.15) and (3.17)). We define

$$(M_{ij}) = \frac{(p-1)^n - \gamma_1 \dots \gamma_n}{(p-1)^n} A\Big(\frac{\gamma_1}{p-1}, \dots, \frac{\gamma_n}{p-1}\Big)^{-1}. \tag{5.4}$$

To analyze (5.2) it is necessary to distinguish the two cases $p > N$ and $p < N$.

(i) Suppose that $p > N$. In this case applying Theorem 4.1 to (5.2), we conclude that system (5.1) possesses increasing radial solutions $(u_1(|x|), \dots, u_n(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, $0 < \rho_i < \frac{p-N}{p-1}$, $i = 1, \dots, n$, if and only if

$$0 < \sum_{j=1}^n M_{ij}(p + \nu_j) < (p - N)\Big(1 - \frac{\gamma_1\gamma_2 \cdots \gamma_n}{(p-1)^n}\Big), \quad i = 1, \dots, n. \tag{5.5}$$

In this case the ρ_i's are uniquely determined by

$$\rho_i = \frac{(p-1)^{n-1}}{(p-1)^n - \gamma_1 \dots \gamma_n} \sum_{j=1}^n M_{ij}(p + \nu_j), \quad i = 1, \dots, n, \tag{5.6}$$

and moreover the asymptotic behavior of any such solution as $|x| \to \infty$ is governed by the unique growth law

$$u_i(|x|) \sim |x|^{\rho_i} \Big[\prod_{j=1}^n \Big(\frac{\varphi_j(|x|)}{(p - N - (p-1)\rho_j)\rho_j^{p-1}}\Big)^{M_{ij}}\Big]^{\frac{(p-1)^{n-1}}{(p-1)^n - \gamma_1 \cdots \gamma_n}}, \quad |x| \to \infty, \tag{5.7}$$

for $i = 1, \dots, n$, where $\varphi_i \in \mathrm{SV}$ is the slowly varying part of f_i; that is, $f_i(t) = t^{\nu_i}\varphi_i(t)$.

(ii) Suppose that $p < N$. In this case from Theorem 4.2 applied to (5.2) it follows that system (5.1) possesses decreasing radial solutions $(u_1(|x|), \dots, u_n(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, $\frac{p-N}{p-1} < \rho_i < 0$, $i = 1, \dots, n$, if and only if

$$(p - N)\Big(1 - \frac{\gamma_1\gamma_2 \cdots \gamma_n}{(p-1)^n}\Big) < \sum_{j=1}^n M_{ij}(p + \nu_j) < 0, \quad i = 1, \dots, n. \tag{5.8}$$

In this case ρ_i are uniquely determined by (5.6) and the asymptotic behavior of any such solution as $|x| \to \infty$ is governed by the unique decay law

$$u_i(|x|) \sim |x|^{\rho_i} \Big[\prod_{j=1}^{n} \Big(\frac{\varphi_j(|x|)}{(N-p+(p-1)\rho_j)(-\rho_j)^{p-1}} \Big)^{M_{ij}} \Big]^{\frac{(p-1)^{n-1}}{(p-1)^n - \gamma_1 \cdots \gamma_n}},$$

as $|x| \to \infty$, for $i = 1, \dots, n$, where φ_i is the slowly varying part of f_i.

Consider the particular case of (5.1) in which $f_i(t) \equiv c_i > 0$, i.e.,

$$\operatorname{div}\Big(|\nabla u_i|^{p-2} \nabla u_i\Big) + c_i |u_{i+1}|^{\gamma_i - 1} u_{i+1} = 0, \quad i = 1, \dots, n, \quad u_{n+1} = u_1. \tag{5.9}$$

In this case $\nu_i = 0$ for all i, and so (5.5) and (5.8) are always violated. Therefore system (5.9) cannot admit intermediate radial solutions $(u_1(|x|), \dots, u_N(|x|))$ such that $u_i \in \mathrm{RV}(\rho_i)$, where ρ_i satisfy

$$\begin{gathered} 0 < \rho_i < \frac{p-N}{p-1}, \quad i = 1, \dots, n, \quad \text{if } p > N; \\ \frac{p-N}{p-1} < \rho_i < 0, \quad i = 1, \dots, n, \quad \text{if } p < N. \end{gathered}$$

5.2. **Nonlinear metaharmonic equations.** Now we consider the nonlinear metaharmonic equation

$$\Delta^m u = (-1)^m g(|x|) |u|^{\gamma - 1} u, \quad x \in \Omega_R, \tag{5.10}$$

where $N \ge 3$, $m \ge 2$ and $\gamma > 0$ are constants, and $g(t)$ is a positive continuous function on $[a, \infty)$ which is regularly varying of index ν. We are interested in radial positive solutions u of (5.10) such that u and $(-1)^i \Delta^i u$, $i = 1, \dots, m-1$, are regularly varying of negative indices. It is clear that seeking such solutions of (5.10) is equivalent to seeking radial regularly varying solutions of negative indices of the system

$$\begin{gathered} \Delta u_i + u_{i+1} = 0, \quad i = 1, \dots, m-1, \\ \Delta u_m + g(|x|) |u_{m+1}|^{\gamma - 1} u_{m+1} = 0, \quad x \in \Omega_R, \end{gathered} \tag{5.11}$$

where $u_{m+1} = u_1$. This system is equivalent to the system of ordinary differential equations

$$\begin{gathered} (t^{N-1} u_i')' + t^{N-1} u_{i+1} = 0, \quad i = 1, \dots, m-1, \\ (t^{N-1} u_m')' + t^{N-1} g(t) |u_{m+1}|^{\gamma - 1} u_{m+1} = 0, \quad t \ge R, \end{gathered} \tag{5.12}$$

which is a special case of (1.1) with

$$\begin{gathered} \alpha_1 = \ldots = \alpha_m = 1, \quad \beta_i = \ldots = \beta_{m-1} = 1, \quad \beta_m = \gamma, \\ \lambda_1 = \ldots = \lambda_m = N-1, \quad \mu_1 = \ldots = \mu_{m-1} = N-1, \quad \mu_m = N-1+\nu. \end{gathered}$$

We assume that $\gamma < 1$. The $m \times m$-matrix (3.15) associated with (5.12) reads $A(1, \dots, 1, \gamma)$. Define the matrix (M_{ij}) by

$$(M_{ij}) = (1-\gamma) A(1, \dots, 1, \gamma)^{-1}. \tag{5.13}$$

As is easily checked, $M_{ij} = 1$ for $1 \le i \le j \le m$ and $M_{ij} = \gamma$ for $1 \le j < i \le m$.

Since $\lambda_i = N-1 > 1 = \alpha_i$ for all i, Theorem 4.2 can be utilized to determine the structure of decreasing regularly varying solutions $(u_1, \dots, u_m) \in \mathrm{RV}(\rho_1, \dots, \rho_m)$,

$2-N<\rho_i<0$, of the cyclic system (5.12). The regularity indices ρ_i should be given by (3.18) which in the present situation reduce to

$$\rho_i = \frac{2m+\nu}{1-\gamma} - 2(i-1), \quad i=1,\dots,m, \tag{5.14}$$

from which we see that all ρ_i are admissible if and only if

$$2m-N < \frac{2m+\nu}{1-\gamma} < 0 \iff -2m+(2m-N)(1-\gamma) < \nu < -2m. \tag{5.15}$$

Clearly, (5.15) makes sense only if $N>2m$, in which case it is concluded that equation (5.10) possesses radial positive solutions $u(|x|) \in \mathrm{RV}(\rho_1)$, where

$$\rho_1 = \frac{2m+\nu}{1-\gamma} \in (2-N, 0), \tag{5.16}$$

such that $(-1)^i \Delta^i u(|x|) \in \mathrm{RV}(\rho_{i+1})$, $2-N<\rho_{i+1}<0$, for $i=1,\dots,m-1$. Furthermore, the asymptotic behavior of any such solution $u(|x|)$ is governed by the formula

$$u(|x|) \sim |x|^{\rho_1} \Big[\frac{\psi(|x|)}{[\prod_{j=1}^{i-1}(N-2+\rho_j)(-\rho_j)]^{\gamma} \prod_{j=i}^{m}(N-2+\rho_j)(-\rho_j)} \Big]^{\frac{1}{1-\gamma}}, \tag{5.17}$$

as $|x|\to\infty$, where $\psi(t)$ denotes the slowly varying function such that $g(t)=t^{\nu}\psi(t)$. We remark that the particular case of (5.10),

$$\Delta^m u = (-1)^m c|u|^{\gamma-1}u, \quad x \in \Omega_R\,,$$

where $c>0$ is a constant, can by no means possess radial solutions $u(|x|) \in \mathrm{RV}(\rho_1)$ with $2-N<\rho_1<0$.

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