EXISTENCE OF SOLUTIONS TO THE CAHN-HILLIARD/ALLEN-CAHN EQUATION WITH DEGENERATE MOBILITY

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Abstract. This article we study the Cahn-Hilliard/Allen-Cahn equation with degenerate mobility. Under suitable assumptions on the degenerate mobility and the double well potential, we prove existence of weak solutions, which can be obtained by considering the limits of Cahn-Hilliard/Allen-Cahn equations with non-degenerate mobility.

1. Introduction

In this article, we consider a scalar Cahn-Hilliard/Allen-Cahn equation with degenerate mobility

\[ u_t = -\nabla[D(u)\nabla(\Delta u - f(u))] + (\Delta u - f(u)), \quad \text{in } Q_T, \]

where \( Q_T = \Omega \times (0, T) \), \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a \( C^3 \)-boundary \( \partial \Omega \) and \( f(u) \) is the derivative of a double-well potential \( F(u) \) with wells \( \pm 1 \). The mobility \( D(u) \in C(\mathbb{R}; [0, \infty)) \) is in the form

\[ D(u) = |u|^m, \quad \text{if } |u| < \delta, \]
\[ C_0 \leq D(u) \leq C_1 |u|^m, \quad \text{if } |u| \geq \delta, \]

for some constants \( C_0, C_1, \delta > 0 \), where \( 0 < m < \infty \) if \( n = 1, 2 \) and \( \frac{4}{n} < m < \frac{4}{n-2} \) if \( n \geq 3 \).

Equation (1.1) is supplemented by the boundary conditions

\[ u|_{\partial \Omega} = \Delta u|_{\partial \Omega} = 0, \quad t > 0, \]

and the initial condition

\[ u(x, 0) = u_0(x). \]

Equation (1.1) was introduced as a simplification of multiple microscopic mechanisms model [8] in cluster interface evolution. Equation (1.1) with constant mobility has been intensively studied. Karali and Nagase [9] investigated existence of weak solution to (1.1) with \( D(u) \equiv D \) and a quartic bistable potential \( F(u) = (1 - u^2)^2 \). Karali and Nagase [9] only provided existence of the solution for the deterministic case. Then Antonopoulou, Karali and Millet [2] studied the stochastic case. The

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main result of this paper is the existence of a global solution, under a specific sub-linear growth condition for the diffusion coefficient. Path regularity in time and in space is also studied. In addition, Karali and Ricciardi [7] constructed special sequences of solutions to a fourth order nonlinear parabolic equation of the Cahn-Hilliard/Allen-Cahn equation, converging to the second order Allen-Cahn equation.

They studied the equivalence of the fourth order equation with a system of two second order elliptic equations. Karali and Katsoulakis [8] focus on a mean field partial differential equation, which contains qualitatively microscopic information on particle-particle interactions and multiple particle dynamics, and rigorously derive the macroscopic cluster evolution laws and transport structure. They show that the motion by mean curvature is given by \( V = \mu \sigma \kappa \), where \( \kappa \) is the mean curvature, \( \sigma \) is the surface tension and \( \mu \) is an effective mobility that depends on the presence of the multiple mechanisms and speeds up the cluster evolution. This is in contrast with the Allen-Cahn equation where the velocity equals the mean curvature.

During the past few years, many authors have paid much attention to the Cahn-Hilliard equation with degenerate mobility. An existence result for the Cahn-Hilliard equation with a degenerate mobility in a one-dimensional situation has been established by Yin [19]. Elliott and Garcke [5] considered the Cahn-Hilliard equation with non-constant mobility for arbitrary space dimensions. Based on Galerkin approximation, they proved the global existence of weak solutions. Dai and Du [4] improved the results of the paper [5]. Liu [12] proved the existence of weak solutions for the convective Cahn-Hilliard equation with degenerate mobility. The relevant equations or inequalities have also been studied in [10, 11, 13, 14].

Motivated by the above works, we prove the existence of weak solution to (1.1)-(1.4) under a more general range of the double-well potential \( F \). In particular, we assume that for \( s \in \mathbb{R} \), \( F \in C^2(\mathbb{R}) \) satisfies

\[
\begin{align*}
k_0(|s|^{r+1} - 1) \leq F(s) \leq k_1(|s|^{r+1} + 1), \\
|F'(s)| \leq k_2(|s|^r + 1), \\
|F''(s)| \leq k_3(|s|^{r-1} + 1),
\end{align*}
\]

for some constants \( k_0, k_1, k_2, k_3 > 0 \) where \( 1 \leq r < \infty \) if \( n = 1, 2 \) and \( 1 \leq r \leq \frac{n}{n-2} \) if \( n \geq 3 \). What’s more, we need the assumption on the boundary of \( f(u) \),

\[
f(u)|_{\partial \Omega} = 0, \quad t > 0. \tag{1.8}
\]

We can give examples satisfying the condition (1.8), such as \( F(u) = (1 - u^2)^2 \) studied by Karali and Nagase [9], the logarithmic function \( f(u) = -\theta_c u + \frac{\theta}{2} \ln \frac{1 + u}{1 - u}, \) \( u \in (-1, 1), \ 0 < \theta < \theta_c \) [3].

Concerning the Allen-Cahn structure, we rewrite (1.1), (1.3), (1.4) and (1.8) to the form

\[
\begin{align*}
& u_t = \nabla (D(u) \nabla v) - v, \quad \text{in} \ Q_T, \\
& v = -\Delta u + f(u), \quad \text{in} \ Q_T, \\
& u(x, 0) = u_0(x), \quad \text{in} \ \Omega, \\
& u = v = 0, \quad \text{on} \ \partial \Omega. \tag{1.9}
\end{align*}
\]
We consider the free energy functional $E(u)$ defined in [1] given by

$$E(u) := \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + F(u) \right) \, dx. \tag{1.10}$$

For a pair of solution $(u, v)$ of \[1.9\] it holds that

$$\frac{d}{dt} E(u) = \int_\Omega v u \, dx = \int_\Omega v[D(u) \nabla v - v] \, dx = - \int_\Omega (D(u)|\nabla v|^2 + v^2) \, dx \leq 0.$$

**Notation.** Define the usual Lebesgue norms and the $L^2$-inner-product

$$\|u\|_p = \|u\|_{L^p(\Omega)} \quad \text{and} \quad (u, v) = (u, v)_{L^2(\Omega)}.$$

The duality pairing between the space $H^2(\Omega)$ and its dual $(H^2(\Omega))^\prime$ will be denoted using the form $(\cdot, \cdot)$. For simplicity, $2^* := \frac{2n}{n-2}$. $\chi_B$ denotes the characteristic function of $B$.

This paper is organized as follows. In Section 2, we use a Galerkin method to give a existence of weak solution for a positive mobility. Section 3 uses a sequence of non-degenerate solutions to approximate the degenerate case \[1.9\].

## 2. Existence for Positive Mobility

In this section, we study the Cahn-Hilliard/Allen-Cahn equation with a non-degenerate mobility $D_\varepsilon(u)$ defined for an $\varepsilon$ satisfying $0 < \varepsilon < \delta$ by

$$D_\varepsilon(u) := \begin{cases} |u|^m, & \text{if } |u| > \varepsilon, \\ \varepsilon^m, & \text{if } |u| \leq \varepsilon. \end{cases} \tag{2.1}$$

So we consider the problem

$$u_t = \nabla (D_\varepsilon(u) \nabla v) - v, \quad \text{in } Q_T,$$

$$v = -\Delta u + f(u), \quad \text{in } Q_T,$$

$$u(x, 0) = u_0(x), \quad \text{in } \Omega,$$

$$u = v = 0, \quad \text{on } \partial \Omega. \tag{2.2}$$

**Theorem 2.1.** Suppose $u_0 \in H^1(\Omega)$, under assumptions [1.2] and [1.5]–[1.7], for any $T > 0$, there exists a pair of functions $(u_\varepsilon, v_\varepsilon)$ such that

1. $u_\varepsilon \in L^\infty(0, T; H^1_0(\Omega)) \cap C([0, T]; L^p(\Omega)) \cap L^2(0, T; H^3(\Omega))$, where $1 \leq p < \infty$ if $n = 1, 2$ and $2 \leq p < \frac{2n}{n-2}$ if $n \geq 3$,
2. $\partial_t u_\varepsilon \in L^2(0, T; (H^2(\Omega))^\prime)$,
3. $u_\varepsilon(x, 0) = u_0(x)$ for all $x \in \Omega$,
4. $v_\varepsilon \in L^2(0, T; H^1_0(\Omega))$,

which satisfies equation (2.2) in the following weak sense

$$\int_0^T \langle \partial_t u_\varepsilon, \phi \rangle dt + \int_{Q_T} (-\Delta u_\varepsilon + f(u_\varepsilon)) \phi \, dx \, dt = - \int_{Q_T} D_\varepsilon(u_\varepsilon) (\nabla u_\varepsilon - \nabla \Delta u_\varepsilon + F''(u_\varepsilon) \nabla u_\varepsilon) : \nabla \phi \, dx \, dt \tag{2.3}$$

for all test functions $\phi \in L^2(0, T; H^2(\Omega) \cap H^1_0(\Omega))$. In addition, $u_\varepsilon$ satisfies the energy inequality

$$E(u_\varepsilon) + \int_0^t \int_\Omega (D_\varepsilon(u_\varepsilon(x, \tau))|\nabla v_\varepsilon(x, \tau)|^2 + |v_\varepsilon(x, \tau)|^2) \, dx \, d\tau \leq E(u_0), \tag{2.4}$$
for all $t > 0$.

To prove the above theorem, we apply a Galerkin approximation. Let $\{\phi_J\}_{j \in \mathbb{N}}$ be the eigenfunctions of the Laplace operator on $L^2(\Omega)$ with Dirichlet boundary condition, i.e.,

\[
-\Delta \phi_J = \lambda_J \phi_J, \quad \text{in } \Omega,
\]

\[
\phi_J = 0, \quad \text{on } \partial \Omega.
\]

The eigenfunctions $\{\phi_J\}_{j=1}^{\infty}$ form an orthogonal basis for $L^2(\Omega), H^1(\Omega)$ and $H^2(\Omega)$. Hence, for initial data $u_0 \in H^1(\Omega)$, we can find sequences of scalars $(u^0_{N,j}; j = 1, 2, \ldots, N)_{N=1}^{\infty}$ such that

\[
\lim_{N \to \infty} \sum_{j=1}^{N} u^0_{N,j} \phi_J = u_0, \quad \text{in } H^1(\Omega).
\]

Let $V_N$ denote the linear span of $(\phi_1, \ldots, \phi_N)$ and $P_N$ be the orthogonal projection from $L^2(\Omega)$ to $V_N$, that is

\[
P_N \phi := \sum_{j=1}^{N} \left( \int_{\Omega} \phi_J \, dx \right) \phi_J.
\]

Let $u^N(x, t) = \sum_{j=1}^{N} c_j^N(t) \phi_J(x)$, $v^N(x, t) = \sum_{j=1}^{N} d_j^N(t) \phi_J(x)$ be the approximate solution of (2.2) in $V_N$; that is, $u^N, v^N$ satisfy the $g$ system of equations

\[
\begin{align*}
\int_{\Omega} \partial_t u^N \phi_J \, dx &= - \int_{\Omega} D_e(u^N) \nabla v^N \cdot \nabla \phi_J \, dx - \int_{\Omega} v^N \phi_J \, dx, \\
\int_{\Omega} v^N \phi_J \, dx &= \int_{\Omega} \nabla u^N \cdot \nabla \phi_J + f(u^N) \phi_J \, dx,
\end{align*}
\]

\[
u^N(x, 0) = \sum_{j=1}^{N} u^0_{N,j} \phi_J(x),
\]

for $j = 1, \ldots, N$ and $u^0_{N,j} = \int_{\Omega} u_0 \phi_J \, dx$.

This gives an initial value problem for a system of ordinary differential equations for $(c_1, \ldots, c_N)$

\[
\begin{align*}
\partial_t c_j^N(t) &= - \sum_{k=1}^{N} d_k^N(t) \int_{\Omega} D_e \left( \sum_{i=1}^{N} c_i^N(t) \phi_i(x) \right) \nabla \phi_k \nabla \phi_J \, dx - d_j^N(t), \\
d_j^N(t) &= \lambda_j c_j^N(t) + \int_{\Omega} f \left( \sum_{i=1}^{N} c_i^N(t) \phi_i(x) \right) \phi_J \, dx, \\
c_j^N(0) &= u^0_{N,j} = (u_0, \phi_J),
\end{align*}
\]

which has to hold for $j = 1, \ldots, N$.

Define $X(t) = (c_1^N(t), \ldots, c_N^N(t))$, $F(t, X(t)) = (f_1(t, X(t)), \ldots, f_N(t, X(t)))$, where

\[
f_j(t, X(t)) = - \sum_{k=1}^{N} \int_{\Omega} D_e \left( \sum_{i=1}^{N} c_i^N(t) \phi_i(x) \right) \nabla \phi_k \nabla \phi_J \, dx \\
\times \left( \lambda_k c_k^N(t) + \int_{\Omega} f \left( \sum_{i=1}^{N} c_i^N(t) \phi_i(x) \right) \phi_k \, dx \right)
\]
for } j = 1, \ldots, N. \, \text{Then problem } (2.10)-(2.12) \text{ is equivalent to the problem}

\[ X'(t) = F(t, X(t)), \quad X(0) = (u^0_{N,1}, \ldots, u^0_{N,N}). \]

Since the right hand side of the above equation is continuous, it follows from the Cauchy-Peano Theorem [16] that the problem (2.10)-(2.12) has a solution } X(t) \in C^1[0,T_N], \text{ for some } T_N > 0, \text{ i. e., the system } (2.7)-(2.9) \text{ has a local solution.}

To prove the existence of solutions, we need some a priori estimates on } u^N.

\textbf{Lemma 2.2.} For any } T > 0, \text{ we have}

\[ \|u^N\|_{L^\infty(0,T; H^1_0(\Omega))} \leq C, \quad \text{for all } N, \]

\[ \|\partial_t u^N\|_{L^2(0,T; (H^2(\Omega))')} \leq C, \quad \text{for all } N, \]

where } C \text{ independent of } N.

\textit{Proof.} For any fixed } N \in \mathbb{N}^+, \text{ we multiply } (2.7) \text{ by } \partial_t^N \text{ and sum over } j = 1, \ldots, N \text{ to obtain}

\[ \int_\Omega \partial_t u^N v^N dx = - \int_\Omega D_x(u^N) |\nabla v^N|^2 dx - \int_\Omega |v^N|^2 dx. \quad (2.13) \]

Multiply (2.8) by } \partial_t c_j^N \text{ and sum over } j = 1, \ldots, N \text{ to obtain}

\[ \int_\Omega v^N \partial_t u^N dx = \int_\Omega (\nabla u^N \partial_t u^N + f(u^N) \partial_t u^N) dx,
\]

\[ = \frac{d}{dt} \int_\Omega \left( \frac{1}{2} |\nabla u^N|^2 + F(u^N) \right) dx. \]

By (2.13) and the above identity, we have

\[ \frac{d}{dt} \int_\Omega \left( \frac{1}{2} |\nabla u^N|^2 + F(u^N) \right) dx = - \int_\Omega D_x(u^N) |\nabla v^N|^2 dx - \int_\Omega |v^N|^2 dx. \quad (2.14) \]

Replacing } t \text{ by } \tau \text{ in (2.14) and integrating over } \tau \in [0,t], \text{ by (1.5) and the Sobolev embedding theorem we obtain}

\[ \int_\Omega \left( \frac{1}{2} |\nabla u^N(x,t)|^2 + F(u^N(x,t)) \right) dx
\]

\[ + \int_0^t \int_\Omega (D_x(u^N(x,\tau)) |\nabla v^N(x,\tau)|^2 + |v^N(x,\tau)|^2) \, d\tau \, dx
\]

\[ = \int_\Omega \left( \frac{1}{2} |\nabla u^N(x,0)|^2 + F(u^N(x,0)) \right) dx
\]

\[ \leq \frac{1}{2} |\nabla u^N(x,0)|_2^2 + k_1 |u^N(x,0)|_{H^{r+1}_0(\Omega)}^2, \]

\[ \leq \frac{1}{2} |\nabla u^N(x,0)|_2^2 + k_1 C |u^0|_{H^{r+1}_0(\Omega)}^2 + k_1 |\Omega| \leq C. \]

The last inequality follows from } u_0 \in H^1(\Omega). \text{ This implies}

\[ \int_\Omega \left( \frac{1}{2} |\nabla u^N(x,t)|^2 + k_0 |u^N|^r+1 \right) dx
\]

\[ + \int_0^t \int_\Omega (D_x(u^N(x,\tau)) |\nabla v^N(x,\tau)|^2 + |v^N(x,\tau)|^2) \, d\tau \, dx \leq C. \quad (2.15) \]
By (2.15) and Poincaré’s inequality we have
\[ \|u^N\|_{H^1(\Omega)} \leq C, \quad \text{for } t > 0. \]
This estimate implies that the coefficients \{c_j^N : j = 1, \ldots, N\} are bounded in time and therefore a global solution to the system (2.7)-(2.9) exists. In addition, for any \( T > 0 \), we have
\[ u^N \in L^\infty(0, T; H^1_0(\Omega)), \quad \|u^N\|_{L^\infty(0, T; H^1_0(\Omega))} \leq C, \quad \text{for all } N. \tag{2.16} \]
Inequality (2.15) implies
\[ \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} \leq C, \quad \text{for all } N, \tag{2.17} \]
\[ \|v^N\|_{L^2(Q_T)} \leq C, \quad \text{for all } N. \tag{2.18} \]
By the Sobolev embedding theorem, the growth condition (1.2) and (2.1), for \( \varepsilon \), we obtain
\[ \int_\Omega |D_\varepsilon(u^N)|^{n/2} \, dx \leq (C_1 + 1) \int_\Omega |u^N|^{mn/2} \, dx \leq C\|u^N\|_{H^1(\Omega)}^{mn/2} \leq C. \]
If \( |u| \leq \varepsilon \), obviously we obtain the above estimate. This implies
\[ \|D_\varepsilon(u^N)\|_{L^\infty(0, T; L^{n/2}(\Omega))} \leq C, \quad \text{for all } N. \tag{2.19} \]
For any \( \phi \in L^2(0, T; H^2(\Omega)) \), we obtain \( \mathcal{P}_N\phi = \sum_{j=1}^{N} a_j(t)\phi_j \), where \( a_j(t) = \int_\Omega \phi_j \, dx \). Multiplying (2.7) by \( a_j(t) \), summing over \( j = 1, 2, \ldots, N \), by Hölder’s inequality, (2.17), (2.19) and the Sobolev embedding theorem, we have
\[ \left| \int_0^T \int_\Omega \partial_t u^N \phi \, dx \, dt \right| \]
\[ = \left| \int_0^T \int_\Omega \partial_t u^N \mathcal{P}_N\phi \, dx \, dt \right| \]
\[ = \left| \int_0^T \int_\Omega \left( D_\varepsilon(u^N)\nabla v^N \nabla \mathcal{P}_N\phi + v^N \mathcal{P}_N\phi \right) \, dx \, dt \right| \]
\[ \leq \int_0^T \|\sqrt{D_\varepsilon(u^N)}\|_n \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_2 \|\nabla \mathcal{P}_N\phi\|_2 \, dt + \int_0^T \|v^N\|_2 \|\mathcal{P}_N\phi\|_2 \, dt \]
\[ \leq C \int_0^T \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_2 \|\phi\|_{H^2} + \|v^N\|_2 \|\phi\|_{H^2} \, dt \]
\[ \leq C \left( \|\sqrt{D_\varepsilon(u^N)}\nabla v^N\|_{L^2(Q_T)} + \|v^N\|_{L^2(Q_T)} \right) \|\phi\|_{L^2(0, T; H^2(\Omega))} \]
\[ \leq C \|\phi\|_{L^2(0, T; H^2(\Omega))}. \]
Hence,
\[ \|\partial_t u^N\|_{L^2(0, T; H^2(\Omega))} \leq C \quad \text{for all } N. \tag{2.20} \]
The proof is complete. \( \square \)

**Lemma 2.3.** Suppose \( u_0 \in H^1(\Omega) \), under assumptions (1.2) and (1.5)-(1.7), for any \( T > 0 \), there exists a pair of functions \((u_\varepsilon, v_\varepsilon)\) such that
\begin{enumerate}
  \item \( u_\varepsilon \in L^\infty(0, T; H^1_0(\Omega)) \cap C([0, T]; L^p(\Omega)) \), where \( 1 \leq p < \infty \) if \( n = 1, 2 \) and \( 2 \leq p < \frac{2n}{n-2} \) if \( n \geq 3 \),
  \item \( \partial_t u_\varepsilon \in L^2(0, T; (H^2(\Omega))'), \)
  \item \( u_\varepsilon(x, 0) = u_0(x) \) for all \( x \in \Omega \),
\end{enumerate}
\( (4) \) \( v_\varepsilon \in L^2(0,T; H^1_0(\Omega)) \),

which satisfies

\[
\int_0^T (\partial_t u_\varepsilon, \phi) \, dt = -\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot \nabla \phi \, dx \, dt - \int_0^T \int_\Omega v_\varepsilon \phi \, dx \, dt.
\]

**Proof.** Since the embedding \( H^1_0(\Omega) \hookrightarrow L^p(\Omega) \) is compact for \( 1 \leq p < \infty \) if \( n = 1,2 \) and \( 1 \leq p < \frac{2n}{n-2} \) if \( n \geq 3 \), \( L^p(\Omega) \hookrightarrow (H^2(\Omega))' \) is continuous for \( p \geq 1 \) if \( n \leq 3 \), \( p > 1 \) if \( n = 4 \) and \( p \geq \frac{2n}{n+4} \) if \( n \geq 5 \). Using the Aubin-Lions lemma (Lions [17]), we can find a subsequence which we still denote by \( u_\varepsilon \) and \( u_\varepsilon \in L^\infty(0,T; H^1_0(\Omega)) \), such that as \( N \to \infty \)

\[
u_\varepsilon \rightharpoonup u_\varepsilon \quad \text{weak-* in } L^\infty(0,T; H^1_0(\Omega)),
\]

\[
u_\varepsilon \rightharpoonup u_\varepsilon \quad \text{strongly in } C([0,T]; L^p(\Omega)),
\]

\[
u_\varepsilon \rightharpoonup u_\varepsilon \quad \text{strongly in } L^2(0,T; L^p(\Omega)) \text{ and almost everywhere in } Q_T,
\]

\[
\partial_t u_\varepsilon \rightharpoonup \partial_t u_\varepsilon \quad \text{weakly in } L^2(0,T; (H^2(\Omega))'),
\]

where \( 2 \leq p < 2^* \) if \( n \geq 3 \) and \( 1 \leq p < \infty \) if \( n = 1,2 \).

By multiplying \( (2.27) \) by \( a_j(t) \) and integrating \( (2.7) \) over \( t \in [0,T] \), we obtain

\[
\int_0^T \int_\Omega \partial_t u_\varepsilon a_j(t) \phi_j \, dx \, dt = -\int_0^T \int_\Omega D_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \cdot a_j(t) \nabla \phi_j \, dx \, dt - \int_0^T \int_\Omega v_\varepsilon a_j(t) \phi_j \, dx \, dt.
\]

To pass to the limit in \( (2.25) \), we need the convergence of \( v_\varepsilon \) and \( D_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \). By \( (2.27) \) and \( D_\varepsilon(u_\varepsilon) \geq \varepsilon \), then

\[
\|\nabla v_\varepsilon\|_{L^2(Q_T)} \leq C \varepsilon^{-\frac{2}{p}} < \infty, \quad \text{for any } \varepsilon > 0.
\]

This implies that \( \{\nabla v_\varepsilon\} \) is a bounded sequence in \( L^2(Q_T) \), thus there exists a subsequence, not relabeled, and \( \zeta_\varepsilon \in L^2(Q_T) \) such that

\[
\nabla v_\varepsilon \rightharpoonup \zeta_\varepsilon \quad \text{weakly in } L^2(Q_T).
\]

By \( (2.26) \) and Poincaré’s inequality, we have

\[
\|v_\varepsilon\|_{L^2(0,T; H^1_0(\Omega))} \leq C \varepsilon^{-\frac{2}{p}} < \infty, \quad \text{for any } \varepsilon > 0.
\]

Hence we can find a subsequence of \( v_\varepsilon \), not relabeled, and \( v_\varepsilon \in L^2(0,T; H^1_0(\Omega)) \) such that

\[
v_\varepsilon \rightharpoonup v_\varepsilon \quad \text{weakly in } L^2(0,T; H^1_0(\Omega)).
\]

For any \( g \in L^2(0,T; H^1_0(\Omega)) \), by \( (2.26) \) and \( (2.27) \) we have

\[
\lim_{N \to \infty} \int_0^T \int_\Omega \nabla v_\varepsilon \cdot g \, dx \, dt = \int_0^T \int_\Omega \zeta_\varepsilon g \, dx \, dt = \int_0^T \int_\Omega \nabla v_\varepsilon g \, dx \, dt.
\]

Hence \( \zeta_\varepsilon = \nabla v_\varepsilon \) almost all in \( Q_T \) and

\[
\nabla v_\varepsilon \rightharpoonup \nabla v_\varepsilon \quad \text{weakly in } L^2(Q_T).
\]
By (2.18), we can extract a further sequence of $v^N$, not relabeled, and $\eta_\varepsilon \in L^2(Q_T)$ such that
\[
v^N \to \eta_\varepsilon, \quad \text{weakly in } L^2(Q_T).
\tag{2.30}
\]
By (2.28) and (2.30) for any $g \in L^2(Q_T) \subset L^2(0,T;H^{-1}(\Omega))$, we have
\[
\lim_{N \to \infty} \int_0^T \int_\Omega v^N g \, dx \, dt = \int_0^T \int_\Omega v_\varepsilon g \, dx \, dt = \int_0^T \int_\Omega \eta_\varepsilon g \, dx \, dt.
\]
This implies $\eta_\varepsilon = v_\varepsilon$ almost all $Q_T$ and
\[
v^N \to v_\varepsilon, \quad \text{weakly in } L^2(Q_T). \tag{2.31}
\]
Consequently we have the bound
\[
\int_{Q_T} |v_\varepsilon|^2 \, dx \, dt \leq C. \tag{2.32}
\]
For any $t \in [0,T]$, by $D_\varepsilon(u^N) \leq C(1 + |u^N|^m)$, we have
\[
(D_\varepsilon(u^N))^{n/2} \leq C(1 + |u^N|^m)^{n/2} \leq (C(1 + |u^N|))^{mn/2},
\]
where $2 \leq \frac{mn}{2} < 2^*$. By (2.22), $C(1 + |u^N|) \to C(1 + |u_0|)$ in $L^{mn/2}(\Omega)$. Since $D_\varepsilon$ is continuous and (2.23), we obtain
\[
D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{a.e. in } \Omega.
\]
The generalized Lebesgue convergence theorem [1] gives
\[
D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{in } L^{n/2}(\Omega).
\]
This implies
\[
\|D_\varepsilon(u^N) - D_\varepsilon(u_\varepsilon)\|_{n/2} \to 0, \quad \text{as } N \to \infty.
\]
The above estimate holds for any $t \in [0,T]$, and we can take supremum on both sides of the above estimate to obtain
\[
\sup_{t \in [0,T]} \|D_\varepsilon(u^N) - D_\varepsilon(u_\varepsilon)\|_{n/2} \to 0, \quad \text{as } N \to \infty.
\]
This implies
\[
D_\varepsilon(u^N) \to D_\varepsilon(u_\varepsilon), \quad \text{strongly in } C(0,T;L^{n/2}(\Omega)). \tag{2.33}
\]
By $\sqrt{D_\varepsilon(u^N)} \leq C(1 + |u^N|^\frac{m}{2})$, (2.22), (2.23) and the generalized Lebesgue convergence theorem, similarly, we have
\[
\sqrt{D_\varepsilon(u^N)} \to \sqrt{D_\varepsilon(u_\varepsilon)}, \quad \text{strongly in } C(0,T;L^{n}(\Omega)). \tag{2.34}
\]
For any $\varphi \in L^2(0,T;L^{2^*}(\Omega))$, by H"{o}lder’s inequality we have
\[
\begin{align*}
&\left| \int_{Q_T} \left( \sqrt{D_\varepsilon(u^N)} \nabla v^N \varphi - \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \varphi \right) \, dx \, dt \right| \\
&= \left| \int_{Q_T} \left[ \sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)} \right] \nabla v^N \varphi + \sqrt{D_\varepsilon(u_\varepsilon)} \left( |\nabla v^N \varphi - \nabla v_\varepsilon \varphi| \right) \right) \, dx \, dt \\
&\leq \int_0^T \|\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}\|_{n} \|\nabla v^N\|_{2^*} \|\varphi\|_{2^*} \, dt \\
&\quad + \left| \int_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)} \varphi |\nabla v^N - \nabla v_\varepsilon| \, dx \, dt \right|
\end{align*}
\]
By H"older’s inequality and (2.17), we have
\[
\sup_{t \in [0,T]} \| D_\varepsilon(u^N) - \sqrt{D_\varepsilon(u_\varepsilon)} \|_{n} \| \nabla v^N \|_{L^2(Q_T)} \| \varphi \|_{L^2(0,T;L^{2*}(\Omega))} \\
+ \left| \int_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)} |\nabla v^N - \nabla v_\varepsilon| \, dx \, dt \right|
\equiv I + II.
\]

By (2.29) and (2.34), \( I \to 0 \) as \( N \to \infty \). By H"older’s inequality and (2.34) we have
\[
\int_{Q_T} |\sqrt{D_\varepsilon(u_\varepsilon)} \varepsilon|^2 \, dx \, dt \leq \int_0^T \left( \int_{\Omega} (D_\varepsilon(u_\varepsilon))^{n/2} \, dx \right)^{n/2} \left( \int_{\Omega} |\varphi|^{2n/n} \, dx \right)^{n/(n-2)} \frac{n}{n-2} \frac{n}{n-2} \, dt
\leq \sup_{t \in [0,T]} \| \sqrt{D_\varepsilon(u_\varepsilon)} \|_n \int_0^T \| \varphi \|_{L^{2*}(\Omega)}^2 \, dt
\leq C \| \varphi \|_{L^2(0,T;L^{2*}(\Omega))}^2.
\]

This implies
\[
\sqrt{D_\varepsilon(u_\varepsilon)} \varphi \in L^2(Q_T).
\]

Thus \( II \to 0 \) as \( N \to \infty \) by (2.29). Hence
\[
\sqrt{D_\varepsilon(u^N)} \nabla v^N \rightharpoonup \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon, \quad \text{weakly in } L^2(0,T;L^{2*}(\Omega)).
\]

Next we consider the convergence of \( D_\varepsilon(u^N) \nabla v^N \). By (2.17), (2.36) and \( L^2(Q_T) \subset L^2(0,T;L^{2*}(\Omega)) \), we can extract a further sequence, not relabeled, such that
\[
\sqrt{D_\varepsilon(u^N)} \nabla v^N \rightharpoonup \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon, \quad \text{weakly in } L^2(Q_T).
\]

By H"older’s inequality and (2.17), we have
\[
\int_{Q_T} \sqrt{D_\varepsilon(u^N)} \nabla v^N \cdot \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \, dx \, dt
\leq \| \sqrt{D_\varepsilon(u^N)} \nabla v^N \|_{L^2(Q_T)} \| \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \|_{L^2(Q_T)}
\leq C \| \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \|_{L^2(Q_T)},
\]
where \( C \) is independent of \( \varepsilon \). Taking the limit of (2.38) on both sides, by (2.37) we have
\[
\| \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \|_{L^2(Q_T)} \leq C.
\]

For any \( \varphi \in L^2(0,T;L^{2*}(\Omega)) \), by H"older’s inequality we obtain
\[
\left| \int_{Q_T} (D_\varepsilon(u^N) \nabla v^N \varphi - D_\varepsilon(u_\varepsilon) \nabla v_\varepsilon \varphi) \, dx \, dt \right|
\leq \left| \int_{Q_T} (\sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)}) \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v^N \varphi \, dx \, dt \right|
+ \left| \int_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)} \left[ \sqrt{D_\varepsilon(u^N)} \nabla v^N \varphi - \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \varphi \right] \, dx \, dt \right|
\leq \int_0^T \| \sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_\varepsilon)} \|_n \| \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v^N \|_{L^2(Q_T)} \| \varphi \|_{L^{2*}} \, dt
+ \left| \int_{Q_T} \sqrt{D_\varepsilon(u_\varepsilon)} \varphi \left[ \sqrt{D_\varepsilon(u^N)} \nabla v^N - \sqrt{D_\varepsilon(u_\varepsilon)} \nabla v_\varepsilon \right] \, dx \, dt \right|
\begin{equation}
\leq \sup_{t \in [0,T]} \| \sqrt{D_\varepsilon(u^N)} - \sqrt{D_\varepsilon(u_c)} \|_n \| \sqrt{D_\varepsilon(u^N)} \nabla v^N \|_{L^2(Q_T)} \| \phi \|_{L^2(0,T;L^2(\Omega))} \\
+ \left| \int_{Q_T} \sqrt{D_\varepsilon(u_c)} \phi [\sqrt{D_\varepsilon(u^N)} \nabla v^N - \sqrt{D_\varepsilon(u_c)} \nabla v_c] \, dx \, dt \right| \\
= I + II.
\end{equation}

By (2.34) and (2.37), \( I \to 0 \) as \( N \to \infty \). By (2.35) and (2.37), we have \( II \to 0 \) as \( N \to \infty \). Thus

\begin{equation}
D_\varepsilon(u^N) \nabla v^N \rightharpoonup D_\varepsilon(u_c) \nabla v_c, \ \text{weakly in} \ L^2(0,T;L^2(\Omega)).
\end{equation}

For any \( \phi \in L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega)) \), we obtain \( \mathcal{P}_n \phi = \sum_{j=1}^n a_j(t) \phi_j \), where \( a_j(t) = \int_\Omega \phi \phi_j \, dx \). Then \( \mathcal{P}_n \phi \) converges strongly to \( \phi \) in \( L^2(0,T;H^2 \cap H^1_0(\Omega)) \) and \( a_j(t) \in L^2(0,T) \). For \( \phi_j \in H^2(\Omega) \), by Sobolev embedding theorem, we obtain

\[ \| \nabla \phi_j \|_{L^2} \leq C \| \nabla \phi_j \|_{H^1(\Omega)} \leq C. \]

Thus \( a_j(t) \nabla \phi_j \in L^2(0,T;L^2(\Omega)) \) and

\[ a_j(t) \phi_j \in L^2(0,T;H^2(\Omega)) \subset L^2(0,T;H^{-1}(\Omega)). \]

Taking the limit as \( N \to \infty \) on both sides of (2.25), by (2.24), (2.40) and (2.28), we have

\begin{equation}
\int_0^T \langle \partial_t u_c, a_j(t) \phi_j \rangle \, dt \\
= - \int_0^T \int_\Omega D_\varepsilon(u_c) \nabla v_c \cdot a_j(t) \nabla \phi_j \, dx \, dt - \int_0^T \int_\Omega v_c a_j(t) \phi_j \, dx \, dt,
\end{equation}

for all \( j \in N \).

Then we sum over \( j = 1,2,\ldots,n \) on both sides (2.41) to get

\begin{equation}
\int_0^T \langle \partial_t u_c, \mathcal{P}_n \phi \rangle \, dt \\
= - \int_0^T \int_\Omega D_\varepsilon(u_c) \nabla v_c \cdot \nabla \mathcal{P}_n \phi \, dx \, dt - \int_0^T \int_\Omega v_c \mathcal{P}_n \phi \, dx \, dt.
\end{equation}

Since \( \mathcal{P}_n \phi \) converges strongly to \( \phi \) in \( L^2(0,T;H^2(\Omega)) \), thus as \( n \to \infty \),

\[ \int_0^T \| \nabla \mathcal{P}_n \phi - \nabla \phi \|_{L^2}^2 \, dt \leq \int_0^T \| \nabla \mathcal{P}_n \phi - \nabla \phi \|_{H^1}^2 \, dt \\
\leq \int_0^T \| \mathcal{P}_n \phi - \phi \|_{H^2}^2 \, dt \to 0. \]

This implies that \( \nabla \mathcal{P}_n \phi \) converges strongly to \( \nabla \phi \) in \( L^2(0,T;L^2(\Omega)) \). Thus we obtain

\begin{equation}
\mathcal{P}_n \phi \rightharpoonup \phi, \ \text{weakly in} \ L^2(0,T;H^2(\Omega) \cap H^1_0(\Omega)),
\end{equation}

\begin{equation}
\nabla \mathcal{P}_n \phi \rightharpoonup \nabla \phi, \ \text{weakly in} \ L^2(0,T;L^2(\Omega)).
\end{equation}

By \( L^2(0,T;H^1_0(\Omega)) \subset L^2(0,T;H^{-1}(\Omega)) \), we take the limit as \( n \to \infty \) on both sides (2.42), then obtain

\begin{equation}
\int_0^T \langle \partial_t u_c, \phi \rangle \, dt = - \int_0^T \int_\Omega D_\varepsilon(u_c) \nabla v_c \cdot \nabla \phi \, dx \, dt - \int_0^T \int_\Omega v_c \phi \, dx \, dt.
\end{equation}
As for the initial value, by (2.9) as \( N \to \infty \),
\[
u^N(x, 0) \to u_0(x) \quad \text{in} \quad L^2(\Omega).
\]
By (2.22), \( u_\varepsilon(x, 0) = u_0(x) \) in \( L^2(\Omega) \). The proof is complete. \( \square \)

**Proof of Theorem 2.1.** We need only to check that
\[
u^N \in L^2(0,T; H^3(\Omega)), \quad \nabla v_\varepsilon = -\Delta u_\varepsilon + f(u_\varepsilon) \quad \text{and} \quad \nabla u_\varepsilon = -\nabla \Delta v_\varepsilon + \nabla F'(u_\varepsilon) \nabla u_\varepsilon.
\]
First we consider the convergence of \( \nabla u^N \) and \( f(u^N) \).

By (2.21), we have
\[
\int_0^T \| \nabla u^N \|_2^2 \, dt \leq C.
\]
Hence we can find a subsequence of \( u^N \), not relabeled, and \( \upsilon \in L^2(Q_T) \), such that
\[
\nabla u^N \rightharpoonup \upsilon \quad \text{weakly in} \quad L^2(Q_T).
\]
(2.46)

For any \( \phi \in L^2(0,T; H^1_0(\Omega)) \), by integration by parts we have
\[
\lim_{N \to \infty} \int_0^T \int_\Omega \nabla u^N \phi \, dx \, dt = \lim_{N \to \infty} \int_0^T \int_\Omega u^N \nabla \phi \, dx \, dt.
\]
By (2.21), (2.46) and \( \nabla \phi \in L^2(Q_T) \subset L^1(0,T; H^{-1}(\Omega)) \) we have
\[
\int_0^T \int_\Omega \upsilon \phi \, dx \, dt = \int_0^T \int_\Omega u_\varepsilon \nabla \phi \, dx \, dt = \int_0^T \int_\Omega \nabla u_\varepsilon \phi \, dx \, dt.
\]
Hence \( \upsilon = \nabla u_\varepsilon \) almost all in \( \Omega \times [0,T] \) and
\[
\nabla u^N \rightharpoonup \nabla u_\varepsilon \quad \text{weakly in} \quad L^2(Q_T).
\]
(2.47)

By \( |F'(u^N)| \leq C(1 + |u^N|^{\gamma}) \), (2.22), (2.23) and the general dominated convergence theorem, similarly, we have
\[
F'(u^N) \to F'(u_\varepsilon) \quad \text{strongly in} \quad C(0,T; L^q(\Omega)),
\]
for \( 1 \leq q < \infty \) if \( n = 1, 2 \) and \( 2 \leq q < \frac{2n}{r(n-2)} \) if \( n \geq 3 \).

By the growth condition (1.6) and the Sobolev embedding theorem, we obtain
\[
\|f(u^N)\|_{L^2(\Omega)}^2 = \int_\Omega (F'(u^N))^2 \, dx \leq C \int_\Omega (|u^N|^{\gamma} + 1)^2 \, dx \leq 2C \int_\Omega |u^N|^{2 \gamma} \, dx + 2C|\Omega| \leq C\|u^N\|_{H^3(\Omega)}^2 + C.
\]
Thus there exists a \( w \in L^\infty(0,T; L^2(\Omega)) \) such that
\[
F'(u^N) \rightharpoonup w \quad \text{weakly-* in} \quad L^\infty(0,T; L^2(\Omega)).
\]
This implies
\[
\lim_{N \to \infty} \int_0^T \int_\Omega F'(u^N) \, g \, dx \, dt = \int_0^T \int_\Omega w \, g \, dx \, dt,
\]
for any \( g \in L^1(0,T; L^2(\Omega)) \).

By Hölder's inequality, (2.48) and (2.49) we have as \( N \to \infty \)
\[
\left| \int_{Q_T} (F'(u_\varepsilon) - w) \, g \, dx \, dt \right|
\]
Next we show
\[
F'(u_\varepsilon) = w \text{ a.e. in } Q_T \quad \text{and} \quad F'(u_\varepsilon) \rightharpoonup F'(u) \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)).
\] (2.50)

Multiplying (2.8) by \(a_j(t)\) and integrating (2.8) over \(t \in [0,T]\), we obtain
\[
\int_0^T \int_\Omega v^N a_j(t) \phi_j \, dx \, dt = \int_0^T \int_\Omega (\nabla v^N \cdot a_j(t) \nabla \phi_j + F'(u_\varepsilon) a_j(t) \phi_j) \, dx \, dt.
\] (2.51)

For any \(\phi \in L^2(0,T;H^1_0(\Omega))\), we obtain \(P_n \phi = \sum_{j=1}^n a_j(t) \phi_j\), where \(a_j(t) \in L^2(0,T)\). Thus \(a_j(t) \phi_j \in L^2(0,T;H^1_0(\Omega))\) and \(a_j(t) \nabla \phi_j \in L^2(Q_T)\). By (2.28), (2.47) and (2.50), we take the limit as \(n \to \infty\) on both sides of (2.51) to get
\[
\int_0^T \int_\Omega v_a(t) \phi_j \, dx \, dt = \int_0^T \int_\Omega (\nabla v_a(t) \nabla \phi_j + F'(u_\varepsilon) a_j(t) \phi_j) \, dx \, dt,
\] (2.52)

for all \(j \in N\).

Then we sum over \(j = 1, \ldots, n\) on both sides (2.52), and obtain
\[
\int_0^T \int_\Omega v_a(t) \phi \, dx \, dt = \int_0^T \int_\Omega (\nabla v_a \cdot \nabla P_n \phi + F'(u_\varepsilon) P_n \phi) \, dx \, dt.
\] (2.53)

Since \(P_n \phi\) converges strongly to \(\phi\) in \(L^2(0,T;H^1_0(\Omega))\), we have as \(n \to \infty\)
\[
\int_0^T \|\nabla P_n \phi - \nabla \phi\|_2^2 \, dt \leq \int_0^T \|P_n \phi - \phi\|_{H^1}^2 \, dt \to 0.
\]

This implies that \(\nabla P_n \phi\) converges strongly to \(\nabla \phi\) in \(L^2(Q_T)\). Thus we obtain
\[
P_n \phi \rightharpoonup \phi \quad \text{weakly in } L^2(0,T;H^1_0(\Omega))
\] (2.54)
\[
\nabla P_n \phi \to \nabla \phi \quad \text{weakly in } L^2(Q_T).
\] (2.55)

By \(L^2(0,T;H^1_0(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))\) and \(L^\infty(0,T;L^2(\Omega)) \subset L^2(0,T;H^{-1}(\Omega))\), we take the limit as \(n \to \infty\) on both sides (2.53), and we obtain
\[
\int_0^T \int_{Q_T} v_a \phi \, dx \, dt = \int_0^T \int_{Q_T} (\nabla v_a \cdot \nabla \phi + F'(u_\varepsilon) \phi) \, dx \, dt.
\]

Since \(F'(u_\varepsilon) \in L^\infty(0,T;L^2(\Omega))\) and \(v_a \in L^2(0,T;H^1_0(\Omega))\), it follows from regularity theory [1] that \(u_\varepsilon \in L^2(0,T;H^1(\Omega))\). Hence
\[
v_a = -\Delta u_\varepsilon + F'(u_\varepsilon) \quad \text{almost everywhere in } Q_T.
\] (2.56)

Next we show \(F'(u_\varepsilon) \in L^2(0,T;H^1(\Omega))\). By Hölder’s inequality, the Sobolev embedding theorem and (1.7), we have
\[
\int_0^T \int_\Omega |\nabla F'(u_\varepsilon)|^2 \, dx \, dt = \int_0^T \int_\Omega |F''(u_\varepsilon)|^2 |\nabla u_\varepsilon|^2 \, dx \, dt
\]
\[
\leq \int_0^T \left( \int_\Omega |F''(u_\varepsilon)|^{2\times \frac{n}{2}} \, dx \right)^{\frac{2}{2/n}} \left( \int_\Omega |\nabla u_\varepsilon|^{2\times \frac{n}{2}} \, dx \right)^{\frac{n-2}{n}} \, dt
\]
\[ \leq C \int_0^T \left( \int_\Omega \left( 1 + |u_\varepsilon|^r \right)^{\frac{r-1}{n}} dx \right) \frac{2/n}{2-n} \left\| \nabla u_\varepsilon \right\|_{\frac{2n}{n-2}}^2 dt \]

\[ \leq C \int_0^T \left( 1 + \int_\Omega |u_\varepsilon|^{(r-1)n} dx \right) \frac{2/n}{2-n} \left\| \nabla u_\varepsilon \right\|_{H^1(\Omega)}^2 dt \]

\[ \leq C \int_0^T \left( 1 + \|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^4 \right) \|u_\varepsilon\|_{H^2(\Omega)}^2 dt \]

\[ \leq C(1 + \|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^4) \int_0^T \|u_\varepsilon\|_{H^2(\Omega)}^2 dt \]

\[ \leq C(1 + \|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}^4) \|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))}^2 \leq C. \]

Thus \( \nabla F'(u_\varepsilon) \in L^2(Q_T) \) and \( F'(u_\varepsilon) \in L^2(0,T;H^1(\Omega)) \). Combined with \( u_\varepsilon \in L^2(0,T;H^1(\Omega)) \), by (2.56) and regularity theory we have \( u_\varepsilon \in L^2(0,T;H^3(\Omega)) \) and

\[ \nabla u_\varepsilon = -\nabla u_\varepsilon + F''(u_\varepsilon) \nabla u_\varepsilon, \quad \text{almost everywhere in } Q_T. \] (2.57)

By (2.45), (2.56) and (2.57), we obtain

\[
\int_0^T \langle \partial_t u_\varepsilon, \phi \rangle dt + \int_0^T \int_\Omega (-\Delta u_\varepsilon + F'(u_\varepsilon)) \phi dx dt = - \int_0^T \int_\Omega D_\varepsilon(u_\varepsilon)(-\nabla u_\varepsilon + F''(u_\varepsilon) \nabla u_\varepsilon) \cdot \nabla \phi dx dt, \tag{2.58}
\]

for all \( \phi \in L^2(0,T;H^1(\Omega) \cap H^2_0(\Omega)) \).

Last we show that a weak solution \( u_\varepsilon \) to (2.2) satisfies energy inequality (2.4). Replacing \( t \) by \( \tau \) in (2.14) and integrating over \( \tau \in [0,T] \), we have

\[
E(u_N(x,t)) + \int_0^t \int_\Omega D_\varepsilon(u_N(x,\tau)) \nabla v_N(x,\tau)^2 dx d\tau \\
+ \int_0^t \int_\Omega |v_N(x,\tau)|^2 dx d\tau = E(u_N(x,0)). \tag{2.59}
\]

Next, we pass to the limit in (2.59). First, by mean value theorem and (1.6) we have

\[
\left| \int_\Omega (F(u_N(t)) - F(u_\varepsilon(t))) dx \right| \\
\leq \int_\Omega |F'(\xi)||u_N(t) - u_\varepsilon(t)| dx \\
\leq \int_\Omega C(|u_N(t)|^r + |u_\varepsilon(t)|^r + 1)|u_N(t) - u_\varepsilon(t)| dx,
\]

for \( 1 \leq r < \infty \) if \( n = 1,2 \) and \( 1 \leq r \leq \frac{n}{n-2} \) if \( n \geq 3 \), \( \xi = \lambda u_N(t) + (1-\lambda)u_\varepsilon(t) \) for some \( \lambda \in (0,1) \). By Hölder’s inequality, we have

\[
\int_\Omega |u_N(t)|^r |u_N(t) - u_\varepsilon(t)| dx \leq \|u_N(t) - u_\varepsilon(t)\|^r_2 \|u_N(t)\|^r_2. \tag{2.61}
\]

Since the Sobolev embedding theorem says that \( H^1_0(\Omega) \hookrightarrow L^p(\Omega) \) for \( 1 \leq p \leq 2^* \) and the embedding is compact if \( 1 \leq p < 2^* \), by (2.21), then for a subsequence, not relabeled, we have \( u_N \to u_\varepsilon \) strongly in \( L^\infty(0,T;L^2(\Omega)) \) and \( u_N \) is bounded in \( L^\infty(0,T;L^{2r}(\Omega)) \). Hence, it follows from (2.61) that

\[
\int_\Omega |u_N(t)|^r |u_N(t) - u_\varepsilon(t)| dx \to 0, \tag{2.62}
\]
as $N \to \infty$, for almost all $t \in [0, T]$.

Similarly, we can prove that
\[
\int \Omega \left( |u_\varepsilon(t)|^r + 1 \right) |u^N(t) - u_\varepsilon(t)| \, dx \to 0,
\]
as $N \to \infty$, for almost all $t \in [0, T]$, by (2.60), (2.62) and (2.63), we have
\[
\lim_{N \to \infty} \int \Omega F(u^N(t)) \, dx = \int \Omega F(u_\varepsilon(t)) \, dx.
\]
Since $u^N(x, 0) \to u_0(x)$ strongly in $L^2(\Omega)$, we obtain
\[
\lim_{N \to \infty} \int_{Q_T} \left( \frac{1}{2} |\nabla u^N(x, t)|^2 + F(u^N(x, t)) \right) \, dx d\tau \leq \lim_{N \to \infty} \inf \int_{Q_T} \left( \frac{1}{2} |\nabla u^N(x, t)|^2 + F(u^N(x, t)) + \int_{Q_T} (D_\varepsilon(u_\varepsilon(x, \tau)) |\nabla v^N(x, \tau)|^2 + |v^N(x, \tau)|^2) \, dx d\tau \right) = \lim_{N \to \infty} \inf E(u^N(x, 0)).
\]

Since $u^N(x, 0) \to u_0(x)$ strongly in $H^1(\Omega)$, by (2.65) we have
\[
\lim_{N \to \infty} E(u^N(x, 0)) = \int \Omega \left( \frac{1}{2} |\nabla u_0(x)|^2 + F(u_0(x)) \right) \, dx.
\]

Combining (2.66) with (2.67) gives the energy inequality (2.4). The proof is complete. □

3. Degenerate mobility

This section is devoted to the existence of weak solutions to the equations (1.9). Here we consider the limit of approximate solutions $u_\varepsilon$ defined in section 2. The limiting value $u$ does exist and solves the degenerate Allen-Cahn/Cahn-Hilliard equation in the weak sense.

**Theorem 3.1.** Suppose $u_0 \in H^1(\Omega)$, under assumptions (1.2) and (1.5)-(1.7), for any $T > 0$, problem (1.9) has a weak solution $u : Q_T \to \mathbb{R}$ satisfying

1. $u \in L^\infty(0, T; H^1_0(\Omega)) \cap C([0, T]; L^p(\Omega)) \cap L^2(0, T; H^2(\Omega))$, where $1 \leq p < \infty$ if $n = 1, 2$ and $2 \leq p < \frac{2n}{n-2}$ if $n \geq 3$,
2. $\partial_t u \in L^2(0, T; (H^2(\Omega))')$,
3. $u(x, 0) = u_0(x)$ for all $x \in \Omega$,

which satisfies (1.9) in the following weak sense:
(1) Define \( P \) as the set where \( D(u) \) is non-degenerate, that is
\[
P := \{(x, t) \in Q_T : |u| \neq 0\}.
\]
There exists a set \( A \subset Q_T \) with \(|Q_T \setminus A| = 0\) and a function \( \zeta : Q_T \to \mathbb{R}^n \) satisfying \( \chi_{A \cap P} D(u) \zeta \in L^2(0, T; L^{\frac{2n}{n+2}}(\Omega)) \), such that
\[
\int_0^T \langle \partial_t u, \phi \rangle \, dt = \int_0^T \int_\Omega (-\Delta u + f(u)) \phi \, dx \, dt
\]
for all test functions \( \phi \in L^2(0, T; H^2(\Omega) \cap H_0^1(\Omega)) \).

(2) For each \( j \in \mathbb{N} \), there exists \( E_j := \{(x, t) \in Q_T; u_i \to u \text{ uniformly, } |u| > \delta_j \text{ for } \delta_j > 0\} = T_J \times S_J \) such that
\[
u \in L^2(T_J; H^3(S_J)),
\]
\( \zeta = -\nabla \Delta u + E''(u)\nabla u, \) in \( E_j \).

In addition, \( u \) satisfies the energy inequality
\[
E(u) + \int_{Q_t \cap A \cap P} D(u(x, \tau)) |\zeta(x, \tau)|^2 \, dx \, d\tau
+ \int_{Q_t} |-\Delta u + f(u)|^2 \, dx \, d\tau \leq E(u_0),
\]
for all \( t > 0 \).

Proof. We consider a sequence of positive numbers \( \varepsilon_i \) monotonically decreasing to 0 as \( i \to \infty \). Fix \( u_0 \in H^1(\Omega) \), for any fixed \( \varepsilon_i \), here, for the sake of simplicity, we write \( u_i := u_{\varepsilon_i} \) and \( D_i(u_i) := D_{\varepsilon_i}(u_{\varepsilon_i}) \). By Theorem 2.1 there exists a function \( u_i \) such that

1. \( u_i \in L^\infty(0, T; H^1_0(\Omega)) \cap C([0, T]; L^p(\Omega)) \cap L^2(0, T; H^2(\Omega)) \), where \( 1 \leq p < \infty \) if \( n = 1, 2 \) and \( 2 \leq p < \frac{2n}{n+2} \) if \( n \geq 3 \),
2. \( \partial_t u_i \in L^2(0, T; (H^2(\Omega))') \),

\[
\int_0^T \langle \partial_t u_i, \phi \rangle \, dt = -\int_0^T \int_\Omega D_i(u_i) \nabla v_i \cdot \nabla \phi \, dx \, dt - \int_0^T \int_\Omega v_i \phi \, dx \, dt
\]
for all \( v_i = -\Delta u_i + f(u_i) \), almost everywhere in \( Q_T \).

By the arguments in the proof of Theorem 2.1, the bounds on the right hand side of (2.16), (2.20), (2.30), and (2.32) depend only on the growth conditions of the mobility and potential, so there exists a constant \( C > 0 \) independent of \( \varepsilon_i \) such that

\[
\|u_i\|_{L^\infty(0, T; H^1_0(\Omega))} \leq C, \tag{3.5}
\]
\[
\|
\partial_t u_i\|_{L^2(0, T; (H^2(\Omega))')} \leq C, \tag{3.6}
\]
\[
\|\sqrt{D_i(u_i)} \nabla v_i\|_{L^2(Q_T)} \leq C, \tag{3.7}
\]
\[
\|v_i\|_{L^2(Q_T)} \leq C. \tag{3.8}
\]
where $1 \leq i$ as enable us to find a subsequence, not relabeled, and $u \in L^\infty(0, T; H^1_0(\Omega))$ such that as $i \to \infty$,

$$u_i \to u, \quad \text{weak-* in } L^\infty(0, T; H^1_0(\Omega)), \quad (3.9)$$

$$u_i \to u, \quad \text{strongly in } C(0, T; L^p(\Omega)), \quad (3.10)$$

$$u_i \to u, \quad \text{strongly in } L^2(0, T; L^p(\Omega)) \text{ and almost all in } Q_T, \quad (3.11)$$

$$\partial_t u_i \to \partial_t u, \quad \text{weakly in } L^2(0, T; (H^2(\Omega))'), \quad (3.12)$$

where $1 \leq p < \infty$ if $n = 1, 2$ and $2 \leq p < \frac{2n}{n-2}$ if $n \geq 3$.

By (3.7) and (3.8), there exists $\xi, \eta \in L^2(Q_T)$ such that

$$\nabla D_i(u_i) \nabla v_i \to \xi, \quad \text{weakly in } L^2(Q_T), \quad (3.13)$$

$$v_i \to \eta, \quad \text{weakly in } L^2(Q_T). \quad (3.14)$$

Next we show the convergence of $D_i(u_i) \nabla v_i$ and $\eta = -\Delta u + f(u)$ a.e. $Q_T$. Similar to having (2.33) and (2.34), by the uniform convergence of $D_i \to D$, we obtain

$$D_i(u_i) \to D(u), \quad \text{strongly in } C(0, T; L^{n/2}(\Omega)), \quad (3.15)$$

$$\nabla D_i(u_i) \to \nabla D(u), \quad \text{strongly in } C(0, T; L^n(\Omega)). \quad (3.16)$$

For any $\varphi \in L^2(0, T; L^{2^*}(\Omega))$, by Hölder’s inequality, we have

$$\left| \iint_{Q_T} \left( D_i(u_i) \nabla v_i \varphi - \sqrt{D(u)} \xi \varphi \right) \, dx \, dt \right|$$

$$\leq \left| \iint_{Q_T} \left[ \sqrt{D_i(u_i)} - \sqrt{D(u)} \right] \sqrt{D_i(u_i)} \nabla v_i \varphi \, dx \, dt \right|$$

$$+ \left| \iint_{Q_T} \sqrt{D(u)} \left[ \sqrt{D_i(u_i)} \nabla v_i \varphi - \xi \varphi \right] \, dx \, dt \right|$$

$$\leq \int_0^T \| \sqrt{D_i(u_i)} - \sqrt{D(u)} \|_n \| \sqrt{D_i(u_i)} \nabla v_i \|_2 \| \varphi \|_{2^*} \, dt$$

$$+ \left| \iint_{Q_T} \sqrt{D(u)} \varphi \left[ \sqrt{D_i(u_i)} \nabla v_i - \xi \right] \, dx \, dt \right|$$

$$\leq \sup_{\varphi \in [0, T]} \| \sqrt{D_i(u_i)} - \sqrt{D(u)} \|_n \| \sqrt{D_i(u_i)} \nabla v_i \|_{L^2(Q_T)} \| \varphi \|_{L^2(0, T; L^{2^*}(\Omega))}$$

$$+ \left| \iint_{Q_T} \sqrt{D(u)} \varphi \left[ \sqrt{D_i(u_i)} \nabla v_i - \xi \right] \, dx \, dt \right|$$

$$=: I + II.$$

By (3.16) and (3.7), $I \to 0$ as $N \to \infty$. By Hölder’s inequality and the boundedness of $D(u)$ in $C(0, T; L^{n/2}(\Omega))$ we have

$$\iint_{Q_T} |\sqrt{D(u)} \varphi|^2 \, dx \, dt \leq \int_0^T \left( \int_\Omega (D(u))^{n/2} \, dx \right)^{n/2} \left( \int_\Omega |\varphi|^{2^*} \, dx \right)^{\frac{n-2}{n}} \, dt$$

$$\leq \sup_{\varphi \in [0, T]} \| D(u) \|_n \int_0^T \| \varphi \|_{L^{2^*}(\Omega)}^2 \, dt$$

$$\leq C \| \varphi \|_{L^2(0, T; L^{2^*}(\Omega))}^2.$$
This implies
\[ \sqrt{D(u)} \varphi \in L^2(Q_T). \] (3.17)
By (3.13), thus \( II \to 0 \) as \( N \to \infty \), this implies
\[ D_i(u_i) \nabla u_i \rightharpoonup \sqrt{D(u)} \xi \quad \text{weakly in } L^2(0,T;L^\frac{2}{\alpha+2}(\Omega)). \] (3.18)
By (3.4), for any \( \phi \in L^2(0,T;H^1_0(\Omega)) \subset L^2(Q_T) \) we have
\[
\int_Q v_i \phi \, dx \, dt = -\int_Q \Delta u_i \phi \, dx \, dt + \int_Q f(u_i) \phi \, dx \, dt \\
= \int_Q \nabla u_i \nabla \phi \, dx \, dt + \int_Q f(u_i) \phi \, dx \, dt.
\] (3.19)
Recalling that the convergence of \( \nabla u_i \) and \( f(u_i) \) are similar to get (2.47) and (2.50), we have
\[ \nabla u_i \rightharpoonup \nabla u, \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)), \] (3.20)
\[ f(u_i) \rightharpoonup f(u), \quad \text{weak-* in } L^\infty(0,T;L^2(\Omega)). \] (3.21)
By (3.20), (3.21) and \( L^2(0,T;H^1_0(\Omega)) \subset L^1(0,T;L^2(\Omega)) \), taking the limits of (3.19) on both sides, we have
\[ \int_Q \eta \phi \, dx \, dt = \int_Q \nabla u \nabla \phi \, dx \, dt + \int_Q f(u) \phi \, dx \, dt. \]
Since \( f(u) \in L^\infty(0,T;L^2(\Omega)) \) and \( \eta \in L^2(Q_T) \), by regularity theory we see that \( u \in L^2(0,T;H^2(\Omega)) \) and
\[ \eta = -\Delta u + f(u), \quad \text{almost everywhere in } Q_T. \] (3.22)
By (3.12), (3.18) and (3.22), taking the limits of (3.3), we have
\[
\int_0^T \langle \partial_t u, \phi \rangle \, dt = -\int_0^T \int_\Omega \sqrt{D(u)} \nabla \phi \, dx \, dt - \int_0^T \int_\Omega [-\Delta u + f(u)] \phi \, dx \, dt. \] (3.23)
As for the initial value, since \( u_i(x,0) = u_0(x) \) in \( L^2(\Omega) \), by (3.10) we have \( u(x,0) = u_0(x) \).

Now we consider the weak convergence of \( \nabla v_i \). Choose a sequence of positive numbers \( \delta_j \) that monotonically decreases to 0 as \( j \to \infty \). By (3.11) and Egorov’s theorem, for every \( \delta_j > 0 \), there exists a subset \( B_j \subset Q_T \) with \( |Q_T\setminus B_j| < \delta_j \) such that
\[ u_i \rightharpoonup u, \quad \text{uniformly in } B_j. \]
Define \( A_1 = B_1, A_2 = B_1 \cup B_2, \ldots, A_j = B_1 \cup B_2 \cup \cdots \cup B_j. \) Then
\[ A_1 \subset A_2 \subset \cdots \subset A_j \subset A_{j+1} \subset \cdots \subset Q_T. \] (3.24)
Thus the limit of \( \{A_j\} \) exists, then we have \( \lim_{j \to \infty} A_j = \cup_{j=1}^\infty A_j := A \) and \( |Q_T\setminus A| = 0 \).
Define \( P_j := \{(x,t) \in Q_T; |u| > \delta_j\} \). Then
\[ P_1 \subset P_2 \subset \cdots \subset P_j \subset P_{j+1} \subset \cdots \subset Q_T. \] (3.25)
Thus the limit of \( \{P_j\} \) exists, then we have \( \lim_{j \to \infty} P_j = \cup_{j=1}^\infty P_j := P \). For each \( j \), we define
\[ E_j := A_j \cap P_j, \quad \text{where } |u| > \delta_j \text{ and } u_i \to u \text{ uniformly}, \]
\[ G_j := A_j \setminus P_j, \quad \text{where } |u| \leq \delta_j \text{ and } u_i \to u \text{ uniformly}. \]
Thus we obtain $A_J = E_J \cup G_J$. By (3.24) and (3.25), we have
\[ E_1 \subset E_2 \subset \cdots \subset E_J \subset E_{J+1} \subset \cdots \subset Q_T. \]
Thus the limit of $\{E_J\}$ exists, then we have $\lim_{j \to \infty} E_J = \bigcup_{j=1}^{\infty} E_J = A \cap P := E$.

For any $\psi \in L^2(0, T; L^2(\Omega))$,
\[
\int_Q \int D_i(u_i) \nabla v_i \psi \, dx \, dt
= \int_{Q \setminus A_J} D_i(u_i) \nabla v_i \psi \, dx \, dt + \int_{G_J} D_i(u_i) \nabla v_i \psi \, dx \, dt
+ \int_{E_J} D_i(u_i) \nabla v_i \psi \, dx \, dt.
\]
As $i \to \infty$, by (3.18) we obtain
\[
\lim_{i \to \infty} \int_Q D_i(u_i) \nabla v_i \psi \, dx \, dt = \int_Q \sqrt{D(u)} \xi \psi \, dx \, dt,
\]
\[
\lim_{i \to \infty} \int_{Q \setminus A_J} D_i(u_i) \nabla v_i \psi \, dx \, dt = \int_{Q \setminus A_J} \sqrt{D(u)} \xi \psi \, dx \, dt.
\]
By $|Q \setminus A| = 0$, taking the limit of (3.28) as $j \to \infty$, we have
\[
\lim_{j \to \infty} \lim_{i \to \infty} \int_{Q \setminus A_J} D_i(u_i) \nabla v_i \psi \, dx \, dt = 0.
\]
To analyze the second and third terms of (3.26), we write $u_{j-1,i} := u_i$ and $v_{j-1,i} := v_i$ in $A_J$, then we have
\[ u_{j-1,i} \to u, \quad \text{uniformly in } A_J \text{ for all } j \in N. \]
This implies that there exists an index $N_J \in N^+$ such that for all $i \geq N_J$,
\[ |u_{j-1,i} - u| < \frac{\delta_J}{2}. \]
We can easily get the following result:
\[
|u_{j-1,i}| \geq \frac{\delta_J}{2}, \quad \text{if } (x, t) \in E_J,
|u_{j-1,i}| \leq 2\delta_J, \quad \text{if } (x, t) \in G_J.
\]
Considering the limit of the second term of (3.26), by Hölder’s inequality and (3.7) we have
\[
\left| \int_{G_J} D_{j-1,i} (u_{j-1,i}) \nabla v_{j-1,i} \psi \, dx \, dt \right|
\leq \sup_{(x,t) \in G_J} \left| D_{j-1,i} (u_{j-1,i}) \right| \left| \int_{Q_T} \sqrt{D_{j-1,i} (u_{j-1,i})} \nabla v_{j-1,i} ||\psi||_L^2(Q_T) \right| dx \, dt
\leq \sup_{(x,t) \in G_J} \left| D_{j-1,i} (u_{j-1,i}) \right| \left\| \sqrt{D_{j-1,i} (u_{j-1,i})} \nabla v_{j-1,i} \right\|_{L^2(Q_T)} \left\| \psi \right\|_{L^2(Q_T)}^2 \]
\[
\leq C \sup_{(x,t) \in G_J} \left| D_{j-1,i} (u_{j-1,i}) \right| \left( \left| \Omega \right|^{1/2} \left| \psi \right| \right)_{L^2(0,T;L^2(\Omega))} \]
\leq C \max \{(2 \delta_J)^{m/2}, \varepsilon_{j-1,i}^{m/2} \}.
\]
Taking the limits of (3.31) as \( i \to \infty \) and \( j \to \infty \), we have

\[
\lim_{j \to \infty} \lim_{i \to \infty} \int_{G_j} D_{j-1,i}(u_{j-1,i}) \nabla v_{j-1,i} \psi \, dx \, dt \leq C \lim_{j \to \infty} \lim_{i \to \infty} \max \{(2\delta_j)^{m/2}, \varepsilon_j^{m/2}\} = 0.
\]

(3.32)

By (3.7) and (3.30), we obtain

\[
\int_{D_j} |\nabla v_{j-1,i}|^2 \, dx \, dt \leq \int_{D_j} D_{j-1,i}(u_{j-1,i}) |\nabla v_{j-1,i}|^2 \, dx \, dt \leq \int_{Q_T} D_{j-1,i}(u_{j-1,i}) |\nabla v_{j-1,i}|^2 \, dx \, dt \leq C.
\]

This implies

\[
\int_{D_j} |\nabla v_{j-1,i}|^2 \, dx \, dt \leq C(\delta_j)^{-m}.
\]

So \( \nabla v_{j-1,i} \) is bounded in \( L^2(E_j) \), thus there exists a subsequence, labeled as \( \{\nabla v_{j,i}\} \), and \( \zeta_j \in L^2(E_j) \) such that

\[
\nabla v_{j,i} \rightharpoonup \zeta_j, \text{ weakly in } L^2(E_j).
\]

(3.33)

By \( E_{j-1} \subset E_j \), for any \( g \in L^2(E_j) \), we have \( g \in L^2(E_{j-1}) \) and \( \nabla v_{j-1,i} = \nabla v_{j,i} \) in \( E_{j-1} \). By (3.33) we have

\[
\lim_{i \to \infty} \int_{E_{j-1}} \nabla v_{j,i} g \, dx \, dt = \lim_{i \to \infty} \int_{E_{j-1}} \nabla v_{j-1,i} g \, dx \, dt = \int_{E_{j-1}} \zeta_j g \, dx \, dt = \int_{E_{j-1}} \zeta_{j-1} g \, dx \, dt.
\]

Thus \( \zeta_j = \zeta_{j-1} \) almost everywhere in \( E_{j-1} \). We define

\[
\omega_j := \begin{cases} 
\zeta_j, & \text{if } (x,t) \in E_j, \\
0, & \text{if } (x,t) \in E \setminus E_j.
\end{cases}
\]

So for almost every \( (x,t) \in E \), there exists a limit of \( \omega_j(x,t) \) as \( j \to \infty \). We write

\[
\zeta(x,t) = \lim_{j \to \infty} \omega_j(x,t), \text{ almost everywhere in } E.
\]

Clearly \( \zeta(x,t) = \zeta_j(x,t) \) for almost all \( (x,t) \in E_j \) for all \( j \). Using a standard diagonal argument, we can extract a subsequence such that

\[
\nabla v_{k,N_k} \rightharpoonup \zeta, \text{ weakly in } L^2(E_j) \text{ for all } j.
\]

(3.34)

For any \( \varphi \in L^2(0,T;L^2(\Omega)) \), by Hölder’s inequality we have

\[
\left| \int_{Q_T} \chi_{E_j} \sqrt{D_{k,N_k}(u_{k,N_k})} \nabla v_{k,N_k} \varphi - \chi_{E_j} \sqrt{D(u)} \zeta \varphi \, dx \, dt \right|
\leq \left| \int_{Q_T} \chi_{E_j} \sqrt{D_{k,N_k}(u_{k,N_k}) - \sqrt{D(u)}} \nabla v_{k,N_k} \varphi \, dx \, dt \right|
\leq \sup_{t \in [0,T]} \left\| \sqrt{D_{k,N_k}(u_{k,N_k}) - \sqrt{D(u)}} \right\|_n \int_0^T \| \chi_{E_j} \nabla v_{k,N_k} \|_2 \| \varphi \|_{L^2} \, dt.
\]
Thus by taking the limits of third term of (3.26), we have

\[ I + II. \]

By (3.16) and (3.34), we have \( I \rightarrow 0 \) as \( N \rightarrow \infty \). By (3.17) and (3.34), we have \( II \rightarrow 0 \) as \( N \rightarrow \infty \).

By (3.23) and (3.37), we find that

\[ \text{weak sense} \]

for all \( j \).

From \( L^2 < L^{\frac{2n}{n+1}} \) and (3.13), we see that \( \xi = \sqrt{D(u)} \zeta \) in every \( E_j \) and

\[ \xi = \sqrt{D(u)} \zeta \quad \text{in} \quad E. \tag{3.35} \]

Consequently, by (3.18),

\[ \chi_{E_j} \sqrt{D_{k,Nk}(u_{k,Nk})} \nabla v_{k,Nk} \rightarrow \chi_{E_j} \sqrt{D(u)} \zeta, \quad \text{weakly in} \quad L^2(0,T; L^{\frac{2n}{n+1}}(\Omega)), \]

for all \( j \).

From (3.14) and (3.34), we notice that

\[ \text{weakly in} \quad L^2(0,T; L^{\frac{2n}{n+1}}(\Omega)). \]

Thus by taking the limits of third term of (3.26), we have

\[ \lim_{j \rightarrow \infty} \lim_{k \rightarrow \infty} \int_{E_j} D_{k,Nk}(u_{k,Nk}) \nabla v_{k,Nk} \psi \, dx \, dt \]

\[ = \lim_{j \rightarrow \infty} \int_{E_j} D(u) \zeta \psi \, dx \, dt = \int_{E} D(u) \zeta \psi \, dx \, dt. \tag{3.36} \]

By (3.27), (3.29), (3.32) and (3.36), we have

\[ \int_{Q_T} \sqrt{D(u)} \zeta \psi \, dx \, dt = \int_{E} D(u) \zeta \psi \, dx \, dt. \tag{3.37} \]

By (3.23) and (3.37), we find that \( u \) and \( \zeta \) solve equation (1.9) in the following weak sense

\[ \int_0^T \langle \partial_t u, \phi \rangle \, dt = - \int_{E} D(u) \zeta \nabla \phi \, dx \, dt - \int_0^T \int_{\Omega} [-\Delta u + f(u)] \phi \, dx \, dt, \tag{3.38} \]

for all \( \phi \in L^2(0,T; H^2(\Omega) \cap H^1_0(\Omega)) \).

From (3.14) and (3.34), we notice that \( v_i \) is bounded in \( L^2(T_j; H^1(\Omega)) \), where \( E_j = T_j \times S_j \). So we can extract a further sequence, not relabeled, and

\[ v \in L^2(T_j; H^1(S_j)), \quad v_i \rightarrow v \quad \text{weakly in} \quad L^2(T_j; H^1(S_j)). \tag{3.39} \]

Similar to show \( F'(u_i) \in L^2(0,T; H^1(\Omega)) \) and (3.22). Hence, we have \( F'(u) \in L^2(0,T; H^1(\Omega)) \) and \( v = -\Delta u + f(u) \), a.e. in \( E_j \). By \( v \in L^2(T_j; H^1(S_j)) \) we have \( u \in L^2(T_j; H^3(S_j)) \) and

\[ \nabla v = -\nabla \Delta u + F''(u) \nabla u, \quad \text{almost everywhere in} \quad E_j. \tag{3.40} \]

Obviously we have \( \eta = v, \zeta = \nabla v \), a.e. in \( E_j \). So we obtain the desired relation between \( \zeta \) and \( u \):

\[ \zeta = -\nabla \Delta u + F''(u) \nabla u, \quad \text{in} \quad E_j. \]
Finally, we show that a weak solution $u$ to (1.9) satisfies energy inequality (3.2). By (2.4) we have
\[
\begin{align*}
\int_{\Omega} \left( \frac{1}{2} |\nabla u_{k,N}(x,t)|^2 + F(u_{k,N}(x,t)) \right) dx \\
+ \int_{Q_T \cap E} D_{k,N}(u_{k,N}(x,\tau)) |\nabla u_{k,N}(x,\tau)|^2 dxd\tau \\
+ \int_{Q_T} |v_{k,N}(x,\tau)|^2 dxd\tau \\
\leq \int_{\Omega} \left( \frac{1}{2} |\nabla u_0|^2 + F(u_0) \right) dx.
\end{align*}
\] (3.41)
By having (2.47) and (2.66), similarly we have
\[
\nabla u_{k,N} \rightharpoonup \nabla u, \quad \text{weakly in } L^2(Q_T),
\] (3.42)
\[
\lim_{N \to \infty} \int_{\Omega} F(u_{k,N}(t)) dx = \int_{\Omega} F(u(t)) dx.
\] (3.43)
By (3.42), (3.43), (3.13), (3.35), (3.22), (3.41) and the weak lower semicontinuity of the $L^p$ norms. Then
\[
\begin{align*}
\int_{\Omega} \left( \frac{1}{2} |\nabla u(x,t)|^2 + F(u(x,t)) \right) dx \\
+ \int_{Q_T} |D(u(x,\tau))|\zeta(x,\tau)|^2 dxd\tau \\
+ \int_{Q_T} |\Delta u + f(u)|^2 dxd\tau \\
\leq \liminf_{N \to \infty} \int_{\Omega} \left( \frac{1}{2} |\nabla u_{k,N}(x,t)|^2 + F(u_{k,N}(x,t)) \right) dx \\
+ \liminf_{N \to \infty} \int_{Q_T \cap E} D_{k,N}(u_{k,N}(x,\tau)) |\nabla u_{k,N}(x,\tau)|^2 dxd\tau \\
+ \liminf_{N \to \infty} \int_{Q_T} |v_{k,N}(x,\tau)|^2 dxd\tau \\
\leq \int_{\Omega} \left( \frac{1}{2} |\nabla u_0|^2 + F(u_0) \right) dx.
\end{align*}
\]
This gives the energy inequality (3.2). The proof is complete. □

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