BLOW-UP OF SOLUTIONS FOR VISCOELASTIC EQUATIONS OF KIRCHHOFF TYPE WITH ARBITRARY POSITIVE INITIAL ENERGY

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Abstract. We consider the viscoelastic equation

$$u_{tt}(x, t) - M(\|\nabla u\|^2_2)\Delta u(x, t) + \int_0^t g(t-s)\Delta u(x, s)ds + u_t = |u|^{p-1}u$$

with suitable initial data and boundary conditions. Under certain assumptions on the kernel $g$ and the initial data, we establish a new blow-up result for arbitrary positive initial energy, by using simple analysis techniques.

1. Introduction

The wave equation

$$u_{tt} - \Delta u + h(u_t) = f(u)$$

with suitable initial data and boundary conditions has been extensively studied and several results concerning existence and blow-up have been established (see [1, 2, 10, 16]). Here $h$ represents the friction or damping, and $f$ the source. To describe the nonlinear vibrations of an elastic string, the so-called Kirchhoff equation

$$u_{tt} - M(\|\nabla u\|^2_2)\Delta u + h(u_t) = f(u)$$

was introduced [8], where $M(s) = m_0 + bs^\gamma$ is a positive $C^1$-function ($m_0 > 0, b \geq 0, \gamma > 0, s \geq 0$). In this case the existence and blow-up of solutions have been discussed by many authors (see [5, 0, 14, 15, 21] and the references cited therein).

When we take the viscoelastic materials into consideration, the models (1.1) and (1.2) become

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u)$$

and

$$u_{tt} - M(\|\nabla u\|^2_2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u)$$

respectively, where $g$ represents the kernel of the memory.

For (1.3), many existence and blow-up results have been proved. See in this regard [11, 12, 17, 18, 20]. For example, Messaoudi [11] studied (1.3) with $h(u_t) = |u_t|^{m-2}u_t$ and $f(u) = |u|^{p-2}u$ and proved a blow-up result for solutions with negative initial energy if $p > m \geq 2$ and a global existence result for $2 \leq p \leq m$.

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This result has been improved by the same author in [12] to the case of positive initial energy. In [17], Song and Zhang consider (1.3) with \( h(u_t) = -\Delta u_t \) and \( f(u) = |u|^{p-2}u \) and prove a blow-up result for solutions with positive initial energy by using potential well theory introduced by Payne and Sattinger [16]. Later, Song [15] obtained the blow-up result of (1.3) in the case of \( h(u_t) = |u_t|^{m-2}u_t \).

The model (1.4) states that the dynamic equilibrium of a body depends not only on the present state of deformation, but also on the previous history of the deformation [13]. This model was first studied by Torrejón and Young [19], who proved the existence of weakly asymptotic stable solution for a large analytical datum. Later, Munoz Rivera [13] showed the global existence for small datum and the total energy decays to zero exponentially under some restrictions. In [21], Wu and Tsai studied the model (1.4) with strong damping and nonlinear damping respectively and proved the existence and blow-up of solutions. In [22], a blow-up result of the model (1.4) with \( m_0 = 1 \), \( h(u_t) = a|u_t|^{\nu-2}u + a|u_t|^{m-2}u_t \) and \( f(u) = |u|^{p-2}u \) is obtained under some assumptions on the kernel \( g \), the exponential \( p \) and the initial data. But this result holds only in the case \( 0 \leq E(0) < E_1 \), where \( E(0) \) is the initial energy of the solution and \( E_1 \) is some a positive constant. Recently, by using concavity method, Liu and Liang [9] improved the results of [15] and [22]. Wu and Tsai studied the model (1.4) with strong damping and nonlinear damping respectively and proved the existence and blow-up of solutions. In [22], a blow-up result of the model (1.4) with \( m_0 = 1 \), \( h(u_t) = a|u_t|^{\nu-2}u + a|u_t|^{m-2}u_t \) and \( f(u) = |u|^{p-2}u \) is obtained under some assumptions on the kernel \( g \), the exponential \( p \) and the initial data. But this result holds only in the case \( 0 \leq E(0) < E_1 \), where \( E(0) \) is the initial energy of the solution and \( E_1 \) is some a positive constant. Recently, by using concavity method, Liu and Liang [9] improved the results of [22] to the case of arbitrary positive initial energy. They considered the following initial-boundary value problem

\[
\begin{aligned}
    u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t &= f(u), \\
    (x, t) &\in \Omega \times (0, T), \\
    u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with a smooth boundary \( \partial \Omega \). \( u_0 \) and \( u_1 \) are given initial data. \( M \) and \( g \) are two functions which stated as in [12] and [13]. For this model, they obtained a blow-up result under some basic assumptions on \( f, g, M \) and the initial data \( u_0, u_1 \). (Readers can see [9] Conditions A1-A4, (2.3) and (2.4).) However, we find that [9] conditions (A4) and (2.4) are inessential. Moreover, it is difficult to construct a concrete model according to all the assumptions in [9], especially for (A4) and (2.4). So, motivated by [15] and [22], we try to consider the blow-up properties of the model (1.5) with \( m_0 = 1 \) and \( f(u) = |u|^{p-2}u \). That is, we study the following problem

\[
\begin{aligned}
    u_{tt} - M(\|\nabla u\|_2^2)\Delta u + \int_0^t g(t-s)\Delta u(s)ds + u_t &= |u|^{p-2}u, \\
    (x, t) &\in \Omega \times (0, T), \\
    u(x, t) &= 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
    u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \quad x \in \Omega,
\end{aligned}
\]

where \( M(s) = 1 + bs^\gamma (b \geq 0, \gamma > 0, s \geq 0) \) is a positive \( C^1 \) -function. We hope to get some more concise sufficient conditions.
2. Preliminaries and statement of main result

In this article, \( C \) denotes a generic positive constant. It may be different from line to line. And we use the standard Lebesgue space \( L^p(\Omega) \) with their usual norms \( \| \cdot \|_p \). Moreover, we denote by \( (\cdot, \cdot) \) the usual \( L^2(\Omega) \) inner product.

We first state the general assumptions on \( g \) and \( p \) as follows:

\((A1)\) \( g \in C^1([0, \infty)) \) is a non-negative and non-increasing function satisfying
\[
0 < k := \int_0^\infty g(s) ds < 1. \tag{2.1}
\]

\((A2)\) If the space dimension \( n = 1, 2 \), then \( 2(\gamma + 1) < p < \infty \); If \( n \geq 3 \), then
\[
2(\gamma + 1) < p \leq \frac{2(n - 1)}{n - 2}. \tag{2.2}
\]

To simplify the notation, we set
\[
(\phi \circ \psi)(t) := \int_0^t \phi(t-s) \int_{\Omega} |\psi(t) - \psi(s)|^2 dx ds,
\]
where \( \psi \) may be a scalar, or a vector valued function. A direct computation shows that, for any \( g \in C^1(\mathbb{R}) \) and \( u \in H^2(0, T; L^2(\Omega)) \), the following identity holds:
\[
\int_0^t g(t-s)(\nabla u(s), \nabla u_t(t)) ds
= \frac{1}{2}(g'(\circ \nabla u)(t) - \frac{1}{2}g(t)[\nabla u(t)]^2
- \frac{1}{2} \frac{d}{dt} \{ (g \circ \nabla u)(t) - \left( \int_0^t g(s) ds \right) [\nabla u(t)]^2 \}. \tag{2.3}
\]

Now, we state a local existence theorem that can be established by adopting the arguments of [22].

**Theorem 2.1** (Local solution). Assume that \((A1)\) and \((A2)\) hold. Let \( u_0 \in H_0^2(\Omega) \) and \( u_1 \in H_0^1(\Omega) \) be given. Then, there exists a unique weak solution \( u(t) \) of (1.5) such that
\[
u \in C([0, T]; H_0^2(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \quad u_t \in L^2([0, T]; H_0^1(\Omega)). \tag{2.4}
\]

for a small enough \( T > 0 \).

The energy functional of the solution \( u \) of (1.5) is defined as
\[
E(t) := \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\left(1 - \int_0^t g(s) ds \right)\|\nabla u\|^2 + \frac{b}{2(\gamma + 1)}\|\nabla u\|^{2(\gamma + 1)}
+ \frac{1}{2}(g \circ \nabla u)(t) - \frac{1}{p}\|u\|^p. \tag{2.5}
\]

By [22] and assumption (A1), direct computations yield
\[
E'(t) = \frac{1}{2}(g' \circ \nabla u)(t) - \frac{1}{2}g(t)[\nabla u\|^2 - \|u_t\|_2^2 \leq -\|u_t\|^2 \leq 0. \tag{2.6}
\]

According to [22], we can obtain the following blow-up with negative initial energy:
**Theorem 2.2.** Assume that (A1), (A2) and $k < \frac{2(p-2)}{2p-3}$ hold. If $E(0) < 0$, then for all the initial data $u_0 \in H^2_0(\Omega)$ and $u_1 \in H^1_0(\Omega)$, the corresponding solution $u(x,t)$ of the problem (1.5) blows up in finite time.

Our main result is a blow-up with positive initial energy that reads as follows.

**Theorem 2.3.** Assume that (A1), (A2) and $k < \frac{p(p-2)}{(p-1)^2}$ hold. Moreover, $E(0) > 0$ (maybe large enough) is a given initial energy state. If we choose initial data $u_0 \in H^2_0(\Omega)$ and $u_1 \in H^1_0(\Omega)$ satisfying

\[ \int_\Omega u_0 u_1 dx > \beta E(0), \tag{2.6} \]

where $\beta = \frac{1}{2\epsilon_0}, \epsilon_0 \in (0,1)$ is a positive constant, then the corresponding solution $u(x,t)$ of the problem (1.5) blows up in finite time.

In [9], the kernel $g$ must be the so-called positive type function. But, we do not need that assumption. Moreover, our kernel function space is bigger than the one in [22] since $\frac{p(p-2)}{(p-1)^2} > \frac{2(p-2)}{2p-3}$.

### 3. Proof of Main Result

Assume $u$ is a global solution of problem (1.6). Let

\[ Q(t) = \int_\Omega uu_t dx. \]

Multiplying the first equation of (1.6) by $u$ and integrating over $\Omega$, we get

\[ \int_\Omega uu_t dx + M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 - \int_\Omega \left( \int_0^t g(t-s)\Delta u(s)ds \right) u dx + \int_\Omega uu_t dx = \|u\|_p^p. \]

Then, we easily obtain

\[ Q'(t) = \|u_t\|_2^2 - M(\|\nabla u\|_2^2)\|\nabla u\|_2^2 + \|u\|_p^p \]

\[ - \int_\Omega \left( \int_0^t g(t-s)\Delta u(s)ds \right) u dx - \int_\Omega uu_t dx. \tag{3.1} \]

For the last term on the right side of (3.1), using Cauchy inequality, we deduce that

\[ - \int_\Omega \left( \int_0^t g(t-s)\Delta u(s)ds \right) u dx \]

\[ = \int_0^t g(t-s) \int_\Omega \nabla u(s) \nabla u(t) dxds \]

\[ = \int_0^t g(t-s) \int_\Omega \nabla u(t)(\nabla u(s) - \nabla u(t)) dxds + \int_0^t g(s)ds\|\nabla u\|_2^2 \]

\[ \geq - \frac{p(1-\epsilon)}{2} \left(g \circ \nabla u\right)(t) + \left(1 - \frac{1}{2p(1-\epsilon)}\right) \int_0^t g(s)ds\|\nabla u\|_2^2 \tag{3.2} \]
Now, by assumption (A2), we select 
\[ f \] since 
\[ k < \varepsilon \]
for all \( \varepsilon \in (0, 1) \). By (3.2) and (2.4), we have

\[
Q'(t) \geq \|u_t\|_2^2 - (1 - \int_0^t g(s)ds)\|\nabla u\|_2^2 - b\|\nabla u\|_2^{2(\gamma + 1)} + \|u\|_p^p - \int_\Omega uu_t dx \\
\quad - \frac{p(1 - \varepsilon)}{2} (g \circ \nabla u)(t) - \frac{1}{2p(1 - \varepsilon)} \int_0^t g(s)ds \|\nabla u\|_2^2 \\
= \left( \frac{p(1 - \varepsilon)}{2} + 1 \right) \|u_t\|_2^2 + \left( \frac{p(1 - \varepsilon)}{2} - 1 \right) \left( 1 - \int_0^t g(s)ds \right) \|\nabla u\|_2^2 \quad (3.3) \\
\quad - \frac{1}{2p(1 - \varepsilon)} \int_0^t g(s)ds \|\nabla u\|_2^2 - p(1 - \varepsilon)E(t) + \varepsilon \|u\|_p^p - \int_\Omega uu_t dx \\
+ \left( \frac{bp(1 - \varepsilon)}{2(\gamma + 1)} - b \right) \|\nabla u\|_2^{2(\gamma + 1)}.
\]

Moreover, we note that \( f \) is the first eigenvalue of \( -\Delta \) and 
\[ Q(t) = \left( Q(t) - \frac{E(t)}{2\varepsilon} \right) ' \geq Q'(t) + \frac{1}{2\varepsilon} \|u_t\|_2^2 \]

\[
\quad \geq \left( \frac{p(1 - \varepsilon)}{2} + 1 \right) \|u_t\|_2^2 - p(1 - \varepsilon)E(t) - \frac{\varepsilon}{2} \|u\|_2^2 \\
+ \left( \left( \frac{p(1 - \varepsilon)}{2} - 1 \right) (1 - k) - \frac{k}{2p(1 - \varepsilon)} \right) \|\nabla u\|_2^2 \quad (3.4) \\
\quad \geq \left( \frac{p(1 - \varepsilon)}{2} + 1 \right) \|u_t\|_2^2 - p(1 - \varepsilon)E(t) \\
+ (f(\varepsilon)\lambda_1 - \frac{\varepsilon}{2}) \|u\|_2^2,
\]

where \( \lambda_1 \) is the first eigenvalue of \( -\Delta \) and

\[
f(\varepsilon) = \left( \frac{p(1 - \varepsilon)}{2} - 1 \right) (1 - k) - \frac{k}{2p(1 - \varepsilon)}. \quad (3.5)
\]

Since \( k < \frac{p(p-2)}{(p-1)^2} \) and \( p > 2 \), we deduce that \( 1 - k > \frac{1}{(p-1)^2} \) and

\[
\theta := (p - 2)(1 - k) - \frac{k}{p} > 0.
\]

Moreover, we note that \( f(\varepsilon) \rightarrow \frac{\theta}{2} \) as \( \varepsilon \rightarrow 0^+ \). So, we can select \( \varepsilon \) small enough such that \( f(\varepsilon)\lambda_1 - \frac{\varepsilon}{2} > 0 \). Then, using Cauchy inequality to (3.4), we have

\[
\left( Q(t) - \frac{E(t)}{2\varepsilon} \right) ' \geq h(\varepsilon)Q(t) - p(1 - \varepsilon)E(t) \\
= h(\varepsilon) \left( Q(t) - \frac{p(1 - \varepsilon)}{h(\varepsilon)} E(t) \right), \quad (3.6)
\]
where
\[ h(\varepsilon) = 2 \sqrt{\left( \frac{p(1 - \varepsilon)}{2} + 1 \right) \left( f(\varepsilon) \lambda_1 - \frac{\varepsilon}{2} \right)}. \]

Denote
\[ \varphi(\varepsilon) = \left( \frac{p(1 - \varepsilon)}{2} + 1 \right) \left( f(\varepsilon) \lambda_1 - \frac{\varepsilon}{2} \right). \]

It is easy to see that
\[ f(\varepsilon) \lambda_1 - \frac{\varepsilon}{2} \to 0^+ \]
\[ f(\varepsilon) \lambda_1 - \frac{\varepsilon}{2} \to -\infty, \quad \varphi(\varepsilon) \to -\infty \quad \text{as} \quad \varepsilon \to 1^- .\]

Hence, by the continuity of \( \varphi(\varepsilon) \), there exists \( \tilde{\varepsilon} \in (0, 1) \) such that \( \varphi(\tilde{\varepsilon}) = 0 \) and \( \varphi(\varepsilon) > 0 \) for all \( \varepsilon \in (0, \tilde{\varepsilon}) \). So, we have \( h(\tilde{\varepsilon}) = 2 \sqrt{\varphi(\tilde{\varepsilon})} = 0 \) and \( h(\varepsilon) = 2 \sqrt{\varphi(\varepsilon)} > 0 \) for all \( \varepsilon \in (0, \tilde{\varepsilon}) \). And then, we easily deduce that
\[ \frac{p(1 - \varepsilon)}{h(\varepsilon)} \to \frac{p}{\sqrt{\theta \lambda_1 (p + 2)}}, \quad \frac{1}{2\varepsilon} \to +\infty, \quad \text{as} \quad \varepsilon \to 0^+ , \]
\[ \frac{p(1 - \varepsilon)}{h(\varepsilon)} \to +\infty, \quad \frac{1}{2\varepsilon} \to \frac{1}{2\tilde{\varepsilon}}, \quad \text{as} \quad \varepsilon \to \tilde{\varepsilon}^- . \]

Thus, using the continuity in \( \varepsilon \) of \( \frac{p(1 - \varepsilon)}{h(\varepsilon)} \) and \( \frac{1}{2\varepsilon} \), there exists \( \varepsilon_0 \in (0, \tilde{\varepsilon}) \subset (0, 1) \) such that
\[ \frac{1}{2\varepsilon_0} = \frac{p(1 - \varepsilon_0)}{h(\varepsilon_0)}. \]

Now, let
\[ \beta = \frac{1}{2\varepsilon_0} \quad \text{and} \quad H(t) = Q(t) - \beta E(t). \quad (3.7) \]

By using (2.6), (2.5) and (3.6), we deduce that
\[ H(0) = Q(0) - \beta E(0) > 0, \]
\[ H'(t) \geq Q'(t) \geq h(\varepsilon_0) H(t). \]

Then, we have
\[ H(t) \geq e^{h(\varepsilon_0)t} H(0). \]

Since \( u \) is global, by (2.5) and Theorem 2.2, the energy \( E(t) \) remains nonnegative, i.e., \( 0 \leq E(t) \leq E(0) \) for all \( t \in [0, +\infty) \). So, we deduce that \( Q(t) \geq e^{h(\varepsilon_0)t} H(0) \) and
\[ \|u(t)\|^2 = \|u(0)\|^2 + 2 \int_0^t Q(s)ds \]
\[ \geq \|u(0)\|^2 + 2 \int_0^t e^{h(\varepsilon_0)s} H(0)ds \]
\[ = \|u(0)\|^2 + \frac{2H(0)}{h(\varepsilon_0)} \left( e^{h(\varepsilon_0)t} - 1 \right). \quad (3.8) \]
By (2.5), Theorem 2.2, and Hölder inequality, we obtain
\[
\|u(t)\|_2 \leq \|u(0)\|_2 + \int_0^t \|u(s)\|_2 ds \\
\leq \|u(0)\|_2 + t^{1/2} \left( \int_0^t \|u(s)\|_2^2 ds \right)^{1/2} \\
\leq \|u(0)\|_2 + t^{1/2} (E(0) - E(t))^{1/2} \\
\leq \|u(0)\|_2 + t^{1/2} (E(0))^{1/2}
\]
which contradicts (3.8). \(\square\)

As a simple example, we consider a one-dimension model with \(M(s) = 1 + s\), \(\Omega = [0, 2\pi]\) and \(p = 5\). Let
\[
u_0 = \xi \sin(\eta x), \quad u = \xi \eta^2 \sin(\eta x),
\]
where \(\xi > 0\) and \(\eta\) is a positive integer. Then, we have \(Q(0) = (\nu_0, u_1) = \xi^2 \eta^2 \pi\) and
\[
E(0) = \frac{1}{2} \|u_1\|_2^2 + \frac{1}{2} \|\nabla u_0\|_2^2 + \frac{1}{4} \|\nabla u_0\|^2_2 - \frac{1}{5} \|u_0\|^5_5 \\
= \int_0^{2\pi} |\xi \eta^2 \sin(\eta x)|^2 dx - \frac{1}{5} \int_0^{2\pi} |\xi \sin(\eta x)|^5 dx \\
= \xi^2 \eta^4 \pi - \frac{32}{75} \xi^5.
\]
Now, we choose \(\eta > \sqrt{1/(2\beta)}\) and \(\xi = \sqrt[2]{\frac{75}{32}} \eta^2 \pi (\eta^2 - \frac{1}{2\beta})\). Then, we can deduce that
\(Q(0) = 2\beta E(0) > \beta E(0)\).
According Theorem 2.3, the corresponding solution blows up in finite time.

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