REPRODUCTIVE SOLUTIONS FOR THE G-NAVIER-STOKES AND G-KELVIN-VOIGHT EQUATIONS

LUIS FRIZ, MARKO ANTONIO ROJAS-MEDAR, MARÍA DRINA ROJAS-MEDAR

Abstract. This article presents the existence of reproductive solutions of g-Navier-Stokes and g-Kelvin-Voight equations. In this way, for weak solutions, we reach basically the same result as for classic Navier-Stokes equations.

1. Introduction

On one hand, in this work we consider the g-Navier-Stokes equation
\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \text{in } ]0, T[ \times \Omega,
\]
\[
\frac{1}{g}(\nabla (gu)) = \nabla g \cdot u + \nabla \cdot u = 0, \quad \text{in } ]0, T[ \times \Omega,
\]
defined on a domain \( \Omega \subseteq \mathbb{R}^2 \). This system is derived in [10] from the 3-D Navier-Stokes equations
\[
\frac{\partial U}{\partial t} - \nu \Delta U + (U \cdot \nabla)U + \nabla \Phi = f, \quad \text{in } ]0, T[ \times \Omega_g,
\]
\[\nabla \cdot U = 0, \quad \text{in } ]0, T[ \times \Omega_g,
\]
where \( \Omega_g = \{(y_1,y_2,y_3) : (y_1,y_2) \in \Omega, \ 0 \leq y_3 \leq g(y_1,y_2)\} \), with the boundary conditions
\[U \cdot n = 0 \quad \text{on } \partial_{\text{top}} \Omega_g \cap \partial_{\text{bottom}} \Omega_g
\]
being,
\[\partial_{\text{top}} \Omega_g = \{(y_1,y_2,y_3) \in \Omega_g : y_3 = g(y_1,y_2)\},
\]
\[\partial_{\text{bottom}} \Omega_g = \{(y_1,y_2,y_3) \in \Omega_g : y_3 = 0\}.
\]

More precisely, the authors assume that
\[U(y_1,y_2,y_3) = (U_1(y_1,y_2), U_2(y_1,y_2), U_3(y_1,y_2,y_3)),
\]
and they define the following new variables and unknowns
\[y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3 g(x_1,x_2),
\]
\[U_1(y_1,y_2) = u_1(x_1,x_2), \quad U_2(y_1,y_2) = u_2(x_1,x_2), \quad U_3(y_1,y_2,y_3) = u_3(x_1,x_2,x_3)
\]

2010 Mathematics Subject Classification. 35Q35, 76D03.
Key words and phrases. Reproductive solution; g-Navier-Stokes system.
©2016 Texas State University.
Finally, they prove that \( u = (u_1, u_2) \) is solution of the two equation of (1.1) and \( u_3 = x_3 \nabla g \cdot u \). The interested reader can also review [2], [8] and [9]. Although the g-Navier-Stokes system is defined in two dimension domain, we will also study the tridimensional case.

In this article, at first we seek a reproductive solution (or weak periodic solution) of (1.1), i.e. solutions satisfying
\[
u(0, x) = u(T, x), \quad x \in \Omega,
\]
instead of an initial condition. In the case of the Navier-Stokes equation, the study of the reproductive solutions was initiated by Kaniel and Shinbrot in [4], the reader can also see the classical textbook [6] by Lions. In [3] the authors review some results concerning the existence, uniqueness and regularity of reproductive and time periodic solutions of the Navier-Stokes equations and some variants defined in bounded domains. In order to obtain a reproductive solution, they introduce a Galerkin discretization of the problem, proving existence of approximate solution to certain initial conditions. Then, a Leray-Schauder argument, by means of fixed point process, permits to obtain a reproductive Galerkin solution, which converges towards a continuous reproductive solution.

To be more precise, in this work the first purpose is to solve the system
\[
\frac{\partial u}{\partial t} - \nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \quad \text{in } [0, T] \times \Omega,
\]
\[
\frac{1}{g}(\nabla \cdot (gw)) = \nabla g \cdot u + \nabla \cdot u = 0, \quad \text{in } [0, T] \times \Omega,
\]
\[
u(0, x) = u(T, x), \quad x \in \Omega,
\]
\[
u(t, x) = \beta(t, x), \quad \text{on } [0, T] \times \partial \Omega.
\]
(1.3)

Here \( \beta \in C^1(\mathbb{R}, H^{1/2}(\partial \Omega)^n) \) is a \( T \)-periodic function and satisfies the (g-SOC) condition
\[
\int_{\partial \Omega} g \beta \cdot n ds = 0.
\]
(1.4)

This definition is inspired by that given in [7] when \( g \equiv 1 \), the so-called (SOC) condition,
\[
\int_{\partial \Omega} \beta \cdot n ds = 0.
\]
(1.5)

Moreover, in a similar manner to the Navier-Stokes system, we can prove uniqueness of the solution in the bidimensional case.

On the other hand, in this paper we also consider the g-Kelvin-Voight equation
\[
\frac{\partial u}{\partial t} - \frac{\nu}{g}(\nabla \cdot (g \nabla)u) + \frac{\nu}{g}(\nabla g \cdot \nabla)u + \frac{\alpha}{g}(\nabla g \cdot \nabla)u_t + \frac{\alpha}{g}(\nabla g \cdot \nabla)u_t + u \cdot \nabla u + \nabla p = f, \quad \text{in } [0, T] \times \Omega
\]
\[
\frac{1}{g}(\nabla \cdot (gw)) = \nabla g \cdot u + \nabla \cdot u = 0, \quad \text{in } [0, T] \times \Omega
\]
(1.6)

The derivation of this system is analogous to the g-Navier-Stokes. In fact, it is deduced from the Kelvin-Voight system
\[
\frac{\partial U}{\partial t} - \nu \Delta U - \alpha \Delta U_t + (U \cdot \nabla)U + \nabla P = F, \quad \text{in } [0, T] \times \Omega_g,
\]
\[
\nabla \cdot U = 0, \quad \text{in } [0, T] \times \Omega_g.
\]
where $\Omega_g = \{(y_1, y_2, y_3) : (y_1, y_2) \in \Omega, 0 \leq y_3 \leq g(y_1, y_2)\}$. We refer interested readers to the article [5] and the reference given there.

The second purpose of this article is to solve the system

$$
\begin{align*}
\frac{\partial u}{\partial t} - & \nu \frac{g}{g} (\nabla \cdot g \nabla) u + \frac{\nu}{g} (\nabla g \cdot \nabla) u - \frac{\alpha}{g} (\nabla \cdot g \nabla) u_t \\
+ & \frac{\alpha}{g} (\nabla g \cdot \nabla) u_t + u \cdot \nabla u + \nabla p = f, \quad \text{in } |0, T| \times \Omega \\
\frac{1}{g} (\nabla \cdot (g u)) = & \nabla g \cdot u + (\nabla u) = 0, \quad \text{in } |0, T| \times \Omega \\
u(0, x) = & u(T, x), \quad \text{in } \Omega \\
u(t, x) = & 0, \quad \text{in } |0, T| \times \partial \Omega \\
\end{align*}
$$

(1.7)

in other words, we seek a reproductive solution for the $g$-Kelvin-Voight equation.

This article is organized as follows. In section 2 the basic definitions and results are introduced. Section 3 is devoted to proving the existence of the reproductive solution of the $g$-Navier-Stokes system, both for the case $\beta = 0$ and the case $\beta \neq 0$. Finally, in section 4 the existence of the reproductive solution of the $g$-Kelvin-Voight system is proved.

2. Preliminaries

In this section, we introduce notation and spaces to be used later. Let $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$ be a bounded domain with smooth boundary $\partial \Omega$. We assume that $g \in W^{1,\infty}(\Omega)$ satisfies

$$
0 < m_0 \leq g(x) \leq M_0, \quad \forall x \in \Omega, \quad \|\nabla g\|_\infty < \frac{m_0 \lambda_1^{1/2}}{2} \quad (2.1)
$$

where $\lambda_1 > 0$ is the first eigenvalue of the $g$-Stokes operator in $\Omega$ (see [5]), i.e. the spectral problem

$$
\begin{align*}
-\frac{1}{g} (\nabla \cdot g \nabla) w^j + \nabla p^j = & \lambda_j w^j, \quad \text{in } \Omega, \\
\nabla \cdot g w^j = & 0 \quad \text{in } \Omega, \\
w^j = & 0 \quad \text{on } \partial \Omega.
\end{align*}
$$

(2.2)

Problem (2.2) has eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots$ and corresponding eigenfunctions $w^1, w^2, \ldots, w^j, \ldots$ form an orthonormal basis in $\mathbf{H}_g$ and total basis in $\mathbf{V}_g$, where $\mathbf{H}_g$ and $\mathbf{V}_g$ are defined in the following manner:

$$
\begin{align*}
\mathcal{V} &= \{u \in \mathcal{D}(\Omega) : \nabla \cdot (g u) = 0\}, \\
\mathbf{H}_g &= \text{the closure of } \mathcal{V} \text{ in } L^2(\Omega), \\
\mathbf{V}_g &= \text{the closure of } \mathcal{V} \text{ in } H^1_0(\Omega).
\end{align*}
$$

Where $\mathbf{H}_g$ is endowed with the scalar product

$$
(u, v)_g = \int_\Omega (u \cdot v) g dx \quad \text{and} \quad |u|^2 = (u, u)_g.
$$

Notice that this inner product is equivalent to the usual inner product defined in $L^2(\Omega)$. Similarly, we define in $\mathbf{V}_g$ the equivalent inner product:

$$
((u, v))_g = \int_\Omega g \nabla u \cdot \nabla v dx.
$$
Let us recall that \( \beta \) satisfies condition (1.5) if
\[
\int_{\partial \Omega} \beta \cdot n ds = 0.
\]
In this case, Morimoto [7, p. 636] proved the next Lemma.

**Lemma 2.1.** Suppose \( \beta \in C^1(\mathbb{R}, H^{1/2}(\partial \Omega)) \) is \( T \)-periodic and satisfies (SOC). Then for every \( \varepsilon > 0 \), there exists a solenoidal and \( T \)-periodic function \( b \in C^1(\mathbb{R}; H^1(\Omega)) \) such that
\[
\nabla_x \cdot b(t, x) = 0 \quad \text{a.e} \quad x \in \Omega, \quad \forall t \in \mathbb{R},
\]
\[
b(t, x) = \beta(t, x), \quad x \in \partial \Omega, \quad \forall t \in \mathbb{R},
\]
\[
|((u \cdot \nabla)b, u)| \leq \varepsilon \|\nabla u\|^2, \quad \forall u \in V, \forall t \in \mathbb{R}.
\]

Now, if \( \beta \in C^1(\mathbb{R}, H^{1/2}(\partial \Omega)) \) is \( T \)-periodic and satisfies the (1.4) condition:
\[
\int_{\partial \Omega} g \beta \cdot n ds = 0,
\]
we have the following proposition.

**Proposition 2.2.** Suppose \( \beta \in C^1(\mathbb{R}, H^{1/2}(\partial \Omega)) \) is \( T \)-periodic and satisfies (1.4). Then for every \( \varepsilon > 0 \) there exists a \( T \)-periodic function \( \Psi \in C^1(\mathbb{R}; H^1(\Omega)) \) such that:
\[
\nabla_x \cdot (g(x)\Psi(t, x)) = 0 \quad \text{a.e} \quad x \in \Omega, \quad \forall t \in \mathbb{R},
\]
\[
\Psi(t, x) = \beta(t, x), \quad \text{a.e.} \quad x \in \partial \Omega, \quad \forall t \in \mathbb{R},
\]
\[
|((v \cdot \nabla)\Psi, v)_g| \leq C(\Omega, g)(\varepsilon + \|\nabla g\|_{L^\infty} |\nabla \Psi|)|\nabla v|^2, \quad \forall t \in \mathbb{R},
\]
for all \( v \in V_g \).

**Proof.** For \( \varepsilon > 0 \), define \( \Psi(t, x) = \frac{b(t, x)}{g(x)} \), where \( b(t, x) \in C^1(\mathbb{R}; H^1(\Omega)) \) is given by Lemma 2.1. It is clear that \( \Psi \in V_g \) is \( T \)-periodic and \( \Psi = \beta \) on \([0, T] \times \partial \Omega \). We have
\[
((v \cdot \nabla)\Psi, v)_g = \sum_{i,j=1}^{3} \int_{\Omega} v_i \frac{\partial \Psi_j}{\partial x_i} v_j g dx
\]
\[
= \int_{\Omega} \sum_{i,j=1}^{3} \left( \frac{1}{g^2} \right) g v_i \frac{\partial (g \Psi_j)}{\partial x_i} g v_j dx - \int_{\Omega} \sum_{i,j=1}^{3} v_i \frac{\partial g}{\partial x_i} \Psi_j v_j dx
\]
Now, from Lemma 2.1
\[
\left| \int_{\Omega} \sum_{i,j=1}^{3} \left( \frac{1}{g^2} \right) g v_i \frac{\partial (g \Psi_j)}{\partial x_i} g v_j dx \right| \leq \frac{1}{m_0} \|(g \nabla \cdot (g \Psi), g \nabla v)\|
\]
\[
\leq \frac{\varepsilon}{m_0} |\nabla (g \nabla v)|^2
\]
\[
\leq \varepsilon C(\Omega, g) |\nabla v|^2
\]
moreover,
\[
\left| \int_{\Omega} \sum_{i,j=1}^{3} v_i \frac{\partial g}{\partial x_i} \Psi_j v_j \right| \leq |\nabla g|_{L^\infty} |v|_{L^1} |\Psi|_{L^6} |v|
\]
\[
\leq C(\Omega, g) |\nabla g|_{L^\infty} |\nabla \Psi| |\nabla v|^2
\]
Therefore, 
\[ |(v \cdot \nabla)\Psi, v| \leq C(\Omega, g)(\varepsilon + \|\nabla g\|_{L^\infty} |\nabla \Psi|)|\nabla v|^2. \]

\[ \square \]

**Remark 2.3.** Similarly to the case of the Navier-Stokes equation, we can define the trilinear form \( b_g : V_g \times V_g \times V_g \to \mathbb{R} \) by
\[ b_g(u, v, w) = \sum_{i,j=1}^{n} \int_{\Omega} u_i \frac{\partial v_j}{\partial x_i} w_j g dx \]
for every \( u, v, w \in V_g \). It is not difficult (see [12]) to prove that
\[ b_g(u, v, v) = 0, \]
for each \( u, v \in V_g \), moreover (see [5]), if we further assume that \( \Delta g = 0 \) we have
\[ b_g(\nabla g g, v, v) = 0, \]
for all \( v \in V_g \).

Define the g-Laplacian operator as
\[ -\Delta_g u = -\frac{1}{g}(\nabla \cdot g \nabla)u = -\Delta u - \frac{1}{g} \nabla g \cdot \nabla u. \]

Now, we can rewrite the first equation of (1.3) as follows:
\[ \frac{\partial u}{\partial t} - \nu \Delta_g u + \nu \nabla g g \cdot \nabla u + (u \cdot \nabla)u + \nabla p = f. \]

**3. Existence of reproductive and periodic solutions for the g-Navier-Stokes system**

The variational formulation of (1.3) is the following: given \( f \in L^2(0, T; V_g') \) and \( u_0 \in V_g \) to find \( u - \Psi \in L^{\infty}(0, T; H_g) \cap L^2(0, T; V_g) \) such that
\[ \frac{d}{dt}(u - \Psi, v) + \nu((u - \Psi, v)) g + b_g(u - \Psi, u - \Psi, v) \]
\[ + b_g(\nabla g g, u - \Psi, v) + b_g(u - \Psi, \Psi, v) + \nu b_g(\nabla g g, u - \Psi, v) \]
\[ = \langle f, v \rangle - L(\Psi, v) \]
\[ u(0) = u_0 + \Psi(0) \]
for all \( v \in V_g \). Here \( \Psi \) is given in Proposition 2.2, \( b_g \) is the trilinear form given in Remark 2.3 and
\[ L(\Psi, v) = \left( \frac{d\Psi}{dt}, v \right) + \nu((\Psi, v)) g + b_g(\Psi, \Psi, v) + \nu b_g(\nabla g g, \Psi, v). \]

**Definition 3.1.** Let \( u_0 \in H_g \) and \( f \in L^2(0, T; V_g') \). A function \( u \in L^\infty(0, T; H_g) \cap L^2(0, T; V_g) \) is a weak solution of the problem (1.1) with initial data \( u(0) = u_0 \) and boundary data \( u = \beta \) on \( [0, T] \times \partial \Omega \), if \( u \) verifies (3.1) for all \( v \in V_g \).

In the case \( \beta \equiv 0 \), we have the following theorem.

**Theorem 3.2** ([2 thm 6.1]). Assume \( f \in L^2(0, T; V_g') \) and \( u_0 \in H_g \). Then there exists at least a weak solution of the problem (1.1), in the sense of the Definition 3.1. Moreover, \( u \) is weakly continuous from \([0, T]\) into \( H_g \).
Proposition 3.3. If $\Omega \subseteq \mathbb{R}^2$, under the assumptions of Theorem 3.2, the weak solution of (1.1) with initial data $u(0) = u_0$ is unique.

Proof. Let $u_1$ and $u_2$ be two solutions of the problem (3.1) with initial data $u_0$. If we define $w = u_1 - u_2$, then it satisfies the variational formulation

$$\frac{1}{2} \frac{d}{dt}(w, v) + \nu((w, v))_g + \nu\left((\frac{\nabla g}{g} \cdot \nabla) w, v\right)_g = -b_g(u_1, u_1, v) + b_g(u_2, u_2, v)$$

By replacing $v = w$ we get

$$\frac{d}{dt}|w|^2 + 2\nu\|w\|^2 = -2\nu\left((\frac{\nabla g}{g} \cdot \nabla) w, w\right)_g - 2b_g(u_1, u_1, w) + 2b_g(u_2, u_2, w)$$

$$= -2\nu\left((\frac{\nabla g}{g} \cdot \nabla) w, w\right)_g - 2b_g(w, u_1, w);$$

therefore, since

$$-2\nu\left((\frac{\nabla g}{g} \cdot \nabla) w, w\right)_g \leq 2\nu\|\nabla g\|_{\infty} \|w\|^2,$$

by Lemma 2.1, we also have

$$2b_g(w, u_1, w) \leq C\|u_1\|\|w\|\|w\|$$

$$\leq \epsilon\|w\|^2 + C\varepsilon\|u_1\|^2\|w\|^2.$$

Now, for $\varepsilon$ small enough we can obtain

$$\frac{d}{dt}|w|^2 \leq C\varepsilon\|u_1\|^2\|w\|^2;$$

then by using Gronwall’s inequality, we conclude that $w = 0$. \qed

Remark 3.4. After some tedious calculations, it is possible to see that Theorem 3.2 and Proposition 3.3 remain valid even if the $\beta$ is not null.

Our main result is the following.

Theorem 3.5. For any $f \in L^2(0, T; V_g')$ and $\|\nabla g\|_{\infty}$ small enough there exists a weak solution of (1.3) i.e. the weak solution $u \in L^\infty(0, T; H_g) \cap L^2(0, T; V_g')$ has the so-called reproductive property, i.e. a solution of the variational problem (3.1) which satisfies $u(0, x) = u(t, x)$.

Remark 3.6. Note that if $n = 2$ and the external force $f \in L^2(\mathbb{R}; V_g)$ is a $T$-periodic in time function, the above Theorem 3.5 furnishes a $T$-periodic weak solution for (1.3). In fact, it is a strong solution and actually very regular. This is so because we can prove that $u \in C^\infty(\Omega)$ for $t > 0$, where $u$ is solution of the problem (1.1) with initial condition $u_0 \in H_g$. Thus $u_p \in C^\infty(\Omega)$ for $t \in [T, 2T]$ and, by the $T$-periodicity, we conclude that $u_p(t) = u_p(t + T) \in C^\infty(\Omega)$, here $u_p$ is the reproductive solution. In particular, $u_p(0) \in C^\infty(\Omega)$.

3.1. Proof of Theorem 3.5 when $\beta \equiv 0$. Let $\{w^i\}_{i=1}^\infty$ be orthonormal bases in $H_g$ and total bases in $V_g$ obtained in spectral problem (2.2). As $k^{th}$-approximated solution of equation (3.1) we choose

$$u^k(t, x) = \sum_{i=1}^k c_i^k(t)w^i(x)$$ (3.2)
satisfying for all \( i = 1, \ldots, k \), and for all \( t \in (0, T) \) the system of equations

\[
\frac{d}{dt}(u^k, v)_g + \nu((u^k, v))_g + b_g(u^k, u^k, v) + \nu b_g(\nabla g / g, u^k, v) = (f, v)
\]  

\[u^k(0) = P_k u_0\]

for all \( v \in V^k = \{(w^1, w^2, \ldots, w^k)\} \). Taking \( v = u^k \), we have

\[
\frac{d}{dt}|u^k|^2 + 2\nu|\nabla u^k|^2 = (f, u^k) - 2\nu \left( \left( \frac{\nabla g}{g} \cdot \nabla \right) u^k, u^k \right)_g
\]

Therefore, by using the Poincaré inequality,

\[
|v|^2 < \frac{1}{\lambda_1}|\nabla v|^2 \quad \forall v \in H^1_0(\Omega),
\]

we have

\[
\frac{d}{dt}|u^k|^2 + 2\nu|\nabla u^k|^2 \leq \frac{1}{\nu}\|f\|_{V^*}^2 + \nu|\nabla u^k|^2 + 2\nu \frac{\|\nabla g\|_{L^\infty}}{\mu_0 \lambda_1^{1/2}} |\nabla u^k|^2.
\]

Finally, we obtain

\[
\frac{d}{dt}|u^k|^2 + \nu \lambda_0 |u^k|^2 \leq \frac{1}{\nu} \|f\|_{V^*}^2,
\]

where \( \gamma_0 = 1 - \frac{2\|\nabla g\|_{L^\infty}}{\mu_0 \lambda_1^{1/2}} > 0 \) for \( \|\nabla g\|_{L^\infty} \) small. The above inequality implies

\[
\frac{d}{dt}(e^{\nu \lambda_0 t}|u^k|^2) \leq \frac{e^{\nu \lambda_0 t}}{\nu} \|f\|_{V^*}^2.
\]

Integrating from 0 to \( T \) we have

\[
e^{\nu \lambda_0 T}|u^k(T)|^2 \leq |u^k(0)|^2 + \frac{1}{\nu} \int_0^T e^{\nu \lambda_0 t} \|f(t)\|_{V^*}^2.
\]

Next, we show that \( u^k \) is nothing but one fixed point of the operator \( \Phi^k \) defined in what follows. Let \( L^k : [0, T] \to \mathbb{R}^k \) the mapping defined by

\[
L^k(t) = y(t) = (c_1^k(t), \ldots, c_k^k(t)),
\]

where the time dependent functions \( \{c_i^k(t)\}^k_{i=1} \) are the coefficients of the expansion of \( u^k \), as done in (3.3).

Since we have chosen the basis \( \{w^i(x)\}_{i=1}^\infty \) orthonormal in \( H_g \), we have

\[
\|y(t)\|_{L^2} = |u^k(t)| \quad \forall t \in [0, T].
\]

Next, we define the operator \( \Phi^k : \mathbb{R}^k \to \mathbb{R}^k \) as

\[
\Phi^k(x) = y(T)
\]

where \( x = (x_1, x_2, \ldots, x_k) \) and \( y(T) = L^k(T) \) is the vector-coefficients at time \( T \) of the solution of (3.3) with initial condition

\[
\mathbf{u}^k_0(x) = \sum_{i=1}^m x_i w^i(x),
\]

It is not difficult to see that \( \Phi^k \) is continuous and we claim that \( \Phi^k \) has at least one fixed point. It will be a consequence of Leray-Schauder’s Homotopy Theorem. To prove this, it is enough to show that for any \( \lambda \in [0, 1] \), a solution of the equation

\[
\lambda \Phi^k(x(\lambda)) = x(\lambda)
\]  

satisfying for all \( i = 1, \ldots, k \), and for all \( t \in (0, T) \) the system of equations

\[
\frac{d}{dt}(u^k, v)_g + \nu((u^k, v))_g + b_g(u^k, u^k, v) + \nu b_g(\nabla g / g, u^k, v) = (f, v)
\]

\[u^k(0) = P_k u_0\]
has a bound independent of $\lambda$. Since $x(0) = 0$, we restrict the proof to $\lambda \in (0, 1]$. In such case (3.7) may be rewritten as

$$\Phi^k(x(\lambda)) = \frac{1}{\lambda}x(\lambda).$$

By the definition of $\Phi^k$ and (3.6), we deduce from (3.9), that

$$e^{\nu \lambda_1 \gamma_0 T} \left\| \frac{1}{\lambda}x(\lambda) \right\|_{\mathbb{R}^k}^2 \leq \left\| x(\lambda) \right\|_{\mathbb{R}^k}^2 + \int_0^T e^{\nu \lambda_1 \gamma_0 T} \left\| f(t) \right\|_V^2 \, dt$$

Since we impose $u^k(0) = u^k(T)$, we obtain

$$\left\| x(\lambda) \right\|_{\mathbb{R}^k}^2 \leq \frac{1}{e^{\nu \lambda_1 \gamma_0 T} - 1} \int_0^T e^{\nu \lambda_1 \gamma_0 T} \left\| f(t) \right\|_V^2 \, dt \equiv M(T, f), \quad (3.8)$$

for all $\lambda \in (0, 1]$. Obviously, this upper bound do not depends on $\lambda \in [0, 1]$ and so we have stated that the operator $\Phi^k$ has at least one fixed point, denoted by $x(1)$ and then there exists a reproductive Galerkin solution $u^k$, namely it satisfies $u^k(0) = u^k(T)$. Note that, from (3.8), we have that $u^k \in L^\infty(0, T; H_g)$, for every $k \in \mathbb{N}$ and it is uniformly bounded.

From (3.4) and by definition of $\gamma_0$ we can obtain the inequality

$$\frac{d}{dt} \left\| u^k \right\|^2 + \nu \gamma_0 \left\| \nabla u^k \right\|^2 \leq \frac{1}{\nu} \left\| f \right\|^2.$$

Since $u^k$ is a Galerkin reproductive solution and by integrating from 0 to $T$ we have

$$\int_0^T \left\| \nabla u^k \right\|^2 dt \leq \frac{1}{\gamma_0 \nu^2} \int_0^T \left\| f \right\|^2 dt = \tilde{M}(T, f). \quad (3.9)$$

In other words, $u^k \in L^2(0, T; V_g') \cap L^\infty(0, T; H_g)$, for each $k \in \mathbb{N}$ and it is uniformly bounded. It is not difficult to prove that $\frac{d}{dt} u^k \in L^2(0, T; V_g')$ and it is uniformly bounded. By using compactness results (see [11]) with the triplets $H_g \hookrightarrow V_g' \hookrightarrow V_g'$ and $V_g \hookrightarrow H_g \hookrightarrow V_g'$, we have that $(u^k)$ is relatively compact in $L^2(0, T; H_g) \cap C([0, T]; V_g')$. Thus, since $u^k(0) = u^k(T)$ and $u^k(0) \to u(0)$, we get that $u(0) = u(T)$ in $V_g'$, but we also have that $u \in C([0, T]; H_g)$, because $u \in L^2(0, T; H_g)$ and $\frac{d}{dt} u \in L^2(0, T; V_g')$ (see [12]), therefore $u(0) = u(T)$ in $H_g$.

3.2. Proof of Theorem 3.5: general case. Let us define $\hat{u} = u - \Psi$, where $\Psi$ is given in Proposition 2.2 which satisfies

$$\frac{\partial \hat{u}}{\partial t} - \nu \Delta \hat{u} + (\hat{u} \cdot \nabla) \hat{u} + (\hat{u} \cdot \nabla) \Psi + (\Psi \cdot \nabla) \hat{u} + \nabla p = f - \frac{\partial \Psi}{\partial t} + \nu \Delta \Psi - (\Psi \cdot \nabla) \Psi \quad \text{in } [0, T] \times \Omega,$$

$$\frac{1}{g} (\nabla (g \hat{u})) = \frac{\nabla g}{g} \cdot \hat{u} + \nabla = 0 \quad \text{in } [0, T] \times \Omega,$$

$$\hat{u}(0, x) = \hat{u}_0(x) \quad \text{in } [0, T] \times \Omega,$$

$$\hat{u}(t, x) = 0 \quad \text{on } [0, T] \times \partial \Omega.$$

Since $\Psi$ is a $T$-periodic function it is only necessary to prove that there exists a reproductive solution of the problem (3.10).
The variational formulation is as follows: Find \( \hat{u} \in L^\infty(0,T;H_g) \cap L^2(0,T;V_g) \) such that for all \( v \in V_g \) we have

\[
\frac{d}{dt}(\hat{u},v) + \nu((\hat{u},v))_g + b_g(\hat{u},\hat{v},v) + b_g(\hat{u},\hat{v},v) + \nu b_g\left(\frac{\nabla g}{g},\hat{u},v\right) = \langle f,v \rangle - L(\Psi,v),
\]

where

\[
L(\Psi,v) = \left(\frac{d\Psi}{dt},v\right)_g + \nu((\Psi,v))_g + b_g(\Psi,\Psi,v) + \nu b_g\left(\frac{\nabla g}{g},\Psi,v\right).
\]

After some calculations, we can write

\[
|L(\Psi,v)| \leq \left(\frac{1}{2\varepsilon_1}\left(\frac{d\Psi}{dt}\right)^2 + \frac{\nu\|
abla g\|_\infty^2}{m_0}|\nabla \Psi|^2 + \varepsilon_1 \right)|v|^2 + \frac{1}{2\varepsilon_1}(|\nabla \Psi|^2 + \nu|\nabla \Psi|^2)^2 + \frac{\varepsilon_1}{2}|\nabla v|^2.
\]

Let us put

\[
F = \frac{1}{2\varepsilon_1}\left(\frac{d\Psi}{dt}\right)^2 + \frac{\nu\|
abla g\|_\infty^2}{m_0}|\nabla \Psi|^2 + \frac{1}{2\varepsilon_1}(|\nabla \Psi|^2 + \nu|\nabla \Psi|^2)^2 + \frac{1}{2\varepsilon_1}\|f\|_{V_g'}.
\]

By replacing \( v \) by \( \hat{u} \) in (3.11), we obtain

\[
\frac{d}{dt}|\hat{u}|^2 + 2\nu|\nabla \hat{u}|^2 \leq \varepsilon_1|\hat{u}|^2 + F + \left(\varepsilon_1 C(\Omega,g) + \varepsilon_1 + C(\Omega,g)\|
abla g\|_\infty|\nabla \Psi| + 2\nu\|
abla g\|_\infty m_0^{1/2}\right)|\nabla \hat{u}|^2.
\]

By choosing \( \varepsilon_1 \) and \( \|
abla g\|_\infty \) small enough, we obtain

\[
\frac{d}{dt}|\hat{u}(t)|^2 + C|\hat{u}(t)|^2 \leq F(t),
\]

where \( C > 0 \), we can obtain a reproductive solution by following the same argument as in the proof of the case \( \beta \equiv 0 \).

### 4. Existence of Reproductive Solutions for the g-Kelvin-Voight System

The variational formulation of problem (1.6) is: Given \( f \in L^2(0,T;V'_g) \) and \( u_0 \in H_g \), find \( u \in V_g \) such that

\[
\frac{d}{dt}(u,v) + \nu((u,v)) + \alpha((u_t,v)) + \nu b_g\left(\frac{\nabla g}{g},u,v\right) + \alpha b_g\left(\frac{\nabla g}{g},u_t,v\right) + b_g(u,u,v) = \langle f,v \rangle
\]

\[
u(0) = u_0,
\]

for all \( v \in V_g \).
Definition 4.1. Let $u_0 \in H$ and $f \in L^2(0,T;V^\prime_g)$. A function $u \in L^\infty(0,T;H) \cap L^2(0,T;V_g)$ is a weak solution of the problem (1.6) with initial condition $u(0) = u_0$ if $u$ verifies (4.1) for all $v \in V_g$.

Theorem 4.2 ([5]). If $f \in L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$ $u_0 \in V_g$ and $g$ satisfying (2.1) and $\Delta g = 0$, then there exists a unique weak solution of (1.6).

Remark 4.3. It is possible to prove that the hypothesis that $f$ does not depend on time $t$ can be removed and replaced by $f \in L^2(0,T;V_g')$, and the theorem is still valid.

The main result of this section is the following.

Theorem 4.4. For $\|f\|_{L^2(0,T;V_g')} \leq M$ small enough there exists a weak solution of (1.7) i.e. the weak solution $u \in L^\infty(0,T;H) \cap L^2(0,T;V_g)$ has the so-called reproductive property, i.e. a solution of the variational problem (4.1) which satisfies $u(0,x) = u(T,x)$.

4.1. Proof of Theorem 4.4. In the same manner as in the proof of Theorem 3.5 we define

$$u^k(t,x) = \sum_{i=1}^{k} c^k_i(t)w^i(x) \quad (4.2)$$

as the solution of the variational problem

$$\frac{d}{dt}(u^k,v)_g + \nu((u^k,v)) + \alpha((u^k,v)) + \nu b_g(\nabla g, u^k,v) + \alpha b_g(\nabla g, u^k,v) = (f,v)$$

for all $v \in V^k = \langle \{w^1, \ldots, w^k\} \rangle$. The proof of the following lemma can be found in [5] pp 499-501. For simplicity, we denote

$$y(t) = \|u^k(t)\|_g^2 + (\alpha + \nu)\|\nabla u^k(t)\|_g^2.$$

Lemma 4.5. For $\|\nabla g\|_\infty$ small enough there exist positive constants $\beta$ and $\delta$ such that the function $y(t)$ satisfies

$$\frac{dy}{dt} + \beta y \leq \delta y^2 + C\|f(t)\|_g^2.$$

Proposition 4.6. Let $M_1 > 0$ be such that

$$\delta s < \frac{\beta}{2}, \quad \forall s \in [0,M_1].$$

Let us suppose that $\delta$ satisfies $\|f\|_{L^2(0,T;V_g')} \leq \frac{\beta}{2}M_1$. If $g(0) \leq M_1$, then $y(t) \leq M_1$, for all $t \in [0,T]$.

Proof. From Lemma 4.5 $y$ satisfies the differential inequality

$$y' + (\beta - \delta y)y \leq \|f(t)\|_g^2. \quad (4.3)$$

By hypothesis, there exists $\sigma > 0$ such that

$$\delta s \leq \frac{\beta}{2}, \quad \forall s \in [M_1, M_1 + \sigma]. \quad (4.4)$$

At first, we will prove that

$$y(t) < M_1 + \sigma, \quad \forall t \in [0,T].$$
By contradiction, let $T^* \in [0,T]$ be the first value so that $y(T^*) = M_1 + \sigma$ and $y(t) < M_1 + \sigma$, for all $t \in [0,T^*]$. By (4.4), we have that $\delta y(t) \leq \frac{\sigma}{2}$, for all $t \in [0,T^*]$. From (4.3) and the hypothesis

$$y' + \frac{\beta}{2} y \leq \frac{\beta}{2} M_1,$$

by multiplying by $e^{\frac{\beta}{2} t}$ and integrating in time in $[0,T^*]$ we obtain

$$e^{\frac{\beta}{2} T^*} y(T^*) - y(0) \leq M_1 (e^{\frac{\beta}{2} T^*} - 1),$$

$$e^{\frac{\beta}{2} T^*} y(T^*) \leq y(0) + M_1 e^{\frac{\beta}{2} T^*} - M_1,$$

$$e^{\frac{\beta}{2} T^*} y(T^*) \leq M_1 + M_1 e^{\frac{\beta}{2} T^*} - M_1 \leq M_1.$$

In other words, $y(T^*) \leq M_1$ which is a contradiction and, therefore, $y(t) \leq M_1 + \sigma$ for all $t \in [0,T]$. Furthermore, the inequality (4.5) holds for every $t \in [0,T]$, hence by repeating the same arguments in each interval $[0,t]$, for all $t \in [0,T]$, we get $y(t) \leq M_1$, which completes the proof. \hfill \square 

Now, for $(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$ and $\mathbf{u} = \xi_1 \mathbf{w}^1 + \xi_2 \mathbf{w}^2 + \ldots + \xi_m \mathbf{w}^m$, we define the norm

$$||| (\xi_1, \xi_2, \ldots, \xi_m) |||_{\mathbb{R}^m} = ||u(t)||_g^2 + (\alpha + \nu) ||\nabla u(t)||_g^2.$$

Given $(\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m$, define $\Phi^m : \mathbb{R}^m \to \mathbb{R}^m$ in as

$$\Phi^m (\xi_1, \xi_2, \ldots, \xi_m) = (c_1^m(T), c_2^m(T), \ldots, c_m^m(T)),$$

where $(c_1^m(t), c_2^m(t), \ldots, c_m^m(t))$ are the coefficients of the Galerkin solution (4.2) with initial condition $u_0 = \xi_1 \mathbf{w}^1 + \xi_2 \mathbf{w}^2 + \ldots + \xi_m \mathbf{w}^m$. If we define,

$$\mathcal{B} = \{ (\xi_1, \xi_2, \ldots, \xi_m) \in \mathbb{R}^m : ||| (\xi_1, \xi_2, \ldots, \xi_m) ||| < M_1 \}$$

where $M_1$ is given in Proposition 4.6, from Proposition 4.6 $\Phi^m$ maps $\mathcal{B}$ into $\mathcal{B}$; therefore, by the Brower Fixed-Point Theorem $\Phi^m$ has a fixed point and, consequently there exists a reproductive Galerkin solution $\mathbf{u}^m$. The Theorem follows from the standard compact arguments.

Acknowledgments. L. Friz was partially supported by Grants Fondecyt-Chile 1130456, 1251093/R UBB, 121909 GI/C-UBB and 153209 GI/C-UBB. M. A. Rojas-Medar was partially supported by project MTM2012-32325, Spain, Grant 1120260, Fondecyt-Chile.

References


Luis Friz
Grupo de Matemática Aplicada, Dpto. de Ciencias Básicas, Facultad de Ciencias, Universidad del Bío-Bío, Campus Fernando May, Casilla 447, Chillán, Chile
E-mail address: lfriz@ubiobio.cl

Marko Antonio Rojas-Medar
Instituto de Alta Investigación, Universidad de Tarapacá, Casilla 7D, Arica, Chile
E-mail address: marko.medar@gmail.com

María Drina Rojas-Medar
Dpto. de Matemáticas, Facultad de Ciencias Básicas, Universidad de Antofagasta, Antofagasta, Chile
E-mail address: maria.rojas@uantof.cl