UNIFORM ESTIMATE AND STRONG CONVERGENCE OF MINIMIZERS OF A $p$-ENERGY FUNCTIONAL WITH PENALIZATION

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Abstract. This article concerns the asymptotic behavior of minimizers of a $p$-energy functional with penalization as a parameter $\varepsilon$ approaches zero. By establishing $W^{1,p}$ uniform estimates, we obtain $W^{1,p}$ convergence of the minimizer to a $p$-harmonic map.

1. Introduction

Let $G \subset \mathbb{R}^2$ be a bounded and simply connected domain with smooth boundary $\partial G$, and $B_1 = \{ x \in \mathbb{R}^2; x_1^2 + x_2^2 < 1 \}$. Denote $S^1 = \{ x \in \mathbb{R}^3; x_1^2 + x_2^2 = 1, x_3 = 0 \}$ and $S^2 = \{ x \in \mathbb{R}^3; x_1^2 + x_2^2 + x_3^2 = 1 \}$. Sometimes we write the vector value function $u = (u_1, u_2, u_3)$ as $(u', u_3)$. Let $g = (g', 0)$ be a smooth map from $\partial G$ into $S^1$ satisfying $d = \text{deg}(g', \partial G) \neq 0$. Without loss of generality, we may assume $d > 0$. Consider the energy functional

$$E_\varepsilon(u, G) = \frac{1}{p} \int_G |\nabla u|^p dx + \frac{1}{2 \varepsilon^p} \int_G u_3^2 dx, \quad p > 2$$

with a small parameter $\varepsilon > 0$. From the direct method in the calculus of variations it is easy to see that the functional achieves its minimum in the function class $W^{1,p}_g(G, S^2)$. Obviously, the minimizer $u_\varepsilon$ on $W^{1,p}_g(G, S^2)$ is a weak solution of

$$- \text{div}(|\nabla u|^{p-2} \nabla u) = u|\nabla u|^p + \frac{1}{\varepsilon^p} (uu_3^2 - u_3 e_3), \quad \text{on } G,$$

where $e_3 = (0, 0, 1)$. Namely, for any $\psi \in W^{1,p}_0(G, \mathbb{R}^3)$, $u_\varepsilon$ satisfies

$$\int_G |\nabla u|^{p-2} \nabla u \nabla \psi dx = \int_G u\psi|\nabla u|^p dx + \frac{1}{\varepsilon^p} \int_G \psi(uu_3^2 - u_3 e_3) dx. \quad (1.1)$$

Without loss of generality, we assume $u_3 \geq 0$, otherwise we may consider $|u_3|$ in view of the expression of the functional.

When $p = 2$, the functional $E_\varepsilon(u, G)$ was introduced in the study of some simplified model of high-energy physics, which controls the statics of planer ferromagnets and antiferromagnets (see [10] [18]). The asymptotic behavior of minimizers of $E_\varepsilon(u, G)$ has been considered by Fengbo Hang and Fanghua Lin in [8]. When
the term $\frac{\alpha^2}{4^2}$ replaced by $(1-|u|)^2$ and $S^2$ replaced by $\mathbb{R}^2$, the problem becomes the simplified model of the Ginzburg-Landau theory for superconductors and was well studied in many papers such as [3, 4, 17, 19]. These works enunciate that the study of minimizers of the functional with some penalization terms is connected tightly with the study of harmonic maps with $S^1$-value. When $p > 2$, it also shows an enlightenment, namely, the properties (such as the partial regularity, the properties of singularities) of p-harmonic maps can be seen via studying the asymptotic properties of minimizers of some p-energy functional with penalization (cf. [11, 12, 13, 14, 16, 19]).

In this article, as in [3, 4, 8], we concern with the asymptotic behavior of minimizers of functional $E_\varepsilon(u,G)$ on $W^{1,p}(G, S^2)$ where $p > 2$ as $\varepsilon \to 0$.

**Theorem 1.1 ([15 Theorem 1.1]).** Assume $u_\varepsilon$ is a minimizer of $E_\varepsilon(u,G)$ on $W^{1,p}(G, S^2)$. Then all the zeros of $|u'_\varepsilon|$ are included in finite, disintersected discs $B(x_j^\varepsilon, h\varepsilon)$, $j = 1, 2, \ldots, N_1$ where $N_1$ and $h > 0$ do not depend on $\varepsilon \in (0, 1)$.

As $\varepsilon \to 0$, there exists a subsequence $x_{i_\varepsilon}^\varepsilon$ of the center $x_i^\varepsilon$ and $a_i \in \overline{G}$ such that $x_{i_\varepsilon}^\varepsilon \to a_i$, $i = 1, 2, \ldots, N_1$. Perhaps there may be at least two subsequences converging to the same point, we denote by $a_1, a_2, \ldots, a_N$, $N \leq N_1$, the collection of distinct points in $\{(a_i)_{i=1}^{N_1}\}$. Although the relationship between $N$ and $\varepsilon$ is unknown, the integer $N$ is independent of $\varepsilon \in (0, 1)$. By virtue of Theorem 1.1 we see that all the zeros of $|u'_\varepsilon|$ converge to $a_1, a_2, \ldots, a_N$ as $\varepsilon$ tends to 0. In addition, (2.3) in [15] shows

$$|u'_\varepsilon| \geq 1/2 \quad \text{on } K,$$

where $K$ is an arbitrary compact subset of $G \setminus \cup_{i=1}^{N_1} \{a_i\}$.

**Theorem 1.2 ([15 Theorem 1.2]).** Assume $u_\varepsilon$ is a minimizer of $E_\varepsilon(u,G)$ on $W^{1,p}(G, S^2)$. $K$ is an arbitrary compact subset of $\overline{G} \setminus \cup_{j=1}^{N} \{a_j\}$. Then there exists a subsequence $u_{\varepsilon_k}$ of $u_\varepsilon$ such that as $k \to \infty$,

$$u_{\varepsilon_k} \rightharpoonup u_p = (u'_p, 0), \quad \text{weakly in } W^{1,p}(K, \mathbb{R}^3),$$

where $u'_p$ is a map of the least p-energy $\int_K |\nabla u|^p dx$ in $W^{1,p}(K, \partial B_1)$.

We shall give the uniform $L^p_{\text{loc}}$ estimate of $\nabla u_\varepsilon$ in §3. Recalling the case that the parameter $p$ equals to the dimension 2, we know it is available to estimate the upper bound and the lower bound of $\int |\nabla u_\varepsilon|^2 dx$ since we can use the property of conformal transformation of $\int |\nabla u_\varepsilon|^2 dx$ (the idea of which can be seen in [4, 7, 8, 9]). In fact, when scaling $x = ye^\varepsilon$ in $E_\varepsilon(u,G)$, there is a coefficient $\varepsilon^\lambda$ appearing in the scaled energy functional. when $p = 2$, it can be derived that the exponent $\lambda$ of $\varepsilon$ is zero. Therefore, the estimate of the upper bound

$$E_\varepsilon(u_\varepsilon, G) \leq C_1 \ln \frac{1}{\varepsilon} + C$$

and the lower bound

$$\frac{1}{2} \int_{G \setminus \cup_{i=1}^{N_1} B(a_i, h\varepsilon)} |\nabla u'_\varepsilon|^2 dx \geq C_2 \ln \frac{1}{\varepsilon} - C$$

can be obtained, where $C_1 = C_2 = \pi d$ (cf. [8, §4]). The uniform estimate is deduced at once. When $p > 2$, the property of conformal transformation of $\int |\nabla u_\varepsilon|^p dx$ is invalid. Therefore, $\lambda \neq 0$. It is impossible to derive such results as the case $p = 2$
if the idea of estimating the upper and the lower bounds of $\int |\nabla u_\varepsilon|^p dx$ is adopted. In fact, the upper bound

$$E_\varepsilon(u_\varepsilon, G) \leq C_3 \varepsilon^{2-p} + C$$

and the lower bound

$$\frac{1}{p} \int_{G \setminus \cup_{i=1}^N B(a_i, \varepsilon)} |\nabla u'_\varepsilon|^p dx \geq C_4 \varepsilon^{2-p} - C,$$

are also obtained. However, the relationship between $C_3$ and $C_4$ is not clear except that $C_4$ may be smaller. In [13], a comparison method was used to obtain a uniform estimate where the average functions come into plays.

Here, we use the iteration technique introduced in [12] to obtain the uniform $L^p$ estimate of $\nabla u_\varepsilon$. In fact, the term $\int_K |\nabla u_\varepsilon|^p dx$ of the functional $E_\varepsilon(u_\varepsilon, K)$ can be divided into three terms, $\int_K |\nabla u'_\varepsilon|^p dx$, $\int_K |\nabla u_3|^p dx$ and $\int_K |u_3|^p |\nabla u'_3|^p dx$.

We will prove that $\int_K |\nabla u'_\varepsilon|^p dx + \int_K |\nabla u_3|^p dx + \frac{1}{\varepsilon^p} \int_K u_3^2 dx$ may be bounded by $O(\varepsilon^\lambda)$ with $\lambda > 0$ as $\varepsilon \to 0$. Using this estimate we will prove

$$\int_K |\nabla u_\varepsilon|^p dx \leq C + O(\varepsilon^\lambda).$$

Based on the Theorem 1.2 we will prove in §3 that the $p$-harmonic map $u_\varepsilon$ is a map of least $p$-energy $\int_K |\nabla u|^p dx$, and the convergence is also in strong $W^{1,p}_{loc}$ sense.

**Theorem 1.3.** Assume $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$ on $W^{1,p}_g(G, S^2)$. $K$ is an arbitrary compact subset of $\overline{G} \setminus \cup_{j=1}^N \{a_j\}$. Then there exists a subsequence $u_{\varepsilon_k}$ of $u_\varepsilon$ such that as $k \to \infty$,

$$u_{\varepsilon_k} \to u_p = (u'_p, 0), \quad \text{in} \quad W^{1,p}(K, \mathbb{R}^3),$$

where $u'_p$ is the map in Theorem 1.2.

2. Uniform estimate

The following inverse Hölder inequality will be applied later.

**Proposition 2.1.** Assume that $p > 1$, and $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$ on $W^{1,p}_g(G, S^2)$. Then there exist constants $t, R_0 \in (0, 1/2)$ and $C > 0$ which is independent of $\varepsilon$, such that for any $B_R \subset G$ ($2R < R_0$), we have

$$\left( \int_{B_R} |\nabla u_\varepsilon|^q dx \right)^{1/q} \leq C \left( \int_{B_{2R}} (|\nabla u_\varepsilon|^2 + 1)^{p/2} dx \right)^{1/p}, \quad \forall q \in [p, p + 2t).$$

The above proposition is a corollary from [6] Theorem 4.1, with a rescaling.

**Theorem 2.2.** Let $R > 0$ be a small constant such that $B(x, 2R) \subset G \setminus \cup_{j=1}^N \{a_j\}$. There exist constant $\varepsilon_0 > 0$ and $C_j > 0$, and $R_j = 2R - \frac{R}{|p|+1}$ such that for $j = 2, 3, \ldots, [p]$,

$$E_\varepsilon(u_\varepsilon, B_j) \leq C_j \varepsilon^{j-p}$$

where $\varepsilon \in (0, \varepsilon_0)$, $B_j = B(x, R_j)$, and $[p]$ is the integer part of $p$.

For $j = 2$, the inequality [2.1] follows from [13] Proposition 2.1. Suppose that [2.1] holds for all $j \leq m$. Then we have, in particular,

$$E_\varepsilon(u_\varepsilon, B_m) \leq C_m \varepsilon^{m-p}.$$  \hfill (2.2)

If $m = [p]$, then we are done. Suppose $m < [p]$, we want to prove [2.1] for $j = m+1$. \hfill (2.2)
Applying (1.2) we have $\frac{1}{2} \leq |u'_\varepsilon(y)| \leq 1$, for all $y \in B(x, 2R)$. Using the integral mean value theorem we know that there exists $r \in [R_{m+1/2}, R_m]$ such that

$$E_\varepsilon(u_\varepsilon, B_m \setminus B_{m+1/2}) = C_0(r) \int_{\partial B(x,r)} \frac{1}{p} |\nabla u_\varepsilon|^p + \frac{1}{4\varepsilon^p} u_\varepsilon^3 d\xi,$$

and applying (2.2), we see that

$$\int_{\partial B(x,r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x,r)} u_\varepsilon^2 d\xi \leq C_0^{-1}(r) C_m \varepsilon^{m-p}. \quad (2.3)$$

We denote $B = B(x, r)$, and introduce two propositions.

**Proposition 2.3.** If $\rho_1$ is a minimizer of the functional

$$E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} d\rho + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2 d\rho,$$

on $W^{1,p}_{\rho_1}(B, \mathbb{R}^+ \cup \{0\})$. Then $E(\rho_1, B) \leq C \varepsilon^{m-p+1}$.

**Proof.** Obviously, the minimizer $\rho_1$ exists and satisfies

$$-\text{div}(v^{(p-2)/2} \nabla \rho) = \frac{1}{\varepsilon^p} (1 - \rho) \quad \text{on } B,$$

$$\rho|_{\partial B} = |u'_\varepsilon|, \quad (2.5)$$

where $v = |\nabla \rho|^2 + 1$. Since $1/2 \leq |u'_\varepsilon| \leq 1$, it follows from the maximum principle that on $\partial B$,

$$\frac{1}{2} \leq \rho_1 \leq 1. \quad (2.6)$$

Applying (2.2) and noting $(1 - |u'_\varepsilon|)^2 \leq u_\varepsilon^2$, we see easily that

$$E(\rho_1, B) \leq E(|u'_\varepsilon|, B) \leq CE_\varepsilon(u_\varepsilon, B) \leq C \varepsilon^{m-p}. \quad (2.7)$$

Multiplying (2.4) by $\partial_{\nu} \rho$, where $\rho$ denotes $\rho_1$, and integrating over $B$, we have

$$-\int_{\partial B} v^{(p-2)/2} (\partial_{\nu} \rho)^2 d\xi + \int_B v^{(p-2)/2} \nabla \rho \nabla (\partial_{\nu} \rho) d\rho = \frac{1}{\varepsilon^p} \int_B (1 - \rho) (\partial_{\nu} \rho) d\rho, \quad (2.8)$$

where $\nu$ denotes the unit outside norm vector on $\partial B$. Using (2.7) we obtain

$$\left| \int_B v^{(p-2)/2} \nabla \rho \cdot \nabla (\partial_{\nu} \rho) d\rho \right| \leq C \int_B v^{(p-2)/2} |\nabla \rho|^2 d\rho + \frac{1}{p} \int_B \nu \cdot \nabla (v^{p/2}) d\rho \leq C \varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi. \quad (2.9)$$

Combining (2.3), (2.5) and (2.7) we also have

$$\left| \frac{1}{\varepsilon^p} \int_B (1 - \rho)(\partial_{\nu} \rho) d\rho \right| \leq \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2 d\rho \leq \int_{\partial B} (1 - \rho)^2 d\xi \leq C \varepsilon^{m-p}.$$

Substituting this result and (2.9) into (2.8) yields

$$\left| \int_{\partial B} v^{(p-2)/2} (\partial_{\nu} \rho)^2 d\xi \right| \leq C \varepsilon^{m-p} + \frac{1}{p} \int_{\partial B} v^{p/2} d\xi. \quad (2.10)$$
Applying (2.3), (2.5), (2.10) and the Young inequality, we obtain that for any \( \delta \in (0, 1) \),
\[
\int_{\partial B} v^{p/2} d\xi = \int_{\partial B} v^{(p-2)/2} [1 + (\partial_\nu \rho)^2 + (\partial_\tau \rho)^2] d\xi \\
\leq \int_{\partial B} v^{(p-2)/2} d\xi + \int_{\partial B} v^{(p-2)/2} (\partial_\nu \rho)^2 d\xi \\
+ \left( \int_{\partial B} v^{p/2} d\xi \right)^{p-2/p} \left( \int_{\partial B} (\tau \cdot \nabla |u_\varepsilon|)^p d\xi \right)^{2/p} \\
\leq C(\delta)\varepsilon^{m-p} + \left( \frac{1}{p} + 2\delta \right) \int_{\partial B} v^{p/2} d\xi,
\]
where \( \tau \) denotes the unit tangent vector on \( \partial B \). Therefore, it follows by choosing \( \delta > 0 \) sufficiently small that
\[
\int_{\partial B} v^{p/2} d\xi \leq C\varepsilon^{m-p}.
\] (2.11)

We multiply both sides of (2.4) by \((1 - \rho)\) and integrate over \( B \). Then
\[
\int_B v^{(p-2)/2} |\nabla \rho|^2 dx + \frac{1}{\varepsilon^p} \int_B (1 - \rho)^2 dx = -\int_{\partial B} v^{(p-2)/2} (\nu \cdot \nabla \rho)(1 - \rho) d\xi,
\]
whose left hand side is proportional to \( E(\rho_1, B) \). Thus
\[ E(\rho_1, B) \leq C \int_{\partial B} v^{(p-2)/2} (\nu \cdot \nabla \rho)(1 - \rho) d\xi. \]
Applying Holder’s inequality and (2.3), (2.5), (2.6) and (2.11), we obtain
\[
E(\rho_1, B) \leq C \int_{\partial B} v^{p/2} d\xi \left( \int_{\partial B} (1 - \rho^2)^2 d\xi \right)^{1/p} \leq C\varepsilon^{(m-p)(p-1)/p} \int_{\partial B} u_\varepsilon^{2p} d\xi \leq C\varepsilon^{m-p+1}.
\] (2.12)
The proof is complete. \( \square \)

**Proposition 2.4.** Denote \( h = |u_\varepsilon'| \). Then there is \( t \in (0, 1/2) \) such that for any \( \delta \in (0, 1/2) \),
\[
\frac{1}{p} \int_B |\nabla h|^p dx + \frac{1}{p} \int_B |\nabla u_\varepsilon|^p dx + \frac{1}{4\varepsilon^p} \int_B (1 - h^2)^2 dx \\
\leq C\varepsilon^{m-p+1} + \frac{1}{4\varepsilon^p} \int_B |\nabla u_\varepsilon|^p dx + C \left( \int_{B(x, 2r)} |\nabla u_\varepsilon|^p dx \right)^{1/p} + \frac{1}{p} \int_B (1 - h^2)^2 dx \right)^{t/(p+t)}.
\] (2.13)

**Proof.** Let \( U = (\sqrt{2\rho_1 - \rho_1^2} w, 1 - \rho_1) \) on \( B \); \( U = u_\varepsilon \) on \( G \setminus B \), where \( w = w_\varepsilon = \frac{u_\varepsilon'}{|u_\varepsilon'|} \). Then \( U \in W^{1,p}_g(G, S^d) \). Since \( u_\varepsilon \) is a minimizer of \( E_{\varepsilon}(u, G) \), we have
\[ E_{\varepsilon}(u_\varepsilon, G) \leq E_{\varepsilon}(U, G) = E_{\varepsilon}(U, B) + E_{\varepsilon}(u_\varepsilon, G \setminus B), \]
which means \( E_{\varepsilon}(u_\varepsilon, B) \leq E_{\varepsilon}(U, B) \). Using (2.12) it is not difficult to see that for any \( \delta > 0 \),
\[
\int_B |\nabla \rho_1|^2 |\nabla w|^{p-2} dx \leq \left( \int_B |\nabla \rho_1|^p dx \right)^{2/p} \left( \int_B |\nabla w|^p dx \right)^{\frac{p-2}{p}}
\]
\[ \leq \delta \int_{B} |\nabla u_{\varepsilon}|^{p} \, dx + C\varepsilon^{m+1-p}. \]

By using (2.6) and the mean value theorem,
\[ \int_{B} \left( \frac{1 - \rho_{1}^{2}}{2\rho_{1} - \rho_{1}^{2}} |\nabla \rho_{1}|^{2} + (2\rho_{1} - \rho_{1}^{2})|\nabla u_{\varepsilon}|^{p/2} \right) \, dx - \int_{B} ((2\rho_{1} - \rho_{1}^{2})|\nabla u_{\varepsilon}|^{p/2} \, dx \]
\[ \leq C \int_{B} (|\nabla \rho_{1}|^{p} + |\nabla \rho_{1}|^{2}|\nabla w|^{p-2}) \, dx, \]
and noting \(2\rho - \rho^{2} - 1 = -(1 - \rho)^{2} \leq 0\), we have
\[ E_{\varepsilon}(u_{\varepsilon}, B) \leq E_{\varepsilon}(U, B) \]
\[ \leq \frac{1}{p} \int_{B} (\rho_{1}^{2} - \rho_{1}^{2})|\nabla \rho_{1}|^{2} \, dx + C \int_{B} (|\nabla \rho_{1}|^{p} + |\nabla \rho_{1}|^{2}|\nabla w|^{p-2}) \, dx \]
\[ \leq \frac{1}{p} \int_{B} |\nabla \rho_{1}|^{p} \, dx + \delta \int_{B} |\nabla u_{\varepsilon}|^{p} \, dx + C\varepsilon^{m+1-p} + CE(\rho_{1}, B). \]

From this result and (2.12), we deduce
\[ E_{\varepsilon}(u_{\varepsilon}, B) \leq \frac{1}{p} \int_{B} |\nabla \rho_{1}|^{p} \, dx + C\varepsilon^{m+1-p} + \delta \int_{B} |\nabla u_{\varepsilon}|^{p} \, dx. \quad (2.14) \]

By Jensen’s inequality and (2.14), we obtain
\[ \frac{1}{p} \int_{B} |\nabla h|^{p} \, dx + \frac{1}{p} \int_{B} (h^{p} - 1)|\nabla w|^{p} \, dx + \frac{1}{p} \int_{B} |\nabla u_{\varepsilon}|^{p} \, dx \]
\[ + \frac{1}{4\varepsilon^{p}} \int_{B} (1 - \rho_{1})^{2} \, dx \]
\[ \leq E_{\varepsilon}(u_{\varepsilon}, B) - \frac{1}{p} \int_{B} |\nabla \rho_{1}|^{p} \, dx \]
\[ \leq C\varepsilon^{m-p+1} + \delta \int_{B} |\nabla u_{\varepsilon}|^{p} \, dx. \quad (2.15) \]

Since \(h \geq 1/2\) and Proposition 2.1, there exists a \(t \in (0, 1/2)\) such that
\[ \frac{1}{p} \int_{B} (1 - h^{p})|\nabla w_{\varepsilon}|^{p} \, dx \]
\[ \leq \frac{2p}{p} \int_{B} (1 - h^{p})|\nabla u_{\varepsilon}|^{p} \, dx \]
\[ \leq C \left( \int_{B} |\nabla u_{\varepsilon}|^{p+t} \, dx \right)^{p/(p+t)} \left( \int_{B} (1 - h^{p})^{(p+t)/t} \, dx \right)^{t/(p+t)} \]
\[ \leq C \left( \int_{B(x, 2r)} |\nabla u_{\varepsilon}|^{p} \, dx + 1 \right) \left( \int_{B} (1 - h^{2})^{2} \, dx \right)^{t/(p+t)}. \quad (2.16) \]

Combining this with (2.15) we complete the proof. \(\square\)
Proof of Theorem 2.2.

Step 1. Since \(|u'_{\varepsilon}| \geq \frac{1}{2}\), there exists \(\phi \in W^{1,p}(B(x, 3R),[0, 2\pi))\) such that \(w = \frac{u'_{\varepsilon}}{|u'_{\varepsilon}|} = (\cos \phi, \sin \phi)\). Obviously, \(|\nabla w|^2 = |\nabla \phi|^2\). Substituting this into (1.1) with the test function \((\psi, 0)\) yields

\[
\int_{B(x, 3R)} |\nabla u|^{p-2}(w \nabla h + h \nabla w) \nabla \psi dx
\]

\[
= \int_{B(x, 3R)} h w |\nabla u|^{p} \psi dx + \frac{1}{\varepsilon^p} \int_{B(x, 3R)} h w \psi (1 - h^2) dx
\]

where \(\psi \in W^{1,p}_0(G, \mathbb{R}^2)\). Let \(e^{i\phi} = \cos \phi + i \sin \phi\). Then

\[
\int_{B_{3R}(x)} h e^{i\phi} |\nabla u|^{p} \psi dx + \frac{1}{\varepsilon^p} \int_{B_{3R}(x)} h \psi e^{i\phi} (1 - h^2) dx
\]

\[
= \int_{B_{3R}(x)} |\nabla u|^{p-2}(e^{i\phi} \nabla h + h \varepsilon e^{i\phi} \nabla \phi) \nabla \psi dx.
\]

Taking \(\psi = e^{-i\phi} \zeta\), where \(\zeta \in W^{1,p}_0(B(x, 3R), \mathbb{R}^2)\), we obtain

\[
\frac{1}{\varepsilon^p} \int_{B(x, 3R)} h(1 - h^2) \zeta dx
\]

\[
= \int_{B(x, 3R)} |\nabla u|^{p-2}(\nabla h \nabla \zeta + h(|\nabla \phi|^2 - |\nabla u|^2) \zeta) dx.
\]

\[
0 = \int_{B(x, 3R)} |\nabla u|^{p-2}(h \nabla \phi \nabla \zeta - \zeta \nabla h \nabla \phi) dx.
\]

Taking \(\zeta = h\xi\) in (2.18), where \(\xi \in W^{1,p}_0(B(x, 3R), \mathbb{R}^2)\), we have

\[
0 = \int_{B(x, 3R)} |\nabla u|^{p-2} h^2 \nabla \phi \nabla \xi dx.
\]

Assume \(\rho\) is an arbitrary constant in \((0, 3R/2)\). Let \(\zeta \in W^{1,p}_0(B(x, 2\rho), [0, 1])\), and \(\zeta = 1\) on \(B(x, \rho)\). Taking \(\xi = \phi \zeta^2\) in (2.19) and using the Young inequality, for any \(\eta \in (0, 1)\) we obtain

\[
\int_{B(x, 2\rho)} |\nabla u|^{p-2} h^2 |\nabla \phi|^2 \zeta^2 dx \leq C \int_{B(x, 2\rho)} |\nabla u|^{p-2} h^2 (\eta |\nabla \phi|^2 \zeta^2 + C(\eta)) dx.
\]

Choosing \(\eta\) sufficiently small and noticing \(\zeta = 1\) on \(B(x, \rho)\), we obtain

\[
\int_{B(x, \rho)} |\nabla u|^{p-2} h^2 |\nabla \phi|^2 dx \leq C \left( \int_{B(x, 2\rho)} |\nabla u|^{p} dx \right)^{1-2/p}.
\]
Applying (2.20) with \( \rho = r \) we obtain
\[
\int_B |\nabla u|^p \leq \int_B |\nabla u|^{p-2}(h^2|\nabla \phi|^2 + |\nabla h|^2 + |\nabla u_3|^2)dx
\leq C \left( \int_{B(x,2r)} |\nabla u|^p dx \right)^{1-2/p}
+ \left( \int_B (|\nabla h|^p + |\nabla u_3|^p)dx \right)^{2/p} \left( \int_B |\nabla u|^p dx \right)^{(p-2)/p}
\leq C \left( \int_{B(x,2r)} |\nabla u|^p dx \right)^{1-2/p} + \delta \int_B |\nabla u|^p dx
+ C(\delta) \int_B (|\nabla h|^p + |\nabla u_3|^p)dx.
\] (2.21)

Substituting (2.13) into (2.21) and choosing \( \delta > 0 \) sufficiently small we derive
\[
\int_B |\nabla u|^p dx \leq C \left( \int_{B(x,2r)} |\nabla u|^p dx \right)^{1-2/p} + C_\varepsilon^{m-p+1}
+ C \left( \int_{B(x,2r)} |\nabla u_\varepsilon|^p dx + 1 \right) \left( \int_B (1-h^2)^2 dx \right)^{\varepsilon/(p+\varepsilon)}. \] (2.22)

From (2.2) it follows that
\[
\int_B |\nabla u|^p dx \leq C(\varepsilon^{m-p})^{1-2/p} + C_\varepsilon^{m-p+1} + C_\varepsilon^{m-p+\frac{m}{p+t}} = I_1 + I_2 + I_3. \] (2.23)

**Step 2.** When \( m \leq p/2 \), then \( m + 1 - p \leq (m - p)(1 - 2/p) \). Therefore \( I_1 \leq I_2 \). Let \( k_0 \in N \) be the minimum with the property \( m + 1 \leq (1 + \frac{1}{p+t})^k_0 m \).

In the following we shall improve the exponent \( m - p + \frac{1}{p+t} m \) of \( \varepsilon \) in \( I_3 \) to \( m - p + 1 \). Assume \( \zeta \in C_{\infty}^0(B(x,2R),[0,1]) \) satisfying \( \zeta = 1 \) on \( B_{m+1/2} \) and \( |\nabla \zeta| \leq C \). Taking the test function as \( h\zeta(1-h) \) in (2.17), we have
\[
\frac{1}{\varepsilon^p} \int_B h^2(1-h^2)\zeta(1-h)dx + \int_B |\nabla u|^{p-2}|\nabla h|^2 h\zeta dx + \int_B h^2|\nabla u|^p (1-h)\zeta dx
\leq \int_B |\nabla u|^{p-2} |\nabla h| \nabla \zeta h(1-h)dx + \int_B |\nabla u|^p \zeta (1-h) \leq C \int_B |\nabla u|^p dx.
\]

Noting \( \zeta = 1 \) on \( B_{m+1/2} \), applying \( h \geq 1/2 \) and (2.22), we obtain
\[
\frac{1}{\varepsilon^p} \int_{B_{m+1/2}} (1-h^2)^2 dx \leq C \frac{\varepsilon^p}{p} \int_B h^2(1-h^2)(1-h)\zeta dx \leq C(1 + \varepsilon^{m-p+\frac{m}{p+t}}),
\]
which implies
\[
\int_{B_{m+1/2}} (1-h^2)^2 dx \leq C\varepsilon^{m(1+\frac{m}{p+t})}, \quad \varepsilon \in (0,\varepsilon_0). \] (2.24)

On the other hand, similar to the derivation of (2.14), for \( B_{m+1/2} \) we still conclude that for any \( \delta > 0 \),
\[
\sum_{E\varepsilon(u_\varepsilon,B_{m+1/2})} \leq \frac{1}{p} \int_{B_{m+1/2}} |\nabla u|^p dx + C_\varepsilon^{m-p+1} + \delta \int_{B_{m+1/2}} |\nabla u_\varepsilon|^p dx.
\]
Therefore, (2.15) can be written as
\[
\frac{1}{p} \int_{B_{m+1/2}} |\nabla h|^p dx + \frac{1}{p} \int_{B_{m+1/2}} |\nabla u_3|^p dx + \frac{1}{4\varepsilon^p} \int_{B_{m+1/2}} (1 - h^2)^2 dx \\
\leq C \varepsilon^{m-p+1} + \frac{1}{p} \int_{B_{m+1/2}} (1 - h^p)|\nabla w|^p dx + \delta \int_{B_{m+1/2}} |\nabla u_\varepsilon|^p dx.
\tag{2.25}
\]

To estimate the second term of the right hand side of (2.25), we apply (2.23) and (2.24) to obtain
\[
\frac{1}{p} \int_{B_{m+1/2}} (1 - h^p)|\nabla w|^p dx \leq C \varepsilon^{(m-p+1)\frac{1}{m+1}+m+\frac{1}{m+1}m-p} = C \varepsilon^{m(1+\frac{1}{m+1})^2-p}
\]
by the same way as for (2.16). Substituting this into (2.25) yields
\[
\frac{1}{p} \int_{B_{m+1/2}} (|\nabla h|^p + |\nabla u_3|^p) dx \leq C \varepsilon^{m-p+1} + \varepsilon^{m(1+\frac{1}{m+1})^2-p} + \delta \int_{B_{m+1/2}} |\nabla u_\varepsilon|^p dx.
\]

Using this instead of (2.13) and by the same argument of Step 1 we can improve (2.23) as
\[
\int_{B_{m+1/2}} |\nabla u_\varepsilon|^p dx \leq C + C(\varepsilon^{m-p+1} + \varepsilon^{m(1+\frac{1}{m+1})^2-p}) \leq C \varepsilon^{m(1+\frac{1}{m+1})^2-p}.
\]

Now, we use this inequality replacing (2.23) to discuss, thus (2.24) can be written as
\[
\int_{B_{m+3/4}} (1 - h^2)^2 dx \leq C \varepsilon^{m(1+\frac{1}{m+1})^2}, \quad \varepsilon \in (0, \varepsilon_0).
\]

As a result, it is also follows that, as the derivation of (2.16) and (2.23),
\[
\frac{1}{p} \int_{B_{m+3/4}} (1 - h^p)|\nabla w|^p dx \leq C \varepsilon^{m(1+\frac{1}{m+1})^3-p},
\]
\[
\int_{B_{m+3/4}} |\nabla u_\varepsilon|^p dx \leq C + C(\varepsilon^{m-p+1} + \varepsilon^{m(1+\frac{1}{m+1})^3-p}) \leq C \varepsilon^{m(1+\frac{1}{m+1})^3-p}.
\]

If we do in this way, and noting the definition of $k_0$, we can derive by $k_0$ steps that
\[
\int_{B_{m+1-1/2k_0-1}} |\nabla u_\varepsilon|^p dx \leq C + C(\varepsilon^{m-p+1} + \varepsilon^{m(1+\frac{1}{m+1})k_0-p}).
\]

Thus
\[
\int_{B_{m+1}} |\nabla u_\varepsilon|^p dx \leq \int_{B_{m+1-1/2k_0-1}} |\nabla u_\varepsilon|^p dx \leq C(\varepsilon^{m-p+1} + 1).
\]

This is (2.2) for $j = m + 1$.

**Step 3.** When $m > p/2$, $(m-p)(1-2/p) < m + 1 - p$. Let $k \geq 1$ be an integer such that $(m-p)(1-2/p)^k \leq m + 1 - p < (m-p)(1-2/p)^{k+1}$. Now, $I_1 \geq I_2$ in (2.23). Thus,
\[
\int_{B} |\nabla u|^p dx \leq C(\varepsilon^{m-p})^{1-2/p} + C \varepsilon^{m-p+\frac{m}{p+1}}.
\]

Similar to Step 2, we may improve the exponent $m - p + \frac{m}{p+1}$ of $\varepsilon$ in $I_3$ to $(m-p)(1-2/p)$ since we may find $k_0 > 0$ such that $m(1+\frac{1}{m+1})k_0 > (m-p)(1-2/p)$. 


Namely, there is a constant $r_1 \in (R_{m+1}, r)$ such that
\[ \int_{B(x,r_1)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{(m-p)(1-2/p)}. \]
Therefore, as the derivation of (2.24),
\[ \int_{B(x,2r_1/3)} (1 - h^2)^2 \, dx \leq C_\varepsilon^{(m-p)(1-2/p) + p}. \]
Substituting these into (2.22) we have
\[ \int_{B(x,r_1/2)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{m+1-p} + C \left[ \int_{B(x,r)} |\nabla u_\varepsilon|^p \, dx \right]^{1-2/p} \]
\[ + C \left[ \int_{B(x,r)} |\nabla u_\varepsilon|^p \, dx + 1 \right] \left[ \int_{B(x,r)} (1 - h^2)^2 \, dx \right]^{p/(p+1)} \]
\[ \leq C_\varepsilon^{m+1-p} + C_\varepsilon^{(m-p)(1-2/p)^2} + C_\varepsilon^{(m-p)(1-2/p) + [(m-p)(1-2/p)+p]^{p/(p+1)}}. \]
Noting $(m-p)(1-2/p)^2 < m + 1 - p$, we can see that
\[ \int_{B(x,r_1/2)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{(m-p)(1-2/p)^2} + C_\varepsilon^{(m-p)(1-2/p) + [(m-p)(1-2/p)+p]^{p/(p+1)}}. \]
Using the idea of Step 2, we can improve the exponent $(m-p)(1-2/p)^2 + [(m-p)(1-2/p)+p]^{p/(p+1)}$ of $\varepsilon$ to $(m-p)(1-2/p)^2$. Namely, there is a constant $r_2 \in (R_{m+1}, r_1/2)$ such that
\[ \int_{B(x,r_2)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{(m-p)(1-2/p)^2}. \]
Suppose that for some $l \leq k - 1$,
\[ \int_{B(x,r_l-1)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{(m-p)(1-2/p)^l} \]
holds, where $R_{m+1} < r_{l+1} < r_l/2$ for $l = 2, 3, \ldots, k-1$. Therefore, as the derivation of (2.24),
\[ \int_{B(x,r_{l-1})} (1 - h^2)^2 \, dx \leq C_\varepsilon^{(m-p)(1-2/p)^l + p}. \]
Substituting these inequalities into (2.22) yields
\[ \int_{B(x,r_l)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{m+1-p} + C_\varepsilon^{(m-p)(1-2/p)^{l+1}} + C_\varepsilon^{(m-p)(1-2/p)^l + [(m-p)(1-2/p)^l + p]^{p/(p+1)}} \]
\[ \leq C_\varepsilon^{(m-p)(1-2/p)^{l+1}} + C_\varepsilon^{(m-p)(1-2/p)^l + [(m-p)(1-2/p)^l + p]^{p/(p+1)}}. \]
Similar to Step 2, we may improve again the exponent $(m-p)(1-2/p)^l + [(m-p)(1-2/p)^l + p]^{l/(l+1)}$ of $\varepsilon$ to $(m-p)(1-2/p)^{l+1}$. Namely, it can be seen that
\[ \int_{B(x,r_l)} |\nabla u_\varepsilon|^p \, dx \leq C_\varepsilon^{(m-p)(1-2/p)^{l+1}}. \]
From this result it follows that for $l = k - 1$,
\[
\int_{B(x, r_{k-1})} |\nabla u_\varepsilon|^p \leq C\varepsilon^{(m-p)(1-2/p)k}.
\]
Therefore, as the derivation of (2.24),
\[
\int_{B(x, r_{k-1})} (1 - h^2)^2 dx \leq C\varepsilon^{(m-p)(1-2/p)k} + p.
\]
Combining these with (2.22) we obtain
\[
\int_{B(x, r_{k-1}/2)} |\nabla u_\varepsilon|^p dx \leq C\varepsilon^{m+1-p} + C\varepsilon^{(m-p)(1-2/p)k} + [m-p(1-2/p)^k + p] \frac{1}{\varepsilon^{k-1}}.
\]
As in Step 2 and noting the definition of $k$, we may also improve the exponent of $\varepsilon$ to $m + 1 - p$ finally. Namely, we have
\[
\int_{B(x, r_{k-1}/2)} |\nabla u_\varepsilon|^p dx \leq C\varepsilon^{m+1-p}.
\]
This is (2.2) for $j = m + 1$ and proof of Theorem 2.2 is complete.

**Theorem 2.5.** For an arbitrary compact subset $K$ of $G \setminus \{a_1, a_2, \ldots, a_N\}$. There exists a constant $C > 0$ which does not depend on $\varepsilon \in (0, 1)$ such that $E_\varepsilon(u_\varepsilon, K) \leq C$.

**Proof.** It is sufficient to prove that $E_\varepsilon(u_\varepsilon, B(x, R)) \leq C$, where $B(x, R)$ is the disc in $G \setminus \{a_1, a_2, \ldots, a_N\}$. Theorem 2.2 shows that
\[
E_\varepsilon(u_\varepsilon, B_{[p]}) \leq C\varepsilon^{[p]-p}.
\] (2.26)
Using this and the integral mean value theorem, there exists a constant $r \in [R_{[p]+1/2}, R_{[p]}]$ such that
\[
\int_{\partial B(x, r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x, r)} u_\varepsilon^2 d\xi \leq C(r)\varepsilon^{[p]-p}.
\] (2.27)
Consider the functional
\[
E(\rho, B) = \frac{1}{p} \int_B (|\nabla \rho|^2 + 1)^{p/2} dx + \frac{1}{2\varepsilon^p} \int_B (1 - \rho)^2 dx,
\]
where $B = B(x, r)$. It is easy to prove that the minimizer $\rho_2$ of $E(\rho, B)$ on $W^{1, p}_{[p]}(B, \mathbb{R}^+ \cup \{0\})$ exists. Similar to the proof of proposition 2.3 by (2.26) and (2.27) we can derive
\[
E(\rho_2, B) \leq C\varepsilon^{[p]-p+1}.
\] (2.28)
From this it follows that for any $\delta > 0$,
\[
\int_B |\nabla \rho_2|^2 |\nabla w|^{p-2} dx \leq \delta \int_B |\nabla u_\varepsilon|^p dx + C\varepsilon^{[p]+1-p}.
\]
Since $u_\varepsilon$ is a minimizer of $E_\varepsilon(u, G)$, we have

$$E_\varepsilon(u_\varepsilon, B) \leq E_\varepsilon((\rho_2 w, \sqrt{1 - \rho_2^2}), B)$$

$$\leq \frac{1}{p} \int_B (|\nabla \rho_2^p|^p + |\nabla w|^p) dx + C \int_B (|\nabla \rho_2|^p + |\nabla w|^{p-2}) dx$$

$$+ \frac{1}{4p} \int_B (1 - \rho_2^2)^2 dx. \quad (2.29)$$

Therefore,

$$E_\varepsilon(u_\varepsilon, B) \leq \frac{1}{p} \int_B |\nabla w|^p dx + C_\varepsilon|\rho|^1 + 1 \int_B |\nabla u_\varepsilon|^p dx.$$ 

Combining this with Jensen’s inequality yields

$$\frac{1}{p} \int_B |\nabla h|^p dx + \frac{1}{p} \int_B |\nabla u_\varepsilon|^p dx + \frac{1}{4p} \int_B (1 - h^2)^2$$

$$\leq E_\varepsilon(u_\varepsilon, B) - \frac{1}{p} \int_B |\nabla w|^p dx + \frac{1}{p} \int_B (1 - h^p) |\nabla w|^p dx$$

$$\leq C_\varepsilon|\rho|^1 + 1 + \delta \int_B |\nabla u_\varepsilon|^p dx + \frac{1}{p} \int_B (1 - h^p) |\nabla w|^p dx. \quad (2.30)$$

To estimate the third term of the right hand side, we proceed in the same way of the proof of Proposition 2.4, and use $\frac{1}{p} \int_B (1 - h^2)^2 dx \leq C_\varepsilon|\rho|^{-p}$ which is implied by (2.26). As a result, there exists $t \in (0, 1/2)$ such that

$$\frac{1}{p} \int_B (1 - h^p) |\nabla w|^p dx \leq C_\varepsilon|\rho|^1 + 1 + \delta \int_B |\nabla u_\varepsilon|^p dx.$$ 

Substituting this into (2.30) yields

$$\frac{1}{p} \int_B |\nabla h|^p dx + \frac{1}{p} \int_B |\nabla u_\varepsilon|^p dx + \frac{1}{4p} \int_B (1 - h^2)^2 dx$$

$$\leq C_\varepsilon|\rho|^1 + 1 + \varepsilon|\rho| + \frac{|\nabla u_\varepsilon|^p}{p} + \frac{1}{p} \int_B (1 - h^p) |\nabla w|^p dx.$$ 

This and (2.21) imply that

$$\int_B |\nabla u_\varepsilon|^p dx \leq C_\varepsilon|\rho|^{-p+1} + C_\varepsilon|\rho|^{-p+1 - \frac{1}{p}} + C_\varepsilon|\rho|^{-p+1 - \frac{1}{p}} + C_\varepsilon|\rho|^{-p+1 - \frac{1}{p}} + C, \quad (2.31)$$

as long as we choose $\delta > 0$ sufficiently small. Discussing in the same way to Step 2 and Step 3, we may improve the exponent of $\varepsilon$ in the second and the third terms of the right hand side of (2.31) step by step such that the improved exponent is not smaller than $|\rho| - p + 1$, thus for some $B_{|\rho|+1} \subset B$, there exists $C$ independent of $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0$ sufficiently small such that

$$\int_{B_{|\rho|+1}} |\nabla u_\varepsilon|^p dx \leq C + C_\varepsilon|\rho|^{-1} \leq C.$$ 

The proof is complete. \qed
3. Proof of Theorem 1.3

**Step 1.** Suppose \( B(x_0,2\sigma) \subset [G \setminus \cup_{j=1}^N \{a_j\}] \), where the constant \( \sigma \) may be sufficiently small but independent of \( \varepsilon \). Since theorem 2.5 implies \( E_\varepsilon(u_\varepsilon,B(x_0,2\sigma) \setminus B(x_0,\sigma)) \leq C \), there is a constant \( r \in (\sigma,2\sigma) \) such that

\[
\int_{\partial B(x_0,r)} |\nabla u_\varepsilon|^p d\xi + \frac{1}{\varepsilon^p} \int_{\partial B(x_0,r)} u_\varepsilon^2 d\xi \leq C(r).
\]

Thus, we can find a subsequence \( u_{\varepsilon_k} \) of \( u_\varepsilon \) such that \( u_{\varepsilon_k} \to u_p = (u'_p,0) \) in \( C(\partial B(x_0,r),\mathbb{R}^3) \), where \( u'_p \) is the \( S^1 \)-valued harmonic map, which leads to

\[
u'_{\varepsilon_k} \to u'_p, \quad \text{in} \quad C(\partial B(x_0,r)).
\] (3.1)

**Step 2.** Denote \( B = B(x_0,r) \). It is easy to see the existence of the solution \( w_\varepsilon \) of

\[
\min \left\{ \int_B |\nabla u|^p dx : u \in W^{1,p}_{\varepsilon} \right\}.
\] (3.2)

Theorem 2.5 and \(|u'_{\varepsilon} | \geq 1/2 \) on \( B \) imply \( 2^{-p} \int_B |\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx \leq \int_B |\nabla u_\varepsilon|^p dx \leq C \), and hence

\[
\int_B |\nabla u_\varepsilon|^p dx \leq \int_B |\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^p dx \leq C. \quad (3.3)
\]

From this and \((2.28)\) it follows that \( \int_B |\nabla \rho_2|^2 |\nabla w_\varepsilon|^p-2 dx \leq C\varepsilon^{2(|p|+1-p)/p} \), where \( \rho_2 \) is the minimizer of \( E(\rho,B) \) on \( W^{1,p}_{u'_\varepsilon}(B,\mathbb{R}^+ \cup \{0\}) \). Substituting this result into \((2.29)\) and using \((2.28)\), we obtain

\[
\int_B |\nabla u_\varepsilon|^p dx \leq C\varepsilon^{2(|p|+1-p)/p} + \int_B |\nabla w_\varepsilon|^p dx. \quad (3.4)
\]

**Step 3.** Let \( w_\varepsilon^\tau \) be a solution of

\[
\min \left\{ \int_B \left(|\nabla w|^2 + \tau \right)^{p/2} dx : w \in W^{1,p}_{\varepsilon/|\varepsilon|} \right\}, \quad \tau \in (0,1).
\] (3.5)

Clearly, \( w_\varepsilon^\tau \) also solves

\[-\text{div}(v_\varepsilon^{(p-2)/2} \nabla w) = w|\nabla w|^2 v_\varepsilon^{(p-2)/2}, \quad v_\varepsilon = |\nabla w|^2 + \tau.\] (3.6)

Noticing \( \frac{u'_\varepsilon}{|u'_\varepsilon|} \in W^{1,p}_{\varepsilon/|\varepsilon|} \), we have

\[
\int_B |\nabla w_\varepsilon^\tau|^p dx \leq \int_B \left(|\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^2 + \tau \right)^{p/2} dx
\leq \int_B \left(|\nabla \frac{u'_\varepsilon}{|u'_\varepsilon|}|^2 + 1 \right)^{p/2} dx \leq C
\] (3.7)

by using \((3.3)\), where \( C \) is a constant which is independent of \( \varepsilon, \tau \). Then there exist \( w^* \in W^{1,p}_{\varepsilon/|\varepsilon|} \) and a subsequence of \( w_\varepsilon^\tau \) denoted still by itself such that

\[
\lim_{\tau \to 0} w_\varepsilon^\tau = w^* \quad \text{weakly in} \quad W^{1,p}(B,R^2).
\] (3.8)
Noting the weak lower semi-continuity of \( \int_B |\nabla w|^p \), we have
\[
\int_B |\nabla w^\tau|^p dx \leq \liminf_{\tau \to 0} \int_B |\nabla w^\tau_\varepsilon|^p dx \leq \limsup_{\tau \to 0} \int_B |\nabla w^\tau_\varepsilon|^p dx. \tag{3.9}
\]
The fact that \( w^\tau_\varepsilon \) solves \( \text{(3.5)} \) implies
\[
\limsup_{\tau \to 0} \int_B (|\nabla w^\tau_\varepsilon|^2 + \tau)^{p/2} dx \leq \lim_{\tau \to 0} \int_B (|\nabla w^\tau_\varepsilon|^2 + \tau)^{p/2} dx = \int_B |\nabla w^\tau_\varepsilon|^p dx,
\]
where \( w^\tau_\varepsilon \) is a solution of \( \text{(3.2)} \). This and \( \text{(3.9)} \) lead to
\[
\int_B |\nabla w^\tau|^p dx \leq \liminf_{\tau \to 0} \int_B |\nabla w^\tau_\varepsilon|^p dx \leq \limsup_{\tau \to 0} \int_B |\nabla w^\tau_\varepsilon|^p dx \leq \int_B |\nabla w^\tau_\varepsilon|^p dx. \tag{3.10}
\]
Since \( w^\tau \in W^{1,p}_{w^\tau} (B, \partial B_1) \), we know \( w^\tau \) also solves \( \text{(3.2)} \), namely
\[
\int_B |\nabla w^\tau_\varepsilon|^p dx = \int_B |\nabla w^\tau|^p dx. \tag{3.11}
\]
Combining this with \( \text{(3.10)} \) yields \( \lim_{\tau \to 0} \int_B |\nabla w^\tau_\varepsilon|^p dx = \int_B |\nabla w^\tau|^p dx \), which and \( \text{(3.8)} \) imply that as \( \tau \to 0 \),
\[
\nabla w^\tau_\varepsilon \to \nabla w^\tau \quad \text{in} \quad L^p(B, R^2). \tag{3.12}
\]

**Step 4.** By the same argument as in Step 3, we obtain the following conclusion: Let \( u^\tau \) be a solution of
\[
\min \{ \int_B (|\nabla u|^2 + \tau)^{p/2} dx : u \in W^{1,p}_{u_\varepsilon}(B, \partial B_1) \}, \quad \tau \in (0, 1). \tag{3.13}
\]
Then \( u^\tau \) satisfies
\[
\int_B |\nabla u^\tau|^p dx \leq C, \tag{3.14}
\]
where \( C \) is which is independent of \( \tau \), and \( u^\tau \) solves
\[
-\text{div}(u^\tau)^{(p-2)/2} \nabla u^\tau = u^\tau |\nabla u|^2 u^{(p-2)/2}, \quad u^\tau = |\nabla u|^2 + \tau. \tag{3.15}
\]
As \( \tau \to 0 \), there exists a subsequence of \( u^\tau \) denoted by itself such that
\[
\nabla u^\tau \to \nabla u^* \quad \text{in} \quad L^p(B, R^2), \tag{3.16}
\]
where \( u^* \) is a minimizer of \( \int_B |\nabla u|^p dx \) in \( W^{1,p}_{u_\varepsilon}(B, \partial B_1) \). It is well-known that \( u^* \) is a map of the least p-energy, and also a p-harmonic map.

**Step 5.** From [5] Lemma 1, Page 65], we can write
\[
w^\tau_\varepsilon = (\cos \phi^\tau_\varepsilon, \sin \phi^\tau_\varepsilon), \quad u^\tau = (\cos \psi^\tau, \sin \psi^\tau), \nabla w^\tau_\varepsilon = (\cos \phi^\tau, \sin \phi^\tau), \quad u^* = (\cos \psi^*, \sin \psi^*), \nabla u^\tau_\varepsilon |_{\partial B} = (\cos \phi_\varepsilon, \sin \phi_\varepsilon), \quad u^\tau_\varepsilon |_{\partial B} = (\cos \psi, \sin \psi),
\]
where \( \phi^\tau_\varepsilon, \psi^\tau, \phi^\tau, \psi^* \) belong to \( W^{1,p}(B, R) \), \( \phi^*, \psi \) belong to \( W^{1,p}(\partial B, R) \), and they are all single-valued functions since their degrees around \( \partial B \) are zero. Therefore,
\[
\phi^\tau_\varepsilon |_{\partial B} = \phi_\varepsilon, \quad \psi^\tau |_{\partial B} = \psi, \tag{3.17}
\]
and $|\nabla w^\tau_\epsilon| = |\nabla \phi^\tau_\epsilon|$, $|\nabla w^\tau| = |\nabla \psi^\tau|$. Moreover, by (3.6) and (3.15), we obtain that both $\phi^\tau_\epsilon$ and $\psi^\tau$ satisfy $-\text{div}[(|\nabla \Phi|^2 + \tau)^{(p-2)/2}\nabla \Phi] = 0$. Thus,

$$-\text{div}[(|\nabla \phi^\tau_\epsilon|^2 + \tau)^{(p-2)/2}\nabla \phi^\tau_\epsilon] - (|\nabla \psi^\tau|^2 + \tau)^{(p-2)/2}\nabla \psi^\tau] = 0. \quad (3.18)$$

Multiplying both sides of (3.18) by $\phi^\tau_\epsilon - \psi^\tau$ and integrating over $B$, we obtain

$$-\int_{\partial B} (v_\epsilon^{(\tau-2)/2} \phi_\nu - v^{(\tau-2)/2} \psi_\nu)(\phi - \psi)d\xi + \int_B (v_\epsilon^{\tau(p-2)/2} \nabla \phi - v^{\tau(p-2)/2} \nabla \psi) \nabla (\phi - \psi)dx = 0, \quad (3.19)$$

where $\nu$ denotes the unit outside-norm vector of $\partial B$.

Let $w = w^\tau_\epsilon$ be a solution of (3.5). Integrating both sides of (6) over $B$, we have

$$-\int_{\partial B} v_\epsilon^{(\tau-2)/2} w_\nu d\xi = \int_B w|\nabla w|^2 v_\epsilon^{(\tau-2)/2} dx,$$

this and (3.7) imply

$$|\int_{\partial B} v_\epsilon^{(\tau-2)/2} \phi_\nu d\xi| = |\int_{\partial B} v_\epsilon^{(\tau-2)/2} w_\nu d\xi| \leq \int_B v_\epsilon^{\tau p/2} dx \leq C. \quad (3.20)$$

An analogous discussion shows that for the solution $u = u^\tau$ of (3.13) which is equipped with (3.14), we may also obtain

$$|\int_{\partial B} v^{(\tau-2)/2} \phi_\nu d\xi| = \int_{\partial B} v^{(\tau-2)/2} u_\nu d\xi| \leq \int_B |\nabla u|^p dx \leq C. \quad (3.21)$$

Combining (3.17) with (3.19) and (3.21), we derive

$$\int_B (v_\epsilon^{(\tau-2)/2} \nabla \phi - v^{(\tau-2)/2} \nabla \psi) \nabla (\phi - \psi)dx \leq C \sup_{\partial B} |\phi^\tau_\epsilon - \psi^\tau| = C \sup_{\partial B} |\phi_\epsilon - \psi|,$$

where $C$ is independent of $\epsilon, \tau$. Letting $\tau \to 0$ and applying (3.12) and (3.16), we obtain

$$|\int_B (|\nabla \phi^\tau_\epsilon|^2 (p-2)/2 \nabla \phi^\tau_\epsilon - |\nabla \psi^\tau|^2 (p-2)/2 \nabla \psi^\tau) \nabla (\phi^\tau_\epsilon - \psi^\tau)dx| \leq C \sup_{\partial B} |\phi^\tau_\epsilon - \psi|,$$

which implies $\int_B |\nabla \phi^\tau_\epsilon - \nabla \psi^\tau|^p dx \leq C \sup_{\partial B} |\phi^\tau_\epsilon - \psi|$. Letting $\epsilon \to 0$ and using (3.1), we obtain $\int_B |\nabla \phi^\tau_\epsilon|^p dx \to \int_B |\nabla \psi^\tau|^p dx$. That is,

$$\int_B |\nabla \phi^\tau_\epsilon|^p dx \to \int_B |\nabla \psi^\tau|^p dx. \quad (3.22)$$

**Step 6.** Since $\int_B |\nabla u|^p dx$ is weak lower semi-continuous, from Theorem 1.2 we deduce $\int_B |\nabla u|^p dx \leq \liminf_{\epsilon_\kappa \to 0} \int_B |\nabla u_\epsilon|^p dx$. Combining this result with (3.4), (3.11) and (3.22), we obtain

$$\int_B |\nabla u|^p dx \leq \liminf_{\epsilon_\kappa \to 0} \int_B |\nabla u_\epsilon|^p dx \leq \limsup_{\epsilon_\kappa \to 0} \int_B |\nabla u_\epsilon|^p dx \leq \lim_{\epsilon_\kappa \to 0} \int_B |\nabla u_\epsilon|^p dx = \int_B |\nabla u^\star|^p dx.$$
Recalling the definition of \( u^* \) in Step 4, and noticing \( u'_p \in W^{1,p}_p(B, \partial B_1) \), we know that \( u'_p \) is also a minimizer of \( \int_B |\nabla u|^p \), and
\[
\lim_{\varepsilon_k \to 0} \int_B |\nabla u_{\varepsilon_k}|^p \, dx = \int_B |\nabla u_p|^p \, dx = \int_B |\nabla u^*|^p \, dx.
\]
This result and Theorem 1.2 imply \( \nabla u_{\varepsilon_k} \to \nabla u_p \) in \( L^p(B, \mathbb{R}^3) \) when \( \varepsilon_k \to 0 \).
Combining this with the fact \( u_{\varepsilon_k} \to u_p \) in \( L^p(B, \mathbb{R}^3) \), which is implied by Theorem 1.2, we obtain
\[
u_{\varepsilon_k} \to u_p, \quad \text{in } W^{1,p}(B, \mathbb{R}^3)
\]
as \( \varepsilon_k \to 0 \). Then it is not difficult to complete the proof of this theorem.

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**REFERENCES**


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