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GLOBAL STRUCTURE OF SOLUTIONS TO BOUNDARY-VALUE PROBLEMS OF IMPULSIVE DIFFERENTIAL EQUATIONS

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Abstract. In this article, we study the structure of global solutions to the boundary-value problem

$$\begin{gathered} -x''(t)+f(t,x)=\lambda a x(t), \quad t\in(0,1),\ t\neq\frac{1}{2}, \\ \Delta x|_{t=1/2}=\beta_1 x(\frac{1}{2}), \quad \Delta x'|_{t=1/2}=-\beta_2 x(\frac{1}{2}), \\ x(0)=x(1)=0, \end{gathered}$$

where $\lambda\neq 0$, $\beta_1\geq\beta_2\geq 0$, $\Delta x|_{t=1/2}=x(\frac{1}{2}+0)-x(\frac{1}{2})$, $\Delta x'|_{t=1/2}=x'(\frac{1}{2}+0)-x'(\frac{1}{2}-0)$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $a:[0,1]\to(0,+\infty)$ are continuous. By a comparison principle and spectral properties of the corresponding linear equations, we prove the existence of solutions by using Rabinowitz-type global bifurcation theorems, and obtain results on the behavior of positive solutions for large λ when $f(x)=x^{p+1}$.

1. Introduction

In this article, we study the structure of global solutions to the second-order impulsive differential equation

$$\begin{gathered} -x''(t)+f(t,x)=\lambda a x(t), \quad t\in(0,1),\ t\neq\frac{1}{2}, \\ \Delta x|_{t=1/2}=\beta_1 x(\frac{1}{2}), \quad \Delta x'|_{t=1/2}=-\beta_2 x(\frac{1}{2}), \\ x(0)=x(1)=0, \end{gathered} \tag{1.1}$$

where $\lambda\neq 0$, $\beta_1\geq\beta_2\geq 0$, $\Delta x|_{t=1/2}=x(\frac{1}{2}+0)-x(\frac{1}{2})$, $\Delta x'|_{t=1/2}=x'(\frac{1}{2}+0)-x'(\frac{1}{2}-0)$, and $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, $a:[0,1]\to(0,+\infty)$ are continuous.

Impulsive differential equations arise in the contexts of population dynamics, infectious diseases models, chemical technology and so on; see [4, 7, 12, 21, 27, 28, 34]. Because impulsive equations appear in applied mathematics, they attract a lot of attention. Many authors studied the existence, uniqueness as well as multiplicity of solutions, by using the topological degree theory and variational method; see [1, 5, 10, 11, 14, 15, 19, 20, 25, 26, 29, 31, 32, 33, 35, 36].

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An important tool to study the existence of solutions for differential equations is Rabinowitz global bifurcation theory; see [2, 5, 6, 16, 17, 18, 22, 23, 24]. But up to now, just a few results have shown on the structure of global solutions for impulsive differential equations by Rabinowitz global bifurcation theorems. Liu and O'Regan [15] considered the second-order impulsive differential equation

$$\begin{gathered} x''(t)+ra(t)f(t,x(t))=0, \quad t\in(0,1),\ t\neq t_i,\\ \Delta x|_{t=t_i}=\alpha_i x(t_i-0), \quad i=1,2,\dots,k,\\ x(0)=x(1)=0, \end{gathered} \tag{1.2}$$

in which they converted (1.2) to the form

$$\begin{gathered} y''(t)+\frac{r}{\Pi_{0<t_i<t}(1+\alpha_i)}a(t)f(t,\Pi_{0<t_i<t}(1+\alpha_i)y(t))=0, \quad t\in(0,1),\\ y(0)=y(1)=0. \end{gathered} \tag{1.3}$$

By the known properties of the eigenvalues and eigenfunctions of the linear equation corresponding to (1.3), using bifurcation techniques, they obtained the existence of multiple solutions to (1.2).

As we know, Rabinowitz global bifurcation theory can be used effectively is that the spectral properties of relative linearized operators (especially, nodal zeros of eigenfunctions) are clear. But impulses lead to the complexity of eigenvalues and eigenfunctions of the impulsive linear equations, which have not been analyzed completely. A recent work on this aspect comes from [30] in which Wang and Yan presented the spectral properties of the equation

$$\begin{gathered} -x''(t)=\lambda x(t), \quad t\in(0,1),\ t\neq\frac12,\\ \Delta x|_{t=1/2}=\lambda\beta x(\frac12), \quad \Delta x'|_{t=1/2}=-\lambda\beta x'(\frac12-0),\\ x(0)=x(1)=0, \end{gathered} \tag{1.4}$$

and studied the existence of multiple solutions for the relative nonlinear second-order impulsive differential equations by Rabinowitz global bifurcation theory.

In this article, we consider the nonlinear second-order impulsive differential equation (1.1), in which the impulses are different from (1.4). This article is organized as follows. In section 2, a Comparison Principle is established for the second-order impulsive differential equations. In section 3, we focus on the linear impulsive equation

$$\begin{gathered} -x''(t)=\lambda a(t)x(t), \quad t\in(0,1),\ t\neq\frac12,\\ \Delta x|_{t=1/2}=\beta_1 x(\frac12), \quad \Delta x'|_{t=1/2}=-\beta_2 x(\frac12),\\ x(0)=x(1)=0, \end{gathered}$$

where λ, β_1, β_2, $a(t)$, Δx, $\Delta x'$ are introduced as in (1.1), and present the eigenvalues and eigenfunctions properties. In section 4, we obtain the solutions of (1.1) under various hypotheses on the asymptotic behaviour of f using the global bifurcation theorem from Section 3. Finally, with the principal eigenvalue, we discuss the special autonomous case

$$\begin{gathered} -x''(t) + f(x) = \lambda m x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{\frac{1}{2}} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{\frac{1}{2}} = -\beta_2 x(\frac{1}{2}), \\ x(0) = x(1) = 0, \end{gathered} \tag{1.5}$$

where $f(x) = x^{p+1}$ and $p > 0$, $m > 0$ are real numbers, and give results on the behavior of positive solutions for the large λ. Some ideas come from [7].

2. Preliminaries

Comparison argument in ODE plays an important role in analyzing some properties of solutions. Naturally, it necessary to study the relative Comparison argument to get the properties of solutions of (1.1).

Lemma 2.1. *Suppose that $y(t)$ is the solution of*

$$\begin{gathered} y'' + Q(t)y = 0, \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta y|_{t=1/2} = \beta_1 y(\frac{1}{2}), \quad \Delta y'|_{t=1/2} = -\beta_2 y(\frac{1}{2}), \end{gathered} \tag{2.1}$$

and $w(t)$ is the solution of

$$\begin{gathered} w'' + P(t)w = 0, \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta w|_{t=1/2} = \beta_1 w(\frac{1}{2}), \quad \Delta w'|_{t=1/2} = -\beta_2 w(\frac{1}{2}), \end{gathered} \tag{2.2}$$

where $P(t)$ and $Q(t)$ are continuous on the same interval $[0,1]$ with $Q(t) \leq P(t)$. If α, β are two next null points of $y(t)$, then there must be at least one null point of $w(t)$ on $[\alpha, \beta]$.

Proof. Suppose α, β are two next null points of $y(t)$. We discuss two cases about them.

(1) α, $\beta \in [0, 1/2)$ or α, $\beta \in (\frac{1}{2}, 1]$. In this case, it is easy to obtain the results by Sturm comparison theorem in ODE.

(2) $\alpha < \frac{1}{2} < \beta$. Multiply (2.1) by $w(t)$, (2.2) by $y(t)$ and subtract each other to obtain

$$wy'' - w''y + [Q(t) - P(t)]wy = 0.$$

Integrating the equation above from α to β yields

$$\int_\alpha^\beta (wy'' - w''y)\, dt = \int_\alpha^\beta [P(t) - Q(t)]wy\, dt;$$

therefore,

$$\begin{aligned} &\int_\alpha^{\frac{1}{2}-} d(wy' - w'y) + \int_{\frac{1}{2}+}^\beta d(wy' - w'y) \\ &= \int_\alpha^{\frac{1}{2}-} [P(t) - Q(t)]wy\, dt + \int_{\frac{1}{2}+}^\beta [P(t) - Q(t)]wy\, dt. \end{aligned} \tag{2.3}$$

From the impulsive conditions in (2.1) and (2.2), we have

$$w(\frac{1}{2}+) = (1 + \beta_1)w(\frac{1}{2}-),$$

$$\begin{gathered}w'(\frac{1}{2}+)=w'(\frac{1}{2}-)-\beta_2 w(\frac{1}{2}-),\\ y(\frac{1}{2}+)=(1+\beta_1)y(\frac{1}{2}-),\\ y'(\frac{1}{2}+)=y'(\frac{1}{2}-)-\beta_2 y(\frac{1}{2}-),\end{gathered}$$

which together with $y(\alpha)=y(\beta)=0$ guarantees that the left side of (2.3) satisfies

$$-\beta_1\Big[w(\frac{1}{2}-)y'(\frac{1}{2}-)-w'(\frac{1}{2}-)y(\frac{1}{2}-)\Big]-w(\alpha)y'(\alpha)+w(\beta)y'(\beta). \tag{2.4}$$

Denote

$$F(t)=w(t)y'(t)-w'(t)y(t).$$

Combining (2.1) with (2.2), one has

$$F'(t)=w(t)y''(t)-w''(t)y(t)=[P(t)-Q(t)]w(t)y(t).$$

Suppose that $w(t)$ has no zero in (α,β), and without loss of generality $y(t)$ is positive in (α,β). Then two cases will be discussed according to $w(t)$.

(a) $w(t)>0$, $t\in(\alpha,\beta)$. In this case, the fact that $F'(t)=[P(t)-Q(t)]w(t)y(t)>0$ for $t\in[\alpha,\frac{1}{2})$ implies that $F(t)$ increases on $[\alpha,\frac{1}{2})$. Since

$$F(\alpha)=w(\alpha)y'(\alpha)-w'(\alpha)y(\alpha)=w(\alpha)y'(\alpha)>0,$$

one has $F(\frac{1}{2}-)>F(\alpha)>0$. Then (2.4) is

$$-\beta_1 F(\frac{1}{2})-w(\alpha)y'(\alpha)+w(\beta)y'(\beta)<0,$$

while the right-hand side of (2.3) satisfies

$$\int_\alpha^{1/2-}[P(t)-Q(t)]wy dt+\int_{1/2+}^{\beta}[P(t)-Q(t)]wy dt>0,$$

which is a contradiction.

(b) $w(t)<0$, $t\in(\alpha,\beta)$. Since $F'(t)=[P(t)-Q(t)]w(t)y(t)<0$ for $t\in[\alpha,\frac{1}{2})$, $F(t)$ decreases on $[\alpha,\frac{1}{2})$. With $F(\alpha)=w(\alpha)y'(\alpha)<0$, one has $F(\frac{1}{2}-)<0$. Then (2.4) is

$$-\beta_1 F(\frac{1}{2})-w(\alpha)y'(\alpha)+w(\beta)y'(\beta)>0,$$

while the right-hand side of (2.3) satisfies

$$\int_\alpha^{\frac{1}{2}-}[P(t)-Q(t)]wy dt+\int_{\frac{1}{2}+}^{\beta}[P(t)-Q(t)]wy dt<0,$$

which is a contradiction. The proof is complete. □

For convenience, we introduce main lemma and some symbols in global bifurcation theory. Let $\mathcal{F}:\mathcal{E}\to\mathcal{E}_1$ where $\mathcal{E}$ and $\mathcal{E}_1$ are real Banach spaces and $\mathcal{F}$ is continuous. Suppose that the equation $\mathcal{F}(U)=0$ possesses a simple curve of solutions Ψ given by $\{U(t)|t\in[a,b]\}$. If for some $\tau\in(a,b)$, $\mathcal{F}$ possesses zeros not lying on Ψ in every neighborhood of $U(\tau)$, then $U(\tau)$ is said to be a bifurcation point for $\mathcal{F}$ with respect to the curve Ψ (see [23]).

A special family of such equations has the form

$$u=G(\lambda,u), \tag{2.5}$$

where $\lambda \in \mathbb{R}$, $u \in E$ is a real Banach space with the norm $\|\cdot\|$, and $G : \mathcal{E} \equiv \mathbb{R} \times E \to E$ is compact and continuous. In addition, $G(\lambda, u) = \lambda L u + H(\lambda, u)$, where $H(\lambda, u)$ is $o(\|u\|)$ for u near 0 uniformly on bounded λ intervals and L is a compact linear map on E. A solution of (2.5) is a pair $(\lambda, u) \in \mathcal{E}$. The known curve of solutions $\Theta = \{(\lambda, 0) | \lambda \in \mathbb{R}\}$ will henceforth be referred to as the trivial solutions. The closure of the set of nontrivial solutions of (2.5) will be denoted by Σ. A component of Σ is a maximal closed connected subset.

If there exist $\mu \in \mathbb{R}$ and $0 \neq v \in E$ such that $v = \mu L v$, μ is said to be a real characteristic value of L. The set of real characteristic values of L will be denoted by $\sigma(L)$. The multiplicity of $\mu \in \sigma(L)$ is the dimension of $\cup_{j=1}^{\infty} N((\mu L - I)^j)$ where I is the identity map on E and $N(P)$ denotes the null space of P. Since L is compact, μ is of finite multiplicity. It is well known that if $\mu \in \mathbb{R}$, a necessary condition for $(\mu, 0)$ to be a bifurcation point of (2.5) with respect to Θ is that $\mu \in \sigma(L)$.

Lemma 2.2 ([23])**.** *If $\mu \in \sigma(L)$ is simple, then Σ contains a component $\mathbf{C}_\mu$ that can be decomposed into two subcontinua $\mathbf{C}_\mu^+$, $\mathbf{C}_\mu^-$ such that for some neighborhood $\mathbf{B}$ of $(\mu, 0)$,*

$$(\lambda, u) \in \mathbf{C}_\mu^+ (\mathbf{C}_\mu^-) \cap \mathbf{B}, \quad \text{and} \quad (\lambda, u) \neq (\mu, 0)$$

implies $(\lambda, u) = (\lambda, \alpha v + w)$ where $\alpha > 0 (\alpha < 0)$ and $| \lambda - \mu | = o(1)$, $\|w\| = o(| \alpha |)$, at $\alpha = 0$.

Moreover, each of $\mathbf{C}_\mu^+$, $\mathbf{C}_\mu^-$ either

1. *meets infinity in Σ, or*
2. *meets $(\hat{\mu}, 0)$ where $\mu \neq \hat{\mu} \in \sigma(L)$, or*
3. *contains a pair of points (λ, u), $(\lambda, -\mu)$, $u \neq 0$.*

3. Spectral properties for linear impulsive equations

To apply Rabinowitz global bifurcation theorems on (1.1), we need to study the linear equation corresponding to (1.1). In this section the spectral properties including its eigenvalues, along with their corresponding algebraic multiplicity, and also eigenfunctions structures will be discussed for the linear impulsive problems

$$\begin{gathered} -x''(t) = \lambda a(t) x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{t=1/2} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{t=1/2} = -\beta_2 x(\frac{1}{2}), \\ x(0) = x(1) = 0, \end{gathered} \tag{3.1}$$

where λ, β_1, β_2, $a(t)$, Δx, $\Delta x'$ are introduced as in (1.1).

Denote $PC[0,1] = \{x : [0,1] \to \mathbb{R} : x(t)$ is continuous at $t \neq \frac{1}{2}$, and $x(\frac{1}{2} - 0) = \lim_{t \to \frac{1}{2}^-} x(t)$, and $x(\frac{1}{2} + 0) = \lim_{t \to \frac{1}{2}^+} x(t)$ exist $\}$ with the norm

$$\|x\| = \sup_{t \in [0,1]} |x(t)|,$$

and $PC'[0,1] = \{x \in PC[0,1] : x'(t)$ is continuous at $t \neq \frac{1}{2}$, and $x'(\frac{1}{2} - 0) = \lim_{t \to \frac{1}{2}^-} x'(t)$, and $x'(\frac{1}{2} + 0) = \lim_{t \to \frac{1}{2}^+} x'(t)$ exist $\}$ with the norm

$$\|x\|_1 = \max\{ \sup_{t \in [0,1]} |x(t)|, \sup_{t \in [0,1]} |x'(t)| \}.$$

Let $E = \{x \in PC'[0,1] : x(0) = x(1) = 0\}$. It is well known that E is a Banach space with the norm $\|\cdot\|_1$.

Lemma 3.1 ([9]). *The solution $x(t) \in PC[I,\mathbb{R}] \cap C^2[I',\mathbb{R}]$ of* (3.1) *is the same as the solution $x(t) \in PC'[I,\mathbb{R}]$ of the integral equation*

$$x(t) = \begin{cases} \lambda \int_0^1 G(t,s)a(s)x(s)ds - \big(\beta_1 - \frac{1}{2}\beta_2\big)tx(\frac{1}{2}), & t \in [0,\frac{1}{2}], \\ \lambda \int_0^1 G(t,s)a(s)x(s)ds + \big(\beta_1 + \frac{1}{2}\beta_2\big)(1-t)x(\frac{1}{2}), & t \in (\frac{1}{2},1], \end{cases} \tag{3.2}$$

where $I = [0,1]$, $I' = I \setminus \{\frac{1}{2}\}$, and

$$G(t,s) = \begin{cases} s(1-t), & 0 \le s \le t \le 1, \\ t(1-s), & 0 \le t \le s \le 1. \end{cases}$$

Lemma 3.2. *All eigenvalues of* (3.1) *are in $\mathbb{R}$.*

Proof. Let $\lambda = \alpha + i\gamma$ be an eigenvalue of (3.1) and $u(t) = u_1(t) + iu_2(t)$ be an eigenfunction corresponding to λ. It is sufficient to prove that $\gamma = 0$. Since $\lambda = \alpha + i\gamma$ and $u(t) = u_1(t) + iu_2(t)$ satisfy (3.1), it follows that

$$-u_1'' = a(t)(\alpha u_1 - \gamma u_2), \quad -u_2'' = a(t)(\alpha u_2 + \gamma u_1), \tag{3.3}$$

and

$$\begin{gathered} \Delta u_1|_{t=1/2} = \beta_1 u_1(\frac{1}{2}), \quad \Delta u_1'|_{t=1/2} = -\beta_2 u_1(\frac{1}{2}), \\ \Delta u_2|_{t=1/2} = \beta_1 u_2(\frac{1}{2}), \quad \Delta u_2'|_{t=1/2} = -\beta_2 u_2(\frac{1}{2}), \\ u_1(0) = u_1(1) = 0, \quad u_2(0) = u_2(1) = 0. \end{gathered} \tag{3.4}$$

Multiply (3.3) by $u_2(t)$ and $u_1(t)$ respectively and subtract each other to yield that

$$-u_1''u_2 + u_2''u_1 = -a(t)\gamma(u_1^2 + u_2^2).$$

Integrating the equation above from 0 to 1 and with condition (3.4), we have

$$\beta_1 u_1'(\frac{1}{2}-)u_2(\frac{1}{2}) - \beta_1 u_2'(\frac{1}{2}-)u_1(\frac{1}{2}) = -\int_0^1 a(t)\gamma(u_1^2 + u_2^2)dt. \tag{3.5}$$

Denote $F(t) = u_1'(t)u_2(t) - u_2'(t)u_1(t)$ for $t \in [0,\frac{1}{2})$. Since u_1, u_2 satisfy (3.3), it follows that

$$F'(t) = a(t)\gamma(u_1^2(t) + u_2^2(t)).$$

If $\gamma > 0$, by $a(t) > 0$ and $u_1^2(t) + u_2^2(t) > 0$, one has $F'(t) > 0$. Then $F(0) = u_1'(0)u_2(0) - u_2'(0)u_1(0) = 0$ implies that $F(\frac{1}{2}-) > 0$. The left side of (3.5) is $\beta_1 F(\frac{1}{2}-) > 0$, while the right side of it is negative, which is a contradiction. On the other hand, with the similar discussion, we deduce the contradiction if $\gamma < 0$. Therefore, $\gamma = 0$ and Lemma 3.2 holds. □

Theorem 3.3. *Equation* (3.1) *possesses an increasing sequence of eigenvalues*

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_k < \dots, \lim_{k\to+\infty} \lambda_k = +\infty.$$

And the eigenfunction u_k corresponding to λ_k has exactly $k-1$ nodal zeros in $(0,1)$.

The proof of this theorem will be divided into two parts, in which the methods chosen are different due to the complexity of the impulse. Part 1 mainly shows the existence of principal eigenvalue and eigenfunction by Schauder's fixed point theorem, while Part 2 guarantees the existence of second and subsequent eigenvalues by some tools on uncontinuous Sturm-Liouville systems.

Part 1. Existence of principal eigenvalue and eigenfunction In (3.2), let

$$y(t)=\begin{cases} x(t)+(\beta_1-\frac{1}{2}\beta_2)tx(\frac{1}{2}), & t\in[0,\frac{1}{2}],\\ x(t)-(\beta_1+\frac{1}{2}\beta_2)(1-t)x(\frac{1}{2}), & t\in(\frac{1}{2},1], \end{cases}$$

and then $y(t)\in C^1[0,1]$. Equation (3.2) is transformed into

$$y(t)=\lambda\int_0^1 G(t,s)a(s)H(y(s))ds, \tag{3.6}$$

where

$$H(y(t))=\begin{cases} y(t)-\frac{\beta_1-\frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1-\frac{1}{4}\beta_2}y(\frac{1}{2})t, & t\in[0,\frac{1}{2}],\\ y(t)+\frac{\beta_1+\frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1-\frac{1}{4}\beta_2}y(\frac{1}{2})(1-t), & t\in(\frac{1}{2},1]. \end{cases} \tag{3.7}$$

By Lemma 3.1, the study of (3.1) is the same as that of (3.6). Next we give the main result.

Lemma 3.4. *There exist $\lambda^*>0$ and nonnegative function x^* satisfying* (3.1).

Proof. Let $D:=\{y\in C[0,1]|\int_0^1 y(s)ds=1, y(t)\geq 0, y(t)$ is concave and $y(0)=y(1)=0\}$ and define an operator A:

$$(Ay)(t)=\frac{\int_0^1 G(t,s)a(s)H(y(s))ds}{\int_0^1\int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt},\quad \forall y\in D.$$

We show $(Ay)(t)>0$. In fact, by (3.7), the sign of $(Ay)(t)$ depends on the sign of $H(y(t))$. Since $y\in D$ and $\beta_1\geq\beta_2\geq 0$, it is easy to obtain that for $t\in(\frac{1}{2},1]$

$$y(t)+\frac{\beta_1+\frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1-\frac{1}{4}\beta_2}y(\frac{1}{2})(1-t)>0.$$

Define

$$g(t)=y(t)-\frac{\beta_1-\frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1-\frac{1}{4}\beta_2}y(\frac{1}{2})t,\quad t\in[0,\frac{1}{2}].$$

Since $y(t)$ is concave, together with $g(0)=y(0)\geq 0$ and

$$g(\frac{1}{2})=\frac{1}{1+\frac{1}{2}\beta_1-\frac{1}{4}\beta_2}y(\frac{1}{2})\geq 0,$$

one has $g(t)\geq 0$ for $t\in[0,\frac{1}{2}]$. That is to say $(Ay)(t)\geq 0$ for $t\in[0,1]$. Thus, we conclude that D is a closed convex set and an standard argument shows that $A:D\to D$ is continuous. Furthermore, the following two facts show that $A(D)$ is compact.

(1) $A(D)$ is uniformly bounded. For $y\in D$, since $\int_0^1 y(t)dt\leq\|y\|$, one has $\|y\|\geq 1$. Choose $y(t_0)=\max_{t\in[0,1]}\|y\|$. For $t\in(0,t_0)$, we have $y(t)=y(\frac{t}{t_0}t_0+(1-\frac{t}{t_0})0)\geq\frac{t}{t_0}y(t_0)+(1-\frac{t}{t_0})y(0)=\frac{t}{t_0}\|y\|$. The same argument shows that for $t\in(t_0,1)$, we have $y(t)\geq\frac{1-t}{1-t_0}\|y\|$. Hence,

$$y(t)\geq\frac{t(1-t)}{t_0(1-t_0)}\|y\|\geq t(1-t)\|y\|\geq t(1-t),\quad \forall t\in[0,1]$$

and

$$\int_0^1 y(s)ds\geq\int_0^1\|y\|s(1-s)ds=\|y\|\frac{1}{6},\ \text{i.e.}\|y\|\leq 6.$$

Next we estimate

$$\int_0^1 \int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt, \quad \forall y \in D.$$

As $a(t) > 0$ is continuous, suppose

$$0 < k \le a(t) \le K, t \in [0,1],$$

and denote

$$B_1 = \frac{\beta_1 - \frac{1}{2}\beta_2}{1 + \frac{1}{2}\beta_1 - \frac{1}{4}\beta_2}, B_2 = \frac{\beta_1 + \frac{1}{2}\beta_2}{1 + \frac{1}{2}\beta_1 - \frac{1}{4}\beta_2}.$$

With (3.7) and $y(t) \ge t(1-t)$ for $t \in [0,1]$, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt \\
&= \int_0^{1/2} (1-t) \int_0^t sa(s)\Big(y(s) - B_1 y(\frac{1}{2})s\Big)\,ds\,dt \\
&\quad + \int_0^{1/2} t \int_t^{1/2} (1-s)a(s)\left(y(s) - B_1 y(\frac{1}{2})s\right)\,ds\,dt \\
&\quad + \int_0^{1/2} t \int_{\frac{1}{2}}^1 (1-s)a(s)\left(y(s) + B_2 y(\frac{1}{2})(1-s)\right)\,ds\,dt \\
&\quad + \int_{\frac{1}{2}}^1 (1-t) \int_0^{1/2} sa(s)\left(y(s) - B_1 y(\frac{1}{2})s\right)\,ds\,dt \\
&\quad + \int_{\frac{1}{2}}^1 (1-t) \int_{\frac{1}{2}}^t sa(s)\left(y(s) + B_2 y(\frac{1}{2})(1-s)\right)\,ds\,dt \\
&\quad + \int_{\frac{1}{2}}^1 t \int_t^1 (1-s)a(s)\left(y(s) + B_2 y(\frac{1}{2})(1-s)\right)\,ds\,dt \\
&\ge k\Big[\frac{1}{60} + \frac{5}{384} y(\frac{1}{2})(B_2 - B_1)\Big],
\end{aligned}$$

where

$$B_2 - B_1 = \frac{\beta_2}{1 + \frac{1}{2}\beta_1 - \frac{1}{4}\beta_2} \ge 0$$

and $y \in D$. Hence,

$$\int_0^1 \int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt \ge \frac{k}{60}, \quad y \in D.$$

Since $G(t,s)$ is bounded on $[0,1] \times [0,1]$, by letting $0 \le G(t,s) \le M$, we have

$$\begin{aligned}
|(Ay)(t)| &\le \frac{60}{k}\Big|\int_0^1 G(t,s)a(s)H(y(s))ds\Big| \\
&\le \frac{60}{k} MK \Big|\int_0^1 H(y(s))ds\Big|,
\end{aligned}$$

where

$$\Big|\int_0^1 H(y(s))ds\Big| = \Big(1 + \frac{1}{8} y(\frac{1}{2})(B_2 - B_1)\Big).$$

That implies $A(D)$ is uniformly bounded.

(2) $A(D)$ is equicontinuous. In fact, as discussed in above, it's true for all $y \in D$ that
$$\int_0^1 \int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt \geq \frac{k}{60}.$$
Since $G(t,s)$ is continuous on $[0,1]\times[0,1]$, it is uniformly continuous on $[0,1]\times[0,1]$. So for all $\varepsilon > 0$, there exists $\delta > 0$ such that
$$|G(t,s) - G(t',s)| < C\varepsilon, \quad \forall |t-t'| < \delta,$$
where
$$C = \frac{k}{60K\left(1 + \frac{1}{8}y(\frac{1}{2})(B_2 - B_1)\right)}.$$
And then
$$\begin{aligned}
&|(Ay)(t) - (Ay)(t')| \\
&= \Big|\frac{\int_0^1 G(t,s)a(s)H(y(s))ds}{\int_0^1\int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt} - \frac{\int_0^1 G(t',s)a(s)H(y(s))ds}{\int_0^1\int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt}\Big| \\
&= \Big|\frac{\int_0^1 [G(t,s) - G(t',s)]a(s)H(y(s))ds}{\int_0^1\int_0^1 G(t,s)a(s)H(y(s))\,ds\,dt}\Big| \\
&\leq \frac{C\varepsilon \times 60K}{k}\int_0^1 H(y(s))ds = \varepsilon.
\end{aligned}$$

Arezela-Ascoli theorem and (1) (2) imply that $A(D)$ is compact. According to Schauder fixed point theorem, there exists a $y^* \in D$ satisfying $Ay^* = y^*$, i.e.
$$\frac{\int_0^1 G(t,s)a(s)H(y^*(s))ds}{\int_0^1\int_0^1 G(t,s)a(s)H(y^*(s))\,ds\,dt} = y^*(t). \tag{3.8}$$
Denote $\int_0^1\int_0^1 G(t,s)a(s)H(y^*(s))\,ds\,dt = \frac{1}{\lambda^*}$ and then (3.8) can be written as
$$y^* = \lambda^* \int_0^1 G(t,s)a(s)H(y^*(s))ds.$$
Note Lemma 3.1 and transformation
$$y(t) = \begin{cases} x(t) - \left(\beta_1 - \frac{1}{2}\beta_2\right) t x(\frac{1}{2}), & t \in [0,\frac{1}{2}], \\ x(t) - \left(\beta_1 + \frac{1}{2}\beta_2\right)(1-t)x(\frac{1}{2}), & t \in (\frac{1}{2},1], \end{cases}$$
we define
$$x^*(t) = \begin{cases} y^*(t) - \frac{\beta_1 - \frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1 - \frac{1}{4}\beta_2} y^*(\frac{1}{2})t, & t \in [0,\frac{1}{2}], \\ y^*(t) + \frac{\beta_1 + \frac{1}{2}\beta_2}{1+\frac{1}{2}\beta_1 - \frac{1}{4}\beta_2} y^*(\frac{1}{2})(1-t), & t \in (\frac{1}{2},1], \end{cases}$$
and such x^* satisfies Lemma 3.4.

Part 2. Existence of second and subsequent eigenvalues In this part, we consider uncontinuous Sturm-Liouville equation
$$-x''(t) = \lambda a x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \tag{3.9}$$
with the boundary value condition
$$x(0) = 0, x(1) = 0, \tag{3.10}$$

and the impulsive conditions

$$x(\frac{1}{2}+) = (1+\beta_1)x(\frac{1}{2}-), \tag{3.11}$$

$$x'(\frac{1}{2}+) = x'(\frac{1}{2}-) - \beta_2 x(\frac{1}{2}-). \tag{3.12}$$

For each $\lambda \in R$, let $\zeta_-(t,\lambda)$ and $\zeta_+(t,\lambda)$ be the solutions of (3.9) satisfying the initial condition

$$\begin{gathered} \zeta_-(0) = 0, \zeta_-'(0) = -1, \zeta_+(\frac{1}{2}+) = (1+\beta_1)\zeta_-(\frac{1}{2}-), \\ \zeta_+'(\frac{1}{2}+) = -\beta_2\zeta_-(\frac{1}{2}-) + \zeta_-'(\frac{1}{2}-), \end{gathered} \tag{3.13}$$

and let $\eta_+(t,\lambda)$ and $\eta_-(t,\lambda)$ be the solutions of (3.9) satisfying the initial condition

$$\begin{gathered} \eta_+(1) = 0, \quad \eta_+'(1) = -1, \quad \eta_-(\frac{1}{2}-) = \frac{1}{1+\beta_1}\eta_+(\frac{1}{2}+), \\ \eta_-'(\frac{1}{2}-) = \frac{1}{1+\beta_1}\Big[\beta_2\eta_+(\frac{1}{2}+) + (1+\beta_1)\eta_+'(\frac{1}{2}+)\Big]. \end{gathered} \tag{3.14}$$

Denote the solution of (3.9) with condition (3.13) and (3.14) respectively by

$$\zeta(t,\lambda) = \begin{cases} \zeta_-(t,\lambda), & t \in (0,\frac{1}{2}), \\ \zeta_+(t,\lambda), & t \in (\frac{1}{2},1), \end{cases} \quad \eta(t,\lambda) = \begin{cases} \eta_-(t,\lambda), & t \in (0,\frac{1}{2}), \\ \eta_+(t,\lambda), & t \in (\frac{1}{2},1). \end{cases}$$

□

By Lemma 3.2, we choose λ as a real number. When λ increases, the number of zeros of $\zeta(t,\lambda)$ increases subsequently in view of Lemma 2.1. Let $\rho_-(\lambda)$ be the number of zeros of $\zeta(t,\lambda)$ on $(0,\frac{1}{2})$, $\rho_+(\lambda)$ be the number of zeros of $\zeta(t,\lambda)$ on $(\frac{1}{2},1)$ and $\rho(\lambda) = \rho_-(\lambda) + \rho_+(\lambda)$ be the number of zeros of $\zeta(t,\lambda)$ on $(0,\frac{1}{2}) \cup (\frac{1}{2},1)$. Then we have the following conclusions.

Lemma 3.5.

(1) *There exists $A > 0$ such that $\rho(\lambda) = 0$ for all $\lambda < -A$;*
(2) $\lim_{\lambda\to+\infty} \rho(\lambda) = +\infty$

Proof. Denote $M_- = \min_{t\in[0,\frac{1}{2}]} a(t)$ and $M_+ = \min_{t\in[\frac{1}{2},1]} a(t)$.

(i) When $\lambda < 0$, we first consider the ODE initial value problem

$$\begin{gathered} x''(t) + \lambda M_- x(t) = 0, \quad t \in [0,\frac{1}{2}], \\ x(0,\lambda) = 0, x'(0,\lambda) = -1, \end{gathered} \tag{3.15}$$

and its solution can be written as

$$x_-(t,\lambda) = \frac{-1}{2\sqrt{-\lambda M_-}} e^{\sqrt{-\lambda M_-}t} + \frac{1}{2\sqrt{-\lambda M_-}} e^{-\sqrt{-\lambda M_-}t}.$$

The above expression implies that $x_-(t,\lambda) < 0$ as $\lambda \to -\infty$, which means $\rho_-(\lambda) = 0$ by comparison theorem of ODE.

Next, for the another ODE equation,

$$\begin{gathered} x''(t) + \lambda M_+ x(t) = 0, \quad t \in [\frac{1}{2},1], \\ x\big(\frac{1}{2}+,\lambda\big) = (1+\beta_1)x\big(\frac{1}{2}-,\lambda\big), x'\big(\frac{1}{2}+,\lambda\big) = x'\big(\frac{1}{2}-,\lambda\big) - \beta_2 x\big(\frac{1}{2}-,\lambda\big), \end{gathered} \tag{3.16}$$

the solution is

$$
\begin{aligned}
&x_+(t,\lambda)\\
&= \Big(\frac{1+\beta_1}{2}x_-\big(\frac{1}{2}-,\lambda\big) + \frac{-\beta_2 x_-\big(\frac{1}{2}-,\lambda\big)+x_-'\big(\frac{1}{2}-,\lambda\big)}{2\sqrt{-\lambda M_+}}\Big)e^{\sqrt{-\lambda M_+}(t-\frac{1}{2})}\\
&\quad + \Big(\frac{1+\beta_1}{2}x_-\big(\frac{1}{2}-,\lambda\big) - \frac{-\beta_2 x_-\big(\frac{1}{2}-,\lambda\big)+x_-'\big(\frac{1}{2}-,\lambda\big)}{2\sqrt{-\lambda M_+}}\Big)e^{-\sqrt{-\lambda M_+}(t-\frac{1}{2})},
\end{aligned}
$$

from which we see $x_+(t,\lambda)<0$ as $\lambda\to-\infty$ and we conclude that $\rho_+(\lambda)=0$.

(ii) When $\lambda>0$, denote $M=\min\{M_-,M_+\}$ and consider (3.15). A solution of it has the form

$$
x_-(t,\lambda)=\frac{-1}{\sqrt{\lambda M_-}}\sin\sqrt{\lambda M_-}t,
$$

of which the number of zeros on $(0,\frac{1}{2})$ is $[\frac{\frac{1}{2}-0}{\pi}\sqrt{\lambda M_-}]$ or $[\frac{\frac{1}{2}-0}{\pi}\sqrt{\lambda M_-}]+1$.

Then we turn to (3.16), while the solution is

$$
\begin{aligned}
x_+(t,\lambda) &= (1+\beta_1)x_-\big(\frac{1}{2}-,\lambda\big)\cos\sqrt{\lambda M_+}\big(t-\frac{1}{2}\big)\\
&\quad - \frac{1}{\sqrt{\lambda M_+}}\Big(-\beta_2 x_-\big(\frac{1}{2}-,\lambda\big)+x_-'\big(\frac{1}{2}-,\lambda\big)\Big)\sin\sqrt{\lambda M_+}\big(t-\frac{1}{2}\big)\\
&= C\sin\Big(\sqrt{\lambda M_+}t+\gamma\Big).
\end{aligned}
$$

We know the number of zeros of $x_+(t,\lambda)$ on $(\frac{1}{2},1)$ is $[\frac{1-\frac{1}{2}}{\pi}\sqrt{\lambda M_+}]$ or $[\frac{1-\frac{1}{2}}{\pi}\sqrt{\lambda M_+}]+1$. Notice that

$$
\big[\frac{\frac{1}{2}-0}{\pi}\sqrt{\lambda M_-}\big]+\big[\frac{1-\frac{1}{2}}{\pi}\sqrt{\lambda M_+}\big] > \big[\frac{\frac{1}{2}-0}{\pi}\sqrt{\lambda M}\big]+\big[\frac{1-\frac{1}{2}}{\pi}\sqrt{\lambda M}\big].
$$

By a comparison argument, when $\lambda>M$, the number of zeros of $\zeta(t,\lambda)$ is no less than $[\frac{\frac{1}{2}-0}{\pi}\sqrt{\lambda M}]+[\frac{1-\frac{1}{2}}{\pi}\sqrt{\lambda M}]$, which prove (2). We complete the proof. □

Denote by $\mathfrak{R}_k$ $(k=1,2,\dots)$ the point set on $\mathbb{R}$:

$$
\mathfrak{R}_1=\{\lambda\in\mathbb{R}:\rho(\lambda)=0\},\quad \mathfrak{R}_k=\{\lambda\in\mathbb{R}:\rho(\lambda)\ge k-1>0\}.
$$

According to Lemma 3.5 and the continuity of $\zeta_\pm(t,\lambda)$, $\mathfrak{R}_k$ $(k=1,2,\dots)$ are nonempty and $\mathfrak{R}_k(k>1)$ are closed sets with lower bounds.

Lemma 3.6. *Denote $\mu_k=\min\mathfrak{R}_k(k=2,\dots)$. If $\zeta(1,\mu_k)=0$, then $\zeta(t,\mu_k)$ has exactly $k-1$ zeros in $(0,\frac{1}{2})\cup(\frac{1}{2},1)$.*

Proof. Since $\mu_k=\min\mathfrak{R}_k(k=2,\dots)$, it follows that $\rho(\mu_k)\ge k-1$, which means that $\zeta(t,\mu_k)$ has at least $k-1$ zeros, denoting by $0<t_1<t_2<\dots<t_{k-1}<\cdots$. Now we prove $t_k=1$.

On the contrary, assume $t_k<1$. $\zeta'(t_k,\mu_k)=0$ is not possible in view of uniqueness of initial value problem due to $\zeta(t_k,\mu_k)=0$. Noticing that

$$
-\zeta_\lambda''=a(t)\zeta+\lambda a(t)\zeta_\lambda,\quad \zeta_\lambda(0,\lambda)=\zeta_\lambda'(0,\lambda)=0,
$$

and using the Green formula for $\zeta_\lambda(t,\mu_k)$ and $\zeta(t,\mu_k)$ on $[0,t_k]$, we have

$$
(1+\beta_1)\int_0^{1/2}a(t)\zeta^2(t,\mu_k)dt+\int_{\frac{1}{2}}^{t_k}a(t)\zeta^2(t,\mu_k)dt=\zeta_\lambda(t_k,\mu_k)\zeta'(t_k,\mu_k),
$$

hence

$$\zeta_\lambda(t_k,\mu_k)=\frac{(1+\beta_1)\int_0^{1/2}a(t)\zeta^2(t,\mu_k)dt+\int_{\frac{1}{2}}^{t_k}a(t)\zeta^2(t,\mu_k)dt}{\zeta'(t_k,\mu_k)}\neq 0.$$

It follows by the implicit function theorem that with $\zeta(t,\lambda)=0$, there exists a unique implicit function $t=t(\lambda)$ at the neighbourhood of (t_k,μ_k), which satisfies $t_k=t(\mu_k)$ and

$$\frac{dt}{d\lambda}\mid_{(t_k,\mu_k)}=-\frac{\zeta_\lambda(t_k,\mu_k)}{\zeta'(t_k,\mu_k)}=-\frac{(1+\beta_1)\int_0^{1/2}a(t)\zeta^2(t,\mu_k)dt+\int_{\frac{1}{2}}^{t_k}a(t)\zeta^2(t,\mu_k)dt}{\zeta'^2(t_k,\mu_k)}<0.$$

Namely, $t=t(\lambda)$ decreases with λ. Then we select $\lambda_*<\mu_k$ satisfying $t_*=t(\lambda_*)\in(t_k,b)$ such that $\zeta(t_*,\lambda_*)=0$. By Lemma 2.1, $\zeta(t,\lambda_*)$ has at least $k-1$ zeros in (a,t_*), which implies that $\lambda_*\in\mathfrak{R}_k$. This contradicts with $\mu_k=\min\mathfrak{R}_k$. Consequently, $t_k=1$ and the proof is complete. □

By Lemma 3.6 we deduce the next statements.

Corollary 3.7. $[\mu_k,\mu_{k+1})=\mathfrak{R}_k\backslash\mathfrak{R}_{k+1},\mathfrak{R}_k=\cup_{j=k}^\infty[\mu_j,\mu_{j+1})(k=2,\dots)$.

Corollary 3.8. *$\rho(\lambda)$ is an increased step function: $\rho(\lambda)=k-1(k=1,2,\dots),\lambda\in[\mu_k,\mu_{k+1})$.*

Proof of Theorem 3.3. First it will be proved that (3.1) has only one eigenvalue in each $[\mu_n,\mu_{n+1})(n=2,\dots)$, denoting by λ_n, and the corresponding eigenfunction $\zeta(t,\lambda_n)$ has exactly $n-1$ zeros in $(0,\frac{1}{2})\cup(\frac{1}{2},1)$. Because $\zeta(t,\mu_n)$ satisfies the boundary value condition $\zeta(1,\mu_n)=0$, it follows from Lemma 3.6 that $\lambda_n=\mu_n$.

Next we will show there is at most one eigenvalue of (3.1), denoting by λ_1, except for λ_n $(n=2,\dots)$ mentioned above. And then $\lambda_1\in\mathfrak{R}_1$ and the corresponding eigenfunction has no zero in $(0,1/2)\cup(1/2,1)$. Lemma 3.4 guarantees the existence of λ_1 with $\lambda_*=\lambda_1$, so we just need to show the uniqueness of λ_1. Assume there are two different principal eigenvalues: λ_1, λ_1'. Using the Green's formula on eigenfunctions $\zeta(t,\lambda_1)$ and $\zeta(t,\lambda_1')$, we have

$$\begin{aligned}&(\lambda_1-\lambda_1')\Big(\int_0^{1/2}\zeta(t,\lambda_1)\zeta(t,\lambda_1')dt+\frac{1}{1+\beta_1}\int_{\frac{1}{2}}^1\zeta(t,\lambda_1)\zeta(t,\lambda_1')dt\Big)\\&=\zeta(1,\lambda_1)\zeta'(1,\lambda_1')-\zeta'(1,\lambda_1)\zeta(1,\lambda_1')-\zeta(0,\lambda_1)\zeta'(0,\lambda_1')+\zeta'(0,\lambda_1)\zeta(0,\lambda_1').\end{aligned}$$

The left side of the above equation is not zero, while the right side is zero. That is a contradiction.

During the discussion, it is not difficult to find that zeros of $\zeta_-(t,\lambda)$ and $\zeta_+(t,\lambda)$ are nodal zeros by Sturm-Liouville theory in ODE. The proof is complete. □

We remark that result in Theorem 3.3 can be obtained for the equation

$$\begin{gathered}-x''(t)+q(t)x(t)=\lambda a(t)x(t),\quad t\in(0,1),\ t\neq\frac{1}{2},\\ \Delta x|_{t=1/2}=\beta_1x(\frac{1}{2}),\quad \Delta x'|_{t=1/2}=-\beta_2x(\frac{1}{2}),\\ x(0)=x(1)=0,\end{gathered}$$

where $q(t)\in C[0,1]$, $q(t)\geq 0($ or $q(t)\leq 0)$ and $a(t)$, $\Delta x|_{t=1/2}$ and $\Delta x'|_{t=1/2}$ are defined as before.

Lemma 3.9. *For each $k \geq 1$, the algebraic multiplicity of eigenvalue λ_k is 1.*

Proof. Define operator $K : PC'[0,1] \to PC'[0,1]$ as follows:

$$(Kx)(t) = \begin{cases} \int_0^1 G(t,s)a(s)x(s)ds - \frac{1}{\lambda}\big(\beta_1 - \frac{1}{2}\beta_2\big)tx(\frac{1}{2}), & t \in [0, \frac{1}{2}], \\ \int_0^1 G(t,s)a(s)x(s)ds + \frac{1}{\lambda}\big(\beta_1 + \frac{1}{2}\beta_2\big)(1-t)x(\frac{1}{2}), & t \in (\frac{1}{2}, 1]. \end{cases}$$

We just need to prove $\ker(I - \lambda_k K)^2 \subset \ker(I - \lambda_k K)$.

For any $y \in \ker(I - \lambda_k K)^2$, from $(I - \lambda_k K)^2 y = 0$, one has that $(I - \lambda_k K)y \in ker(I - \lambda_k K)$. Let λ_k be the k^{th} eigenvalue of (3.1) and $u_k(t)$ be the eigenfunction corresponding to λ_k. By Theorem 3.3, there exists a γ satisfying

$$(I - \lambda_k K)y = \gamma u_k(t), \quad t \in [0,1],$$

which implies that y satisfies

$$\begin{gathered} y'' + \lambda_k a(t)y + \gamma\lambda_k a(t)u_k(t) = 0, \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta y|_{t=1/2} = \beta_1 y(\frac{1}{2}), \quad \Delta y'|_{t=1/2} = -\beta_2 y(\frac{1}{2}), \\ y(0) = y(1) = 0. \end{gathered} \tag{3.17}$$

Now we prove $\gamma = 0$. Otherwise, assume $\gamma > 0$ and notice (3.17) when $k = 1$. With $\lambda_1 > 0$, $u_1(t) > 0$ for $t \in (0,1)$, two cases of y will be discussed.

If $y(t)$ has no zero in $(0,1)$, assume $y(t) > 0$ and rewrite (3.17) in the form

$$\begin{gathered} y'' + \lambda_1 a(t)\big[1 + \frac{\gamma u_1(t)}{y(t)}\big]y(t) = 0, \quad t \in (0,1), t \neq \frac{1}{2}, \\ \Delta y|_{t=1/2} = \beta_1 y(\frac{1}{2}), \quad \Delta y'|_{t=1/2} = -\beta_2 y(\frac{1}{2}), \\ y(0) = y(1) = 0. \end{gathered} \tag{3.18}$$

Inequality $1 + \frac{\gamma u_1(t)}{y(t)} > 1$, and Lemma 2.1, guarantee that $y(t)$ has a zero in $(0,1)$, which contradicts with the assumption.

If $y(t)$ has a zero in $(0,1)\backslash\frac{1}{2}$, denote it by t^*, so that $y(t) > 0$ for $t \in (0, t^*)$ and $y(t) < 0$ for $t \in (t^*, 1)$ (such t^* always exists because it is a nodal zero). So (3.17) on $(t^*, 1)$ can be transformed into (3.18). From $1 + \frac{\gamma u_1(t)}{y(t)} < 1$ and Lemma 2.1, $u_1(t)$ has a zero in $(t^*, 1)$, which yields a contradiction.

So $\gamma = 0$ and $y \in ker(I - \lambda_k K)$. We complete the proof. $\square$

4. Multiple solutions for nonlinear impulsive differential Equation

In this section, we consider the problem

$$\begin{gathered} -x''(t) + f(t,x) = \lambda a x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{t=1/2} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{t=1/2} = -\beta_2 x(\frac{1}{2}), \\ x(0) = x(1) = 0, \end{gathered} \tag{4.1}$$

where $\lambda \neq 0$, $\beta_1 \geq \beta_2 \geq 0$, $\Delta x|_{t=1/2} = x(\frac{1}{2}+) - x(\frac{1}{2}-)$, $\Delta x'|_{t=1/2} = x'(\frac{1}{2}+) - x'(\frac{1}{2}-)$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$, $a : [0,1] \to (0, +\infty)$ are continuous. In addition, suppose the following assumptions of $f(t,x)$ hold.

$$f(t,0) = 0, f(t,s) = o(s) \tag{4.2}$$

at the neighbourhood of 0 and uniformly for all $t \in [0,1]$.

$$s \mapsto \frac{f(t,s)}{s}, \quad s \mapsto -\frac{f(t,-s)}{s} \tag{4.3}$$

strictly increase on $\mathbb{R}^+$, for all $t \in [0,1]$.

$$\lim_{s\to\pm\infty} \frac{f(t,s)}{s} = +\infty, \tag{4.4}$$

uniformly for all $t \in [0,1]$. Denote λ_i be the i^{th} eigenvalue of problem

$$\begin{gathered}
-\phi''(t) = \lambda a(t)\phi(t), \quad t \in (0,1), \quad t \neq \frac{1}{2}, \\
\Delta\phi|_{t=1/2} = \beta_1\phi(\frac{1}{2}), \quad \Delta\phi'|_{t=1/2} = -\beta_2\phi(\frac{1}{2}), \\
\phi(0) = \phi(1) = 0.
\end{gathered} \tag{4.5}$$

For the rest of this article, we suppose that the initial value problem

$$\begin{gathered}
-x''(t) + f(t,x) = \lambda a x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\
\Delta x|_{t=1/2} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{t=1/2} = -\beta_2 x(\frac{1}{2}), \\
x(t_0) = x'(t_0) = 0,
\end{gathered}$$

has the unique trivial solution $x \equiv 0$ on $[0,1]$, for any $t_0 \in [0,1]$.

Before giving main results, it is necessary to give the lower and upper solution theorem of (4.1), as a useful tool in the proof of the existence of positive solutions. Now we consider

$$\begin{gathered}
-x''(t) + f(t,x) = \lambda a x(t), \quad t \in (0,1),\ t \neq t_1, \\
\Delta x|_{t_1} = \beta_1 x(t_1), \quad \Delta x'|_{t_1} = -\beta_2 x(t_1), \\
x(0) = x(1) = 0,
\end{gathered} \tag{4.6}$$

where $f : D \to R$ is continuous, $D \subseteq [0,1] \times R$ and $t_1 \in (0,1)$. The solution of (4.6) is defined as $x(t) \in PC([0,1],R) \cap PC''((0,1),R)$ satisfying (4.6), where $PC([0,1],R) = \{x : [0,1] \to R : x(t) \text{ is continuous at } t \neq t_1,\ x(t_1-0) = \lim_{t\to t_1^-} x(t)$ and $x(t_1+0) = \lim_{t\to t_1^+} x(t)$ exist $\}$, and $PC''((0,1),R) = \{x : (0,1) \to R : x''(t)$ is continuous at $t \neq t_1$, $\lim_{t\to t_1^-} x''(t)$ and $\lim_{t\to t_1^+} x''(t)$ exist $\}$.

If $\alpha(t) \in PC([0,1],R) \cap PC''((0,1),R)$ satisfies

$$\begin{gathered}
-\alpha''(t) + f(t,\alpha(t)) \leq \lambda a\alpha(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\
\Delta\alpha|_{t_1} = \beta_1\alpha(t_1), \quad \Delta\alpha'|_{t_1} = -\beta_2\alpha(t_1), \\
\alpha(0) \leq 0, \alpha(1) \leq 0,
\end{gathered}$$

it is a lower solution of (4.6), while the upper solution is defined as $\beta(t)$ satisfying

$$\begin{gathered}
-\beta''(t) + f(t,\beta(t)) \geq \lambda a\beta(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\
\Delta\beta|_{t_1} = \beta_1\beta(t_1), \quad \Delta\beta'|_{t_1} = -\beta_2\beta(t_1), \\
\beta(0) \geq 0, \beta(1) \geq 0.
\end{gathered}$$

Denote

$$D_\alpha^\beta = \{(t,x) \in (0,1) \times R, \alpha(t) \leq x \leq \beta(t), t \in (0,1)\},$$

where $\alpha(t)$, $\beta(t) \in PC([0,1], R)$ with $\alpha(t) \le \beta(t)$ for all $t \in [0,1]$. Then we give the lower and upper solution theorem of (4.6).

Lemma 4.1. *Assume $\alpha(t)$, $\beta(t)$ are the lower and upper solutions respectively with $\alpha(t) \le \beta(t)$ for $t \in (0,1)$, and $D_\alpha^\beta \subseteq D$. Moreover, for each $\lambda \in R$ there exists $h(t,\lambda) \in C([0,1] \times R, R^+)$ satisfying*

$$|H(t,x,\lambda)| := |f(t,x) - \lambda a(t)x(t)| \le h(t,\lambda), \quad \forall (t,x) \in D_\alpha^\beta,$$

$$\int_0^1 s(1-s)h(s,\lambda) < g(\lambda) < +\infty.$$

Then there is at least one solution of (4.6) *denoted by $\bar{x(t)}$ with $\alpha(t) \le \bar{x(t)} \le \beta(t)$.*

Proof. Define auxiliary functions

$$H^*(t,x,\lambda) = \begin{cases} H(t,\alpha(t),\lambda), & x < \alpha(t), \\ H(t,x,\lambda), & \alpha(t) \le x(t) \le \beta(t), \\ H(t,\beta(t),\lambda), & x > \beta(t), \end{cases}$$

where $H(t,x,\lambda) = f(t,x) - \lambda a(t)x(t)$, and

$$I^*(x) = \begin{cases} I(\alpha(t_1)), & x < \alpha(t_1), \\ I(x), & \alpha(t_1) \le x \le \beta(t_1), \\ I(\beta(t_1)), & x > \beta(t_1), \end{cases}$$

where $I(x) = x(t_1)$. It is obvious that

$$|H^*(t,x,\lambda)| < h(t,\lambda), \quad \forall (t,x) \in (0,1) \times R.$$

Then we consider problem

$$\begin{gathered} -x''(t) + H^*(t,x,\lambda) = 0, \quad t \in (0,1),\ t \ne t_1, \\ \Delta x|_{t_1} = \beta_1 I^*(x(t_1)), \quad \Delta x'|_{t_1} = -\beta_2 I^*(x(t_1)), \\ x(0) = x(1) = 0, \end{gathered} \tag{4.7}$$

and show that if $x(t)$ is a solution of (4.7), $\alpha(t) \le x(t) \le \beta(t)$ holds, which implies that $x(t)$ is a solution of (4.6).

On the contrary, if there is a t^* such that $x(t^*) < \alpha(t^*)$, there are three possible cases for t^*.

(1) Assume $t^* < t_1$. Let $r = \inf\{t < t^* : x(s) < \alpha(s), \forall s \in [t,t^*], t \in (0,1)\}$ and $r' = \sup\{t > t^* : x(s) < \alpha(s), \forall s \in [t^*,t], t \in (0,1)\}$. We discuss two cases about r'.

(a) There exists a $t' \in (t^*, t_1]$ satisfying $x(t') = \alpha(t')$. In this case, it's obvious that $r' \le t_1$. Then for every $t \in (r,r')$,

$$x(r) = \alpha(r), x(r') = \alpha(r'), x(t) < \alpha(t), H^*(t,x,\lambda) = H(t,\alpha(t),\lambda).$$

Thus, $x(t)$ satisfies

$$-x''(t) + H(t,\alpha(t),\lambda) = 0, \quad t \in (r,r').$$

On the other hand, since $\alpha(t)$ is a lower solution with

$$-\alpha''(t) + H(t,\alpha(t),\lambda) \le 0, \quad t \in (r,r'),$$

by denoting $z(t) = \alpha(t) - x(t)$, for every $t \in (r,r')$, one has

$$z(r) = z(r') = 0, z''(t) \ge 0.$$

By maximum principle, $z(t) \le 0$ holds for all $t \in (r, r')$. Namely for $t \in (r, r')$, $\alpha(t) \le x(t)$, which contradicts $\alpha(t^*) > x(t^*)$.

(b) For every $t \in [t^*, t_1]$, $x(t) < \alpha(t)$ holds. In this case, noticing $\alpha(t_1 + 0) = \alpha(t_1) + \beta_1\alpha(t_1) > x(t_1) + \beta_1\alpha(t_1) = x(t_1) + \beta_1 I^*(x(t_1)) = x(t_1 + 0)$, we have $r' > t_1$ and for every $t \in (r, r')$,

$$x(r) = \alpha(r), x(r') = \alpha(r'), x(t) < \alpha(t), H^*(t, x, \lambda) = H(t, \alpha(t), \lambda).$$

Since $x(t)$ and $\alpha(t)$ satisfy

$$\begin{aligned} -x''(t) + H(t, \alpha(t), \lambda) = 0, \quad t \in (r, r') \setminus t_1, \\ -\alpha''(t) + H(t, \alpha(t), \lambda) \le 0, \quad t \in (r, r') \setminus t_1, \end{aligned}$$

by letting $z(t) = \alpha(t) - x(t)$, for $t \in (r, r')$, it holds that

$$\begin{gathered} -z''(t) \ge 0, \quad t \in (r, r'),\ t \ne t_1, \\ \Delta z|_{t_1} = 0, \quad \Delta z'|_{t_1} = 0, \\ z(r) = z(r') = 0. \end{gathered}$$

Since $-z''(t) \ge 0$ for $t \in (r, r') \setminus t_1$ and $\Delta z'|_{t_1} = 0$, $z'(t)$ increases and is continuous on (r, r'). Lagrange mean value theorem, together with $z(r) = 0$ and $z(t_1) > 0$ guarantees that there exists a $t' \in (r, t_1)$ with $z'(t') > 0$. Similarly, with $z(t_1 + 0) > 0$ and $z(r') = 0$, there exists a $t'' \in (t_1, r')$ so that $z'(t'') < 0$. This contradicts to the monotonicity of $z'(t)$. According to the discussion above, we find case (1) impossible.

(2) Assume $t^* = t_1$. As before, define $r = \inf\{t < t^* : x(s) < \alpha(s), \forall s \in [t, t^*], t \in (0, 1)\}$ and $r' = \sup\{t > t^* : x(s) < \alpha(s), \forall s \in [t^*, t], t \in (0, 1)\}$. The conclusion on (r, r') is same with the case (b) in (1), which can be excluded.

(3) Assume $t^* > t_1$. The definitions of r, r' are same as before. We divide the discussion of r into two cases.

(a') There exists a $t' \in (t_1, t^*]$ satisfying $x(t') = \alpha(t')$ or $\alpha(t_1 + 0) = x(t_1 + 0)$. This case is similar to the (a) in case (1), which deduces a contradiction.

(b') For every $t \in (t_1, t^*]$, $x(t) < \alpha(t)$ holds, and so $x(t_1 + 0) = x(t_1) + \beta_1 I^*(x(t_1)) < \alpha(t_1 + 0) = \alpha(t_1) + \beta_1 I(\alpha(t_1))$. Then it can be concluded that $x(t_1) < \alpha(t_1)$. This case is similar to (b) in case (1), which is impossible.

Combining cases (1), (2) and (3), we conclude that for $t \in [0, 1]$, $x(t) \ge \alpha(t)$. With the same method, $x(t) \le \beta(t)$ is true for $t \in [0, 1]$. The fact is that if $x(t)$ is the solution of (4.7), it must be the solution of (4.6), which has been proved. Next we will give the existence of solutions for (4.7).

The solution of (4.7) has the form (see [9])

$$\begin{aligned} x(t) = &-\int_0^1 G(t, s)H^*(s, x(s), \lambda)ds + [\beta_1 - (t - t_1\beta_2)] \sum_{0<t_1<t} I^*(x(t_1)) \\ &- t[\beta_1 - (1 - t_1)\beta_2]I^*(x(t_1)), \quad t \in [0, 1], \end{aligned}$$

where

$$G(t, s) = \begin{cases} s(1 - t), & 0 \le s \le t \le 1, \\ t(1 - s), & 0 \le t \le s \le 1. \end{cases}$$

For $x \in PC([0,1],R)$ and $\lambda \in R$, define

$$\begin{aligned}(T_\lambda x)(t) = &-\int_0^1 G(t,s)H^*(s,x(s),\lambda)ds + [\beta_1 - (t-t_1\beta_2)]\sum_{0<t_1<t} I^*(x(t_1))\\ &- t[\beta_1 - (1-t_1)\beta_2]I^*(x(t_1)).\end{aligned}$$

From the dominated convergence theorem, we can check that $T_\lambda : PC([0,1],R) \to PC([0,1],R)$ is a continuous and bounded operator for each $\lambda \in R$. Furthermore, $T_\lambda(PC([0,1],R))$ is compact. Indeed, for $t \in [0,1]\backslash t_1$, we have

$$\begin{aligned}&\big|\frac{d}{dt}(T_\lambda x)(t))\big|\\ &\le \big|\frac{d}{dt}\int_0^t s(1-t)H^* ds\big| + \big|\frac{d}{dt}\int_t^1 t(1-s)H^* ds\big| + M|I^*(x(t_1))|\\ &\le \int_0^t s|H^*(s,x(s),\lambda)|ds + \int_t^1 (1-s)|H^*(s,x(s),\lambda)|ds + M|I^*(x(t_1))|\\ &= \gamma_\lambda(t) + M|I^*(x(t_1))|,\end{aligned}$$

and

$$\begin{aligned}\int_0^1 \gamma_\lambda(s)ds &\le \lim_{t\to 1-}(1-t)\int_0^t sh(s,\lambda)ds + \lim_{t\to 1+}\int_t^1 (1-s)h(s,\lambda)ds\\ &\quad + 2\int_0^1 s(1-s)h(s,\lambda)ds\\ &\le 4\int_0^1 s(1-s)h(s,\lambda)ds\\ &\le g(\lambda) < +\infty,\end{aligned}$$

so that for each $\lambda \in R$, $\gamma_\lambda(t) \in L^1([0,1],R^+)$. According to Schauder's fixed point theorem, T_λ has at least a fixed point $\overline{x} \in PC[0,1]$, which implies $\overline{x}(t)$ is a solution of (4.7). The proof is complete. □

Theorem 4.2. *Assume that conditions* (4.2), (4.3) *and* (4.4) *hold. For* $\lambda > \lambda_1$, *there exists an unique positive solution of* (4.1). *The mapping* $\lambda \mapsto x_\lambda$ *is continuous from* $(\lambda_1, +\infty)$ *to* $PC''([0,1],R)$ *and the branch* $\{(\lambda, x_\lambda), \lambda \in (\lambda_1, +\infty)\}$ *bifurcates from the right of trivial solutions at* $(\lambda_1, 0)$. *Moreover,* x_λ *increases strictly with* λ: *if* $\lambda < \mu$, *then* $x_\lambda < x_\mu$. *Finally,* $x_\lambda \to +\infty$, *uniformly for* t *in the closed interval* $\subseteq (0,1)$ *as* $\lambda \to +\infty$.

Proof. For $\rho \in C([0,1],R^+)$, denote $\nu_i(\rho)$ the i^{th} eigenvalue of problem

$$\begin{gathered}-\phi''(t) + \rho\phi = \nu a(t)\phi(t), \quad t \in (0,1),\ t \ne \frac{1}{2},\\ \Delta\phi|_{t=1/2} = \beta_1\phi(\frac{1}{2}), \quad \Delta\phi'|_{t=1/2} = -\beta_2\phi(\frac{1}{2}),\\ \phi(0) = \phi(1) = 0.\end{gathered} \tag{4.8}$$

Then $\lambda_i = \nu_i(0)$. By Lemma 2.1, if $0 < \rho \le \widehat{\rho}$ and $\rho \not\equiv \widehat{\rho}$, then

$$\nu_i(\rho) < \nu_i(\widehat{\rho}), \quad \forall i \in \mathbb{N}.$$

It is true that there is no nontrivial solution of (4.1), if $\lambda \le \lambda_1$. Indeed, on the contrary, let $x \ge 0$ be a solution of (4.1), and then x satisfies (4.8) with

$\rho(t) = \frac{f(t,x(t))}{x(t)}$ if $x(t) > 0$, $\rho(t) = 0$ if $x(t) = 0$, and $\nu = \lambda$. By condition (4.3), $\rho \geq 0$ and $\rho \not\equiv 0$ hold, so that $\lambda = \nu_1(\rho) > \nu_1(0) = \lambda_1$, which is a contradiction.

We show the existence of solutions of (4.1) for $\lambda > \lambda_1$. Let $\phi_1(t)$ be the prime eigenfunction of (4.5) corresponding to λ_1, so $\phi_1(t) > 0$ for $t \in (0,1)$. First, it will be verified that, for sufficiently small $\varepsilon > 0$, $\varepsilon\phi_1$ is a lower solution of (4.1). Indeed, with $\lambda > \lambda_1$ and condition (4.2), for sufficiently small ε, we have

$$\begin{aligned} -\varepsilon\phi_1''(t) + f(t,\varepsilon\phi_1(t)) - \lambda a(t)\varepsilon\phi_1(t) &= \lambda_1 a(t)\varepsilon\phi_1(t) - \lambda a(t)\varepsilon\phi_1(t) + f(t,\varepsilon\phi_1(t)) \\ &= \Big[(\lambda_1 - \lambda)a(t) + \frac{f(t,\varepsilon\phi_1(t))}{\varepsilon\phi_1(t)}\Big]\varepsilon\phi_1(t) < 0, \end{aligned}$$

and $\Delta\varepsilon\phi_1|_{t=1/2} = \beta_1\varepsilon\phi_1(\frac{1}{2})$, $\Delta(\varepsilon\phi_1)'|_{t=1/2} = -\beta_2\varepsilon\phi_1(\frac{1}{2})$, and $\varepsilon\phi_1(0) = \varepsilon\phi_1(1) = 0$, which infer that $\varepsilon\phi_1$ is a lower solution. Next we will find a upper solution of (4.1). Condition (4.4) implies that there exists a sufficiently grand constant M_λ for each $\lambda \in R$, so that $f(t,s) \geq \lambda a(t)s$ for $s > M_\lambda$ and $t \in [0,1]$. Define

$$x^* = \begin{cases} C_1 t + C_2, & t \in [0,\frac{1}{2}], \\ C_3 t + C_4, & t \in (\frac{1}{2}, 1], \end{cases}$$

where C_1, C_2, C_3 and C_4 can be selected by the following conditions:

$$\begin{gathered} x^*(\frac{1}{2}+) = (1+\beta_1)x^*(\frac{1}{2}-), \\ (x^*)'(\frac{1}{2}+) = (x^*)'(\frac{1}{2}-) - \beta_2 x^*(\frac{1}{2}-), \\ x^*(t) > M_\lambda, \quad \forall t \in [0,1]. \end{gathered}$$

Such x^* defined above satisfies

$$\begin{gathered} -(x^*)''(t) + f(t,x^*) \geq \lambda a x^*(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x^*|_{t=1/2} = \beta_1 x^*(\frac{1}{2}), \quad \Delta(x^*)'|_{t=1/2} = -\beta_2 x^*(\frac{1}{2}), \\ x^*(0) \geq 0, x^*(1) \geq 0, \end{gathered}$$

which means x^* a upper solution of (4.1). It is easy to check that Lemma 4.1 holds, so there exists a positive solution of (4.1) between $\varepsilon\phi_1$ and x^*.

Let x_λ be the maximum positive solution of (4.1) satisfying $\varepsilon\phi_1 \leq x(t) \leq x^*$. Then x_λ is unique. Indeed, assume x be another positive solution of (4.1) with $x < x_\lambda$. Combining this with condition (4.3), we have

$$\frac{f(\cdot,x)}{x} < \frac{f(\cdot,x_\lambda)}{x_\lambda},$$

while

$$\lambda = \nu_1\Big(\frac{f(\cdot,x)}{x}\Big) = \nu_1\Big(\frac{f(\cdot,x_\lambda)}{x_\lambda}\Big).$$

This is a contradiction.

Assume $\lambda_1 < \lambda < \mu$ and x_λ, x_μ be the positive solutions of (4.1) corresponding to λ, μ respectively. Then

$$\begin{aligned} -x_\lambda''(t) + f(t,x_\lambda) - \mu a x_\lambda(t) &= \lambda a(t)x_\lambda(t) - \mu a(t)x_\lambda(t) \\ &= (\lambda - \mu)a(t)x_\lambda(t) < 0, \end{aligned}$$

which indicates that x_λ is a lower solution of (4.1) corresponding to μ. By the uniqueness of solutions, one has $x_\lambda < x_\mu$, namely the monotone of x_λ with λ. The prior estimation of M_λ and uniqueness of solution of (4.1), along with Lemma 2.2, mean that the mapping of $\lambda \mapsto x_\lambda$ is continuous from $(\lambda_1, +\infty)$ to $PC''([0,1], R)$ and the branch of $\{(\lambda, x_\lambda), \lambda \in (\lambda_1, +\infty)\}$ bifurcates from the right of trivial solutions at $(\lambda_1, 0)$.

Then we prove $x_\lambda \to +\infty$, as $\lambda \to +\infty$ uniformly for t in the closed interval $\subseteq (0,1)$. For all $\gamma > 0$, define

$$\Lambda(\gamma) = \lambda_1 \|a\|_{C^0[0,1]} + \max_{t\in[0,1]} \frac{f(t,\gamma)}{\gamma}.$$

Then for all $\gamma > 0$, $\Lambda(\gamma) < +\infty$ and $\Lambda(\gamma)$ increases with γ. Here ϕ_1 is also the prime eigenfunction of (4.5) and satisfies $\|\phi_1\|_1 = 1$. For all $\lambda \geq \Lambda(\gamma)$, $\gamma\phi_1$ is a lower solution of (4.1). In fact, condition (4.3) implies that $f(t, \gamma\phi_1) \leq (f(t,\gamma) \setminus \gamma)\gamma\phi_1$, by which we compute

$$-(\gamma\phi_1)'' + f(t, \gamma\phi_1) \leq \lambda_1 a(t) + \frac{f(t,\gamma)}{\gamma}\gamma\phi_1 \leq \Lambda(\gamma)\gamma\phi_1 \leq \lambda\gamma\phi_1.$$

So $\gamma\phi_1$ is a lower solution of (4.1) with $\gamma\phi_1 \leq x_\lambda$, and for all $\gamma > 0$ there exists $\Lambda(\gamma) > 0$ telling that

$$\lambda \geq \Lambda(\gamma) \Rightarrow \gamma\phi_1 \leq x_\lambda, \quad t \in (0,1).$$

Since $\phi_1 > 0$ for $t \in (0,1)$, it can be deduced that $x_\lambda \to +\infty$, as $\lambda \to +\infty$ uniformly for t in the closed interval $\subseteq (0,1)$. The proof is complete. $\square$

For convenience of the next proof, we give some symbols here. Let $E = \{x \in PC'[0,1] : x(0) = x(1) = 0\}$. It is well known that E is a Banach space with the norm $\|\cdot\|_1$. Denote S_k be the set of functions in E which have exactly $k-1$ simple nodal zeros in $(0,1)$ (by a nodal zero we mean the function changes sign at the zeros and at a simple nodal zero, the derivative of the function is nonzero) and denote

$$S_k^+ = \{x \in S_k; x'(0) > 0\}, \quad S_k^- = -S_k^+.$$

Finally, let $\Phi_k^\pm = \mathbb{R} \times S_k^\pm$ and $\Phi_k = \mathbb{R} \times S_k$. It is easy to show that for any positive integer k, S_k, S_k^+ and S_k^- are open in E. With the fact that (4.5) is the linear equation of (4.1) at the neighbourhood of $x = 0$, global bifurcation theorem of Rabinowitz can be applied to (4.1).

Theorem 4.3. *Suppose that f satisfies conditions* (4.2), (4.3) *and* (4.4). *Then for $\lambda \leq \lambda_1$, Equation* (4.1) *admits trivial solution $x_0 = 0$. For $\lambda_k < \lambda \leq \lambda_{k+1}$, Equation* (4.1) *possesses at least $2k+1$ solutions: $x_0 = 0$, $x_1^\pm, \dots x_k^\pm$, with $x_j^+ \in S_j^+$ and $x_j^- \in S_j^-$.*

Proof. The existence of a solution $x \in S_k^\pm$ for $\lambda > \lambda_k$ is obtained easily. In fact, $x(t)$ is a solution of (4.1) if and only if $x(t)$ is a solution of equation $x = \lambda Lx + H(t,x)$, where L is defined as

$$(Lx)(t) = \begin{cases} \int_0^1 G(t,s)a(s)x(s)ds - \frac{1}{\lambda}\big(\beta_1 - \frac{1}{2}\beta_2\big)tx(\frac{1}{2}), & t \in [0, \frac{1}{2}], \\ \int_0^1 G(t,s)a(s)x(s)ds + \frac{1}{\lambda}\big(\beta_1 + \frac{1}{2}\beta_2\big)(1-t)x(\frac{1}{2}), & t \in (\frac{1}{2}, 1], \end{cases}$$

and $H(t,x) = \int_0^1 G(t,s)f(s,x(s))ds$ satisfying $\lim_{x\to 0} H(t,x)/x = 0$.

The global bifurcation theorem of Rabinowitz can be applied for this problem and we just need to prove $\mathbf{C}_k^{\pm} \cap (\lambda \times E) \neq \emptyset$. By Lemma 2.2 and Lemma 3.9, there exists one unbounded continua $\mathbf{C}_k$ of solutions of (4.1), which bifurcates from the right of the trivial solutions at $(\lambda_k, 0)$ and satisfies

$$(\lambda_k, 0) \in \mathbf{C}_k \subset \Phi_k \cup \{(\lambda_k, 0)\}.$$

Moreover, $\mathbf{C}_k = \mathbf{C}_k^+ \cup \mathbf{C}_k^-$ and $\mathbf{C}_k^+ \cap \mathbf{C}_k^- = (\lambda_k, 0)$, while $\mathbf{C}_k^{\pm}$ are two unbounded continua in $R \times E$ satisfying

$$\mathbf{C}_k^{\pm} \subset \Phi_k^{\pm} \cup \{(\lambda_k, 0)\}.$$

By Theorem 4.2, we know that $\|x\|_1 \leq M(\lambda)$ so that $\mathbf{C}_k^{\pm} \subset \{(\lambda, x); \lambda \geq \lambda_k, \|x\|_1 \leq M(\lambda)\}$ and the projection of $\mathbf{C}_k^{\pm}$ on R is unbounded belonging to $(\lambda_k, +\infty)$. So $\mathbf{C}_k^{\pm} \cap (\lambda \times E) \neq \emptyset$, and there exists at least $2k+1$ solutions of (4.1) in $S_k^{\pm}$ for $\lambda \geq \lambda_k$. □

5. Behavior of positive solutions of autonomous impulsive differential equation for large λ

First, we study the existence of the positive solutions to

$$\begin{gathered} -x''(t) + x^{p+1} = \lambda m x(t), \quad t \in (0,1),\ t \neq \frac{1}{2}, \\ \Delta x|_{\frac{1}{2}} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{\frac{1}{2}} = -\beta_2 x(\frac{1}{2}), \\ x(0) = x(1) = 0, \end{gathered} \tag{5.1}$$

where $p > 0$, $m > 0$ are real numbers. This is a special case of (1.1) with $f(t,x) = x^{p+1}$ and $a(t) = m$, so we can use the results in section 4 on it. The behavior of the positive solutions of (5.1), as $\lambda \to \infty$, will be discussed, on the basis of the comparison argument and the property of principal eigenvalue of linear equation of (5.1).

By Theorem 3.3, we denote σ_i the i^{th} eigenvalue of

$$\begin{gathered} -x''(t) = \lambda x(t), \quad t \in (0,1), \quad t \neq \frac{1}{2}, \\ \Delta x|_{\frac{1}{2}} = \beta_1 x(\frac{1}{2}), \quad \Delta x'|_{\frac{1}{2}} = -\beta_2 x(\frac{1}{2}), \\ x(0) = x(1) = 0. \end{gathered} \tag{5.2}$$

From Theorem 4.2, (5.1) admits a positive solution, if and only if $\lambda > \sigma_1/m$. Let $u_{\lambda,m}$ be such positive solution. In order to study the behavior of $u_{\lambda,m}$ as $\lambda \to \infty$, we use the change of variable

$$u = \lambda^{1/p} v, \tag{5.3}$$

which transforms (5.1) into

$$\begin{gathered} -\frac{1}{\lambda} v'' = mv - v^{p+1}, \quad t \in (0,1), \quad t \neq \frac{1}{2}, \\ \Delta v|_{\frac{1}{2}} = \beta_1 v(\frac{1}{2}), \quad \Delta v'|_{\frac{1}{2}} = -\beta_2 v(\frac{1}{2}), \\ v(0) = v(1) = 0. \end{gathered} \tag{5.4}$$

So the problem of analyzing the behavior of $u_{\lambda,m}$ as $\lambda \to \infty$ is equivalent to analyzing the behavior of the unique positive solution of (5.4), which is denoted by $\theta_{\lambda,m}$, as $\lambda \to \infty$. The behavior of $\theta_{\lambda,m}$ in the interior of $[0,1]$ is given by the following conclusion.

Theorem 5.1. *Let $J \subset (0,1)$ be a compact interval. Then*

$$\lim_{\lambda\to+\infty} \theta_{\lambda,m} = m^{1/p} \tag{5.5}$$

uniformly in J.

Proof. Given $\lambda > \frac{\sigma_1}{m}$, $-\theta''_{\lambda,m}(t_0) \geq 0$ at the point $t_0 \in I$ where $\theta_{\lambda,m}$ achieves its maximum and hence

$$\theta_{\lambda,m} \leq m^{1/p}, \quad t \in I. \tag{5.6}$$

Then it is sufficient to show that given $\varepsilon > 0$, there exists $\lambda(\varepsilon)$ such that

$$m^{1/p} - \varepsilon \leq \theta_{\lambda,m}(t), \quad t \in J, \tag{5.7}$$

for all $\lambda \geq \lambda(\varepsilon)$. To show this, we discuss as follows. Consider $t_0 \in J'$ and $J' = [\alpha, \beta] \subset I$. Without loss of generality, we assume $\frac{1}{2} \in J'$. Then $\sigma_1^{J'}$ (resp. $\varphi^{J'} > 0$) will stand for the principal eigenvalue (resp. eigenfunction) of (5.2) with boundary conditions $\varphi^{J'}(\alpha) = \varphi^{J'}(\beta) = 0$. We consider $\varphi^{J'}(t)$ normalized so that $\|\varphi^{J'}\|_1 = 1$. Let $\lambda(t_0, \varepsilon) > 0$ be such that

$$\big[m - \frac{\sigma_1^{J'}}{\lambda(t_0,\varepsilon)}\big]^{1/p} \geq m^{1/p} - \frac{\varepsilon}{2}. \tag{5.8}$$

Lemma 2.1 shows that the mapping $\lambda \to \theta_{\lambda,m}$ is strictly increasing as far as $\lambda > \frac{\sigma_1^{J'}}{m}$. Thus,

$$\theta_{\lambda,m} \geq \theta_{\lambda(t_0,\varepsilon),m} \geq \theta^{J'}_{\lambda(t_0,\varepsilon),m}, \quad \lambda \geq \lambda(t_0,\varepsilon), \tag{5.9}$$

where $\theta^{J'}_{\lambda(t_0,\varepsilon),m}$ is the unique positive solution to

$$\begin{gathered} -\frac{1}{\lambda(t_0,\varepsilon)} v'' = mv - v^{p+1}, \quad t \in J',\ t \neq \frac{1}{2}, \\ \Delta v|_{\frac{1}{2}} = \beta_1 v(\frac{1}{2}), \quad \Delta v'|_{\frac{1}{2}} = -\beta_2 v(\frac{1}{2}), \\ v(\alpha) = v(\beta) = 0. \end{gathered} \tag{5.10}$$

It follows easily that $[m - \frac{\sigma_1^{J'}}{\lambda(t_0,\varepsilon)}]^{1/p}\varphi^{J'}$ is a lower solution of (5.10). Thus, due the convexity of $v \to v^{1/p}$ and using the maximum principle, we find that

$$[m - \frac{\sigma_1^{J'}}{\lambda(t_0,\varepsilon)}]^{1/p}\varphi^{J'} \leq \theta^{J'}_{\lambda(t_0,\varepsilon),m}.$$

From the inequality and (5.9) it follows that

$$\theta_{\lambda,m} \geq \big[m - \frac{\sigma_1^{J'}}{\lambda(t_0,\varepsilon)}\big]^{1/p}\varphi^{J'}, \quad \lambda \geq \lambda(t_0,\varepsilon).$$

Hence, from (5.8) the following holds

$$\theta_{\lambda,m} \geq \big[m^{1/p} - \frac{\varepsilon}{2}\big]\varphi^{J'},$$

for all $\lambda \geq \lambda(t_0, \varepsilon)$ and $t \in J'$. Since $\varphi^{J'}(t_0) = 1$ and the local continuity of $\varphi^{J'}$, there exists $J \subset J' \subset I$ such that

$$\varphi^{J'}(t) \geq \frac{m^{1/p} - \varepsilon}{m^{1/p} - \frac{\varepsilon}{2}}, \quad t \in J.$$

Thus,

$$\theta_{\lambda,m} \geq m^{1/p} - \varepsilon, \quad t \in J,$$

for all $\lambda \geq \lambda(t_0, \varepsilon)$. Then we complete the proof. □

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