Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 60, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABLE SOLITARY WAVES FOR ONE-DIMENSIONAL SCHRÖDINGER-POISSON SYSTEMS

GUOQING ZHANG, WEIGUO ZHANG, SANYANG LIU

Abstract. Based on the concentration compactness principle, we shoe the existence of ground state solitary wave solutions for one-dimensional Schrödinger-Poisson systems with large L^2-norm in the energy space. We also obtain orbital stability for ground state solitary waves.

1. Introduction

Consider the one-dimensional Schrödinger-Poisson system

$$\begin{gathered} i\partial_t\psi + \partial_{xx}\psi + W\psi + b|\psi|^{p-2}\psi = 0, \quad (t,x) \in \mathbb{R}^{1+1}, \\ -\partial_{xx}W = |\psi|^2, \quad (t,x) \in \mathbb{R}^{1+1}, \\ \psi(0,x) = \psi_0(x), \end{gathered} \tag{1.1}$$

where $p > 3$, b is a real constant. The self-consistent Poisson potential W is explicitly given by

$$W_\psi(t,x) = -\frac{1}{2}(|x| * |\psi(t,x)|^2) = -\frac{1}{2}\int_{-\infty}^{+\infty} |x-y||\psi(t,y)|^2 dy.$$

Problem (1.1) can be reduced to the nonlinear nonlocal Schrödinger equation

$$\begin{gathered} i\partial_t\psi + \partial_{xx}\psi - \frac{1}{2}(|x| * |\psi(t,x)|^2)\psi + b|\psi|^{p-2}\psi = 0, \quad (t,x) \in \mathbb{R}^{1+1}, \\ \psi(0,x) = \psi_0(x). \end{gathered} \tag{1.2}$$

The model equation (1.2) appears in various frameworks, such as wave propagation in fibre optics to biophysics [8], one-dimensional reduction of electron density in plasma physics [2].

Recently, one-dimensional (1D) Schrödinger-Poisson system have been studied extensively. In 2005, Stimming [14] obtained the global existence result for (1.2) by using the semi-group theory. In 2007, De Leo, Rial [7] studied the global well-posedness and smoothing effect of (1.2). In 2011, Masaki [12] proved that (1.2) is globally well-posed in the energy space, by means of perturbation methods.

2010 *Mathematics Subject Classification.* 35J50, 35Q55, 37K45.
Key words and phrases. Solitary waves; orbital stability; Schrödinger-Poisson system.

Submitted November 9, 2015. Published February 29, 2016.

We are interested in the search of solitary wave solutions of (1.2), i.e., solutions to (1.2) in the form

$$\psi(t,x)=e^{-i\lambda t}u(x),\ \lambda\in\mathbb{R},$$

and u solving

$$-\partial_{xx}u+\frac{1}{2}(|x|*|u|^2)u-b|u|^{p-2}u=\lambda u,\quad \lambda\in\mathbb{R}. \tag{1.3}$$

As $b=0$, based on the rearrangement inequality, Choquard, Stubble [6] proved the existence and uniqueness result of ground states for (1.3). Hartmann, Zakvzewski [9] obtained the analytic solitary wave solutions which is approximated by a Gaussian, and soloved (1.3) numerically.

In this article, we look for solutions u with a priori prescribed large L^2-norm by using the concentration compactness principle and the constraint minimization method. Notice that the Schrödinger-Poisson system in three dimensional space, Catto, Dolbeault, Sanchez, Soler [3] reviewed some recent results and open problems concerning the existence of solitary wave solutions in the frame work of the concentration compactness principle.

This article is organized as follows. In Section 2, we give some preliminary results and state our main theorems. In Section 3, we prove the existence of ground state solitary wave solutions with sufficiently large L^2-norm for (1.3).

2. Preliminary results and main theorems

For any $1\le q<+\infty$, $L^q(\mathbb{R})$ is the usual Lebesgue space endowed with the norm $|u|_q^q=\int_{-\infty}^{+\infty}|u|^q dx$. $H^1(\mathbb{R})$ is the usual Sobolev space with the norm $\|u\|_{H^1(\mathbb{R})}^2=\int_{-\infty}^{+\infty}(|\partial_x u|^2+|u|^2)dx$. Consider the natural functional space $X=\{u: u\in H^1(\mathbb{R}), \sqrt{|x|}u\in L^2(\mathbb{R})\}$. The energy space X [12] is a Hilbert space with norm given by

$$\|u\|_X^2=\|u\|_{H^1(\mathbb{R})}^2+\int_{-\infty}^{+\infty}|x|u^2(x)dx=\|u\|_{H^1(\mathbb{R})}^2+|u|_*^2,$$

where $|u|_*^2=\int_{-\infty}^{+\infty}|x|u^2(x)dx$. By Rellich's criterion [13], we have the following result.

Lemma 2.1. *X is compactly embedded in $L^q(\mathbb{R})$ for all $q\in[2,+\infty)$.*

Masaki [12] proved the following lemma in 2011.

Lemma 2.2. *When $b>0$, $3\le p<6$, $\psi_0(x)\in X$, problem (1.1) is globally well-posed in the energy space X.*

We consider the symmetric bilinear form

$$(u,v)\mapsto B_0(u,v)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|x-y|u(x)v(y)\,dx\,dy,$$

and define the functional $V:H^1(\mathbb{R})\to\mathbb{R}\cup\{+\infty\}$ as

$$V(u)=B_0(u^2,u^2)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|x-y|u^2(x)u^2(y)\,dx\,dy.$$

Lemma 2.3. *Let $\{u_n\}$ be a sequence in $L^2(\mathbb{R})$ such that $u_n\to u$ in $L^2(\mathbb{R})\setminus\{0\}$, $\{v_n\}$ be a bounded sequence in $L^2(\mathbb{R})$ and $\sup_{n\in\mathbb{N}}B_0(u_n^2,v_n^2)<\infty$. Then there exist $n_0\in\mathbb{N}$, $C>0$ such that $|u_n|_*<C$ for $n\ge n_0$.*

Proof. From the assumptions and Egorov's Theorem, that there exist $n_0 \in \mathbb{N}$, R, $\delta > 0$ and $A \subset B_R(0)$ such that $|A| > 0$ and $u_n^2(x) \geq \delta$ for all $n \geq n_0$. Since

$$|x - y| \geq \frac{|y|}{2} \geq \sqrt{|y|} \quad \text{for all } x \in B_R(0) \text{ and } y \in \mathbb{R} \setminus B_{2R}(0),$$

we have

$$\begin{aligned} B_0(u_n, v_n) &\geq \int_{\mathbb{R}\setminus B_{2R}(0)} \int_A |x - y||u_n(x)|^2 |v_n(y)|^2 \, dx \, dy \\ &\geq \frac{\delta|A|}{2} \int_{\mathbb{R}\setminus B_{2R}(0)} |y||v_n(y)|^2 dy \\ &\geq \frac{\delta|A|}{2} (|v_n|_*^2 - 2R|v_n|_2^2). \end{aligned}$$

Hence, we have $|u_n|_* < C$ for all $n \geq n_0$ because $B_0(u_n, v_n)$ and $|v_n|_2^2$ are bounded. $\square$

Remark 2.4. (1) From Lemma 2.3, we obtain that if $B_0(u_n, v_n) \to 0$ and $|v_n|_2 \to 0$ as $n \to \infty$, then $|u_n|_* \to 0$ as $n \to \infty$.

(2) From Lemma 2.1 and Lemma 2.3, it is easy to obtain that if $u_n \rightharpoonup u$ weakly in X, then we have $B_0(u_n^2, (u_n - u)u) \to 0$ as $n \to \infty$.

Now, for problem (1.3), we consider the functionals $I, N : X \to \mathbb{R}$ defined by

$$\begin{aligned} I(u) &= \frac{1}{2} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx + \frac{1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y| u^2(x) u^2(y) \, dx \, dy - \frac{b}{p} \int_{-\infty}^{+\infty} |u|^p dx \\ &= \frac{1}{2} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx + \frac{1}{4} V(u) - \frac{b}{p} \int_{-\infty}^{+\infty} |u|^p dx, \end{aligned}$$

and

$$N(u) = \Big(\int_{-\infty}^{+\infty} |u|^2 dx \Big)^{1/2} = |u|_2.$$

From [6, 13], we obtain that the functionals I, N are well-defined on X.

Lemma 2.5. *The functional I is of class C^1 on X.*

Proof. Let $\{u_n\}$ be a sequence in X converging to some $u \in X$, we obtain that $\{u_n\}$ is bounded and

$$\begin{aligned} &|V(u_n) - V(u)| \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y||u_n(x)|^2 |u_n^2(y) - u^2(y)| \, dx \, dy \\ &\quad + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x - y||u_n^2(x) - u^2(x)||u(y)|^2 \, dx \, dy \\ &\leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (|x| + |y|)|u_n(x)|^2 |u_n(y) - u(y)||u_n(y) + u(y)| \, dx \, dy \\ &\quad + \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} (|x| + |y|)|u_n(x) - u(x)||u_n(x) + u(x)||u(y)|^2 \, dx \, dy \\ &\leq |u_n(x)|_*^2 |u_n - u|_2^2 |u_n + u|_2^2 + |u_n|_2^2 |u_n - u|_* |u_n + u|_* \\ &\quad + |u_n|_2^2 |u_n - u|_* |u_n + u|_* + |u|_*^2 |u_n - u|_2 |u_n + u|_2 \end{aligned}$$

$$\leq C\|u_n - u\|_X^2,$$

for some $C > 0$. So, we obtain that $V(u_n) \to V(u)$ as $n \to \infty$. By a simple calculation, we have

$$V'(u)v = 4\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y|u(x)^2 u(y)v(y)\,dx\,dy, \quad \forall v \in X.$$

When $u_n \to u$ in X, we can argue as before and obtain

$$\begin{aligned}
&|V'(u_n)v - V'(u)v| \\
&= 4\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y|(u_n^2(x)u_n(y) - u^2(x)u(y))v(y)\,dx\,dy \\
&\leq 4[\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x|+|y|)(u_n(x)-u(x))(u_n(x)+u(x))|u_n(y)||v(y)|\,dx\,dy \\
&\quad + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (|x|+|y|)(u_n(y)-u(y))|u(x)|^2|v(y)|\,dx\,dy] \\
&\leq 4(|u_n-u|_*|u_n+u|_*|u_n|_2^2|v|_2 + |u_n-u|_2|u|_*^2|v|_2) \\
&\leq 4C\|u_n-u\|_X\|v\|_X, \quad \forall v \in X.
\end{aligned}$$

In conclusion, we obtain that $V(u)$ is C^1 on X. Since $3 \leq p < 6$, by Lemma 2.1, we obtain that $|u|_p^p$ is C^1 on X. Hence, the functional I is of class C^1 on X.

On the other hand, it is easy to obtain that $N(u)$ is C^1 on X by Lemma 2.1. □

Inspired by the papers [3, 8], we look for the solution of the problem (1.3) with a priori prescribed L^2-norm. The natural way is to consider the constrained critical points of the functional I on the set

$$B_M = \{u \in X : |u|_2 = M\}.$$

So by a solution of (1.3) we mean a couple $(\lambda_M, u_M) \in \mathbb{R} \times X$, where λ_M is the Lagrange multiplier associated with the critical point u_M on B_M. From a physical point of view, the most interesting case is the existence of solutions for (1.3) with minimal energy (ground state solutions), that is the minimizers of

$$I_M = \inf_{u \in B_M} I(u). \tag{2.1}$$

Functionals I, N are translation invariant, i.e., for every $z \in \mathbb{R}$,

$$I(u(\cdot + z)) = I(u), \quad N(u(\cdot + z)) = N(u).$$

Therefore, the concentration compactness principle [10, 11] is the natural framework for the study of the existence of a minimizer, and for the analysis of the minimizing sequence of (2.1). It is known that, in this kind of problems, the main difficulty is the lack of compactness of the minimizing sequences $\{u_n\}$ in B_M; indeed, two possible bad scenarios are possible: (1) (Vanishing)

$$u_n \rightharpoonup 0; \tag{2.2}$$

(2) (Dichotomy)

$$u_n \rightharpoonup \bar{u} \neq 0 \quad \text{and} \quad 0 < |\bar{u}|_2 < M. \tag{2.3}$$

By the concentration compactness principle, we obtain the strict inequalities

$$I_M < I_{M'} + I_{\sqrt{M^2 - M'^2}} \quad \text{for all } M, M', \quad \text{and} \quad 0 < M' < M,$$

as necessary and sufficient conditions for the precompactness of the minimizing sequences $\{u_n\}$ in the problem (2.1). Now, we state our main theorems in this paper.

Theorem 2.6. *Let $b > 0$, $3 < p < 6$. Then the minimizing sequences for* (2.1) *is precompact in X up to translations with prescribed large L^2-norm. In particular, there exists a couple $(\lambda_M, u_M) \in \mathbb{R} \times X$ solution of* (1.3)*, i.e., problem* (1.1) *has a ground state solitary wave solution.*

Theorem 2.7. *Let $b > 0$, $3 < p < 6$. Then the set*

$$S_M = \{e^{i\theta}u,\ \theta \in [0, 2\pi),\ |u|_2 = M,\ I(u) = I_M\},$$

is orbitally stable, i.e., the ground state solitary wave solution of (1.1) *is orbitally stable.*

The definition of orbital stability is recalled in Definition 3.3 below.

3. Existence and orbital stability

In this case, our aim is to discuss the applicability of concentration compactness principle to the minimizing problem (2.1) for proving the existence of ground state solutions of (1.3). The next result is the Gagliardo-Nirenberg inequality in one-dimensional space, see [4, p. 9].

Lemma 3.1. *For all $u \in H^1(\mathbb{R})$, we have*

$$|u|_q \le C|u|_2^{(1-\delta)}|\partial_x u|_2^{\delta}, \tag{3.1}$$

where $2 \le q < \infty$, $\delta = \frac{1}{2} - \frac{1}{q}$, the constant C only depends on q and δ.

By Lemma 3.1, we obtain that for every $M > 0$, the functional I is bounded from below on B_M. Indeed, from (3.1) and positive property of $V(u)$, we have

$$\begin{aligned} I(u) &\ge \frac{1}{2}\int_{-\infty}^{+\infty} |\partial_x u|^2 dx - \frac{b}{p}\int_{-\infty}^{+\infty} |u|^p dx \\ &\ge \frac{1}{2}|\partial_x u|_2^2 - \frac{b}{p} CM \frac{p(1-\delta)}{2} |\partial_x u|_2^{p\delta}. \end{aligned}$$

Since $3 < p < 6$, we have $p\delta = \frac{p}{2} - 1 < 2$. Hence, we have

$$I(u) \ge \frac{1}{2}\int_{-\infty}^{+\infty} |\partial_x u|^2 dx + o(1),$$

which concludes the proof. Moreover, we also obtain that I is coercive on B_M.

Notice that if we set $u_\theta(x) = \theta^{(1-\frac{\alpha}{2})}u(\frac{x}{\theta^\alpha})$, $\theta > 0$, α is a real number, we have

$$\int_{-\infty}^{+\infty} |u_\theta|^2 dx = |u_\theta|_2^2 = \theta^2|u|_2^2,$$

$$\int_{-\infty}^{+\infty} |\partial_x u_\theta|^2 dx = \theta^{(2-2\alpha)}\int_{-\infty}^{+\infty} |\partial_x u|^2 dx,$$

$$\begin{aligned} &\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y||u_\theta(x)|^2|u_\theta(y)|^2\,dx\,dy \\ &= \theta^{(4+\alpha)}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y||u(x)|^2|u(y)|^2\,dx\,dy, \end{aligned} \tag{3.2}$$

$$\int_{-\infty}^{+\infty} |u_\theta|^p dx = \theta^{(1-\frac{\alpha}{2})p+\alpha} \int_{-\infty}^{+\infty} |u|^p dx.$$

Lemma 3.2. *If $b > 0$, $3 < p < 6$, then there exists $M_1 > 0$, such that*

$$I_{M'} < 0, \quad \text{for all } M' \in (M_1, +\infty),$$
$$I_M < I_{M'} + I_{\sqrt{M^2 - M'^2}},$$

for all $M > M_1$ and $0 < M' < M$.

Proof. By (3.2), we have

$$\begin{aligned}
I(u_\theta) &= \frac{\theta^{(2-2\alpha)}}{2} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx + \frac{\theta^{(4+\alpha)}}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x-y||u(x)|^2 |u(y)|^2 \, dx \, dy \\
&\quad - \frac{b}{p} \theta^{(1-\frac{\alpha}{2})p+\alpha} \int_{-\infty}^{+\infty} |u|^p dx \\
&= \theta^2 [I(u) + \frac{\theta^{(-2\alpha)} - 1}{2} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx \\
&\quad + \frac{\theta^{(2+\alpha)} - 1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x-y||u(x)|^2 |u(y)|^2 \, dx \, dy \\
&\quad - \frac{b}{p} (\theta^{(1-\frac{\alpha}{2})p+\alpha-2} - 1) \int_{-\infty}^{+\infty} |u|^p dx] \\
&= \theta^2 (I(u) + g(\theta, u)),
\end{aligned}$$

where

$$\begin{aligned}
g(\theta, u) = \frac{\theta^{(-2\alpha)} - 1}{2} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx \\
+ \frac{\theta^{(2+\alpha)} - 1}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x-y||u(x)|^2 |u(y)|^2 \, dx \, dy \\
- \frac{b}{p} (\theta^{(1-\frac{\alpha}{2})p+\alpha-2} - 1) \int_{-\infty}^{+\infty} |u|^p dx.
\end{aligned}$$

Let $\alpha = -2/3$, we have

$$\begin{aligned}
I(u_\theta) = \frac{1}{2} \theta^{\frac{10}{3}} \int_{-\infty}^{+\infty} |\partial_x u|^2 dx + \frac{1}{4} \theta^{\frac{10}{3}} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |x-y||u(x)|^2 |u(y)|^2 \, dx \, dy \\
- \frac{b}{p} \theta^{(\frac{4}{3}p - \frac{2}{3})} \int_{-\infty}^{+\infty} |u|^p dx, \quad \forall u_\theta \in X.
\end{aligned}$$

Hence, we obtain that $I(u_\theta) < 0$ for a sufficiently large θ which proves the first case because $\frac{4}{3}p - \frac{2}{3} > \frac{10}{3}$ for $3 < p < 6$.

Claim: $I_{\theta M'} < \theta^2 I_{M'}$ for θ sufficiently large. Indeed, let $\{u_n\}$ be a minimizing sequence in $B_{M'}$ with $I_{M'} < 0$. Since $I_{M'}(u_n) < 0$, we have

$$0 < C_1 < \int_{-\infty}^{+\infty} |\partial_x u_n|^2 dx < C_2, \quad 0 < C_3 < \int_{-\infty}^{+\infty} |u_n|^p dx < C_4.$$

When $\alpha = -2/3$, we have

$$g(\theta, u_n) = \frac{(\theta^{\frac{4}{3}} - 1)}{2} \int_{-\infty}^{+\infty} |\partial_x u_n|^2 dx$$

$$
\begin{aligned}
&+\frac{(\theta^{\frac{4}{3}}-1)}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|x-y||u_n(x)|^2|u_n(y)|^2\,dx\,dy\\
&-\frac{b}{p}(\theta^{(\frac{4}{3}p-\frac{8}{3})}-1)\int_{-\infty}^{+\infty}|u_n|^p dx,
\end{aligned}
$$

with $\frac{4}{3}p-\frac{8}{3}>\frac{4}{3}$, as $3<p<6$. With a simple computation we obtain that

$$
\frac{d}{d\theta}g(\theta,u_n)|_{\theta=1}<0 \text{ and } \frac{d^2}{d\theta^2}g(\theta,u_n)<0, \quad \text{for all } \theta>1.
$$

In conclusion, we obtain that $g(\theta,u_n)<0$ for all $\theta>1$ and $I_{\theta M'}<\theta^2 I(u_n)=\theta^2 I_{M'}$.

From the claim, we obtain that for M' sufficiently large,

$$
\begin{aligned}
I_M &= I_{\frac{M}{M'}M'} < \frac{M^2}{M'^2} I_{M'}\\
&= \frac{(M^2-M'^2+M'^2)}{M'^2} I_{M'}\\
&= I_{M'} + \frac{(M^2-M'^2)}{M'^2} I_{\frac{M'}{\sqrt{M^2-M'^2}}\sqrt{M^2-M'^2}}\\
&< I_{M'} + I_{\sqrt{M^2-M'^2}}.
\end{aligned}
$$

Hence, we complete the proof. □

Proof of Theorem 2.6. Since the functional I is bounded below and coercive on B_M, we obtain that the minimizing sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^2)$. By Lemma 3.2, we obtain that $I_M<0$ on B_M for sufficiently large M. Hence, we have that

$$
\sup_{x\in\mathbb{R}}\int_{B_2(x)}|u_n|^2dx>0 \quad textand \quad u_n \rightharpoonup u\neq 0 \text{ in } H^1(\mathbb{R}^2).
$$

and the vanishing case does not hold. On the other hand, by Lemma 3.2, $I_M < I_{M'} + I_{\sqrt{M^2-M'^2}}$ for all $M'>M_1$ and $0<M'<M$, and the dichotomy case does not hold. Hence, from concentration compactness principle, we obtain that

$$
\text{there exists } x_n\in\mathbb{R} \text{ such that } \tilde{u}_n=u_n(y-x_n)\in X,\ n\in\mathbb{N}, \tag{3.3}
$$

is precompact in $H^1(\mathbb{R})$ and converges strongly to some function $u\in H^1(\mathbb{R})\backslash\{0\}$. We may also assume that $\tilde{u}_n\to u$ pointwise almost everywhere in $\mathbb{R}$.

Claim 1: $|\tilde{u}_n|_*$ is bounded in n. Indeed, since $\{\tilde{u}_n\}$ is a minimizing sequence for I_M on B_M, by the version of Ekeland Variational principle in [p.122]w1, we obtain that there exists $\lambda\in\mathbb{R}$ such that

$$
I'(\tilde{u}_n)-\lambda N'(\tilde{u}_n)\to 0, \quad I(\tilde{u}_n)-\lambda N(\tilde{u}_n)\to I_M-\lambda M^2 \quad \text{as } n\to\infty. \tag{3.4}
$$

Hence, we have

$$
B_0(\tilde{u}_n^2,\tilde{u}_n^2)=V(\tilde{u}_n)=V(u_n)=o(1)+\lambda\|u_n\|^2_{L^2(\mathbb{R})}+b\|u_n\|^p_{L^p(\mathbb{R})}-\|u_n\|^2_{H^1(\mathbb{R})}
$$

as $n\to\infty$, and the right-hand side of this equality remains bounded in n. So, we obtain that $B_0(\tilde{u}_n^2,\tilde{u}_n^2)$ is bounded in n. By Lemma 2.3 and Remark 2.4, we have $|\tilde{u}_n|_*$ is bounded in n. Hence, by the definition of the norm $\|\tilde{u}_n\|_X$, we obtain that $\|\tilde{u}_n\|_X<\infty$.

Claim 2: $\|\tilde{u}_n-u\|_X\to 0$ as $n\to\infty$. Indeed, by (3.4), we have

$$
I'(\tilde{u}_n)(\tilde{u}_n-u)-\lambda N'(\tilde{u}_n)(\tilde{u}_n-u)\to 0 \quad \text{as } n\to\infty.
$$

By a simple calculation, we obtain that

$$
\begin{aligned}
&\|\tilde{u}_n\|_{H^1(\mathbb{R})}^2 - \|u\|_{H^1(\mathbb{R})}^2 + V(\tilde{u}_n)(\tilde{u}_n - u) - b\int_{-\infty}^{+\infty} |\tilde{u}_n|^{p-2}\tilde{u}_n(\tilde{u}_n - u)dx \\
&- \lambda \int_{-\infty}^{+\infty} |\tilde{u}_n|(\tilde{u}_n - u)dx = o(1).
\end{aligned}
$$

Since $\|\tilde{u}_n\|_X$ is bounded in X, we obtain that $\tilde{u}_n \rightharpoonup u$ weakly in X. By Lemma 2.1, we obtain that $\tilde{u}_n \to u$ strongly in $L^s(\mathbb{R})$ for $s \in [2, \infty)$. Hence, we have

$$
V'(\tilde{u}_n)(\tilde{u}_n - u) = B_0(\tilde{u}_n^2, \tilde{u}_n(\tilde{u}_n - u)) = B_0(\tilde{u}_n^2, (\tilde{u}_n - u)^2) + B_0(\tilde{u}_n^2, u(\tilde{u}_n - u)) \to 0
$$

as $n \to \infty$. By Lemma 2.3, we obtain that $|\tilde{u}_n - u|_* \to 0$ as $n \to \infty$. Hence, we obtain that $\|\tilde{u}_n - u\|_X \to 0$ as $n \to \infty$.

From Claim 2, we obtain that the minimizing sequence $\{\tilde{u}_n\}$ of (2.1) is precompact in X with prescribed large L^2-norm. So there exists a couple $(\lambda_M, u_M) \in \mathbb{R}\times X$ solution of (1.3), and (1.1) has a ground state solitary wave solution. Let

$$
S_M = \{e^{i\theta}u(x),\, \theta \in [0, 2\pi),\, |u|_2 = M,\, I(u) = I_M\}.
$$

□

Definition 3.3 (c3)**.** We say S_M is orbitally stable if for every $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that if $\psi_0 \in X$ satisfies $\inf_{v\in S_M} \|v - \psi_0\|_X < \delta(\varepsilon)$, then we have

$$
\sup_{t>0} \inf_{v\in S_M} \|\psi(t,x) - v(x)\|_X < \varepsilon,
$$

where $\psi(t,\cdot)$ is the solution of (1.1) with initial datum ψ_0.

Proof of Theorem 2.7. By [Theorem 1.5]m1, we obtain the solution of (1.1) conserves $|\psi|_{L^2(\mathbb{R})}^2$ and the energy

$$
\begin{aligned}
E(t,\psi) = \frac{1}{2}|\partial_x\psi|_2^2 &+ \frac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y||\psi(t,x)|^2|\psi(t,y)|^2\,dx\,dy \\
&- \frac{b}{p}\int_{-\infty}^{+\infty} |\psi(t,x)|^p dx,
\end{aligned}
$$

i.e.,

$$
\int_{-\infty}^{+\infty} |\psi(t,x)|^2 dx = \int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx,
$$

and

$$
\begin{aligned}
&\frac{1}{2}|\partial_x\psi|_2^2 + \frac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y||\psi(t,x)|^2|\psi(t,y)|^2\,dx\,dy - \frac{b}{p}\int_{-\infty}^{+\infty} |\psi|^p dx \\
&= \frac{1}{2}|\partial_x\psi_0|_2^2 + \frac{1}{4}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |x-y||\psi_0(x)|^2|\psi_0(y)|^2\,dx\,dy - \frac{b}{p}\int_{-\infty}^{+\infty} |\psi_0|^p dx.
\end{aligned}
$$

Suppose by contradiction that there exists a M such that S_M is not orbitally stable. Hence, there exist a subsequence $\{\psi_n(0,x)\}$ and $\{t_n\} \in \mathbb{R}$ such that

$$
\inf_{v\in S_M} \|\psi_n(0,x) - v(x)\|_X \to 0, \quad \inf_{v\in S_M} \|\psi_n(t_n,x) - v(x)\|_X \ge \varepsilon \quad \text{as } n \to \infty.
$$

Then, we obtain that there exists $u_M \in X$ minimizer of I_M and $\theta \in [0, 2\pi]$ such that $v = e^{i\theta}u_M$,

$$
|\psi_n(0,x)|_2 \to |v|_2 = M, \quad I(\psi_n(0,x)) \to I(v) = I_M \quad \text{as } n \to \infty.
$$

Actually, we can assume that $\psi_n(0,x) \in B_M$ (there exists $\alpha_n = \frac{M}{|\psi_n(0,x)|_2} \to 1$ so that $\alpha_n \psi_n(0,x) \in B_M$ and $I(\alpha_n \psi_n(0,x)) \to I_M$, i.e., we can replace $\psi_n(0,x)$ with $\alpha_n \psi_n(0,x)$). So, we have $\{\psi_n(0,x)\}$ is a minimizing sequence for I_M and

$$I(\psi_n(t_n,x)) = I(\psi_n(0,x)),$$

and $\{\psi_n(t_n,x)\}$ is a minimizing sequence for I_M. Since we obtain that every minimizing sequence has a subsequence converging in $X-$norm to a minimum on B_M, and it is a contradiction. □

Acknowledgments. The authors would like to sincerely thank the referee for valuable comments and suggestions. This research was supported by the Shanghai Natural Science Foundation (No. 15ZR1429500).

References

[1] J. Bellazzini, G. Siciliano; *Stable standing waves for a class of nonlinear Schrödinger-Poisson equations*, Z. Angew Math. Phys., 62 (2011), 267-280.
[2] O. Bokanowski, J. L. Lopez, O. Sanchez, J. Soler; *On an exchange interaction model for quantum transport: The Schrödinger-Poisson-Slater system*, Math. Models Methods Appl. Sci., 13 (2003), 1397-1412.
[3] I. Catto, J. Dolbeault, O. Sanchez, J. Solev; *Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the Concentration-Compactness Principle*, Math. Models Methods Appl. Sci., 23 (2013), 1915-1938.
[4] T. Cazenave; *Semilinear Schrödinger Equation, Courant.* in: Lecture Notes in Mathematics, Vol. 10, New York: New York University, 2003.
[5] T. Cazenave, P. L. Lions; *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Commum. Math. Phys., 85 (1982) 549-561.
[6] P. Choquard, J. Stubbe; *The one-dimensional Schrödinger-Newton equations*, Lett. Math. Phys., 81 (2007), 177-184.
[7] M. De Leo, D. Rial; *Well posedness and smoothing effect of Schrödinger-Poisson equation*, J. Math. Phys., 48 (2007) 093509.
[8] A. Hasegawa; *Optical Solitons in Fibres*, Springer, Berlin, 1990.
[9] B. Hartmann, W. J. Zakvzewski; *Soliton solutions of the nonlinear Schrödinger equation with nonlocal Coulomb and Yukawa interactions*, Physics Letters A, 366 (2007), 540-544.
[10] P. L. Lions; *The concentration-compactness principle in the calculus of variations*, the locally compact case I, Ann. I. H. Poincaré-AN, 1 (1984), 109-145.
[11] P. L. Lions; *The concentration-compactness principle in the calculus of variations*, the locally compact case II, Ann. I. H. Poincaré-AN, 1 (1984), 223-283.
[12] S. Masaki; *Energy solution to a Schrödinger-Poisson system in the two-dimensional whole space*, SIAM J. Math. Anal., 43 (2011), 2719-2731.
[13] M. Reed, S. Barry; *Methods of Modern Mathematical physics*, Vol. 4, Analysis of Operators, Academic, London, 1978.
[14] H. P. Stimming; *The IVP for the Schrödinger-Poisson-X_α equation in one dimension*, Math. Models Methods Appl. Sci., 15 (2005), 1169-1180.
[15] W. Willem; *Minimax Theorems*, Progress in Nonlinear Differential Equations and Their Applications, Vol. 24, Birkhaüser, Boston, 1996.

Guoqing Zhang
College of Sciences, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: shzhangguoqing@126.com

Weiguo Zhang
College of Sciences, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: zwgzwm@126.com

Sanyang Liu
College of Mathematics and Statistics, Xidian University, Xi'an 710071 Shanxi, China
E-mail address: liusanyang@126.com