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EXISTENCE OF SOLUTIONS TO FRACTIONAL DIFFERENTIAL EQUATIONS WITH MULTI-POINT BOUNDARY CONDITIONS AT RESONANCE IN HILBERT SPACES

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Abstract. This article is devoted to investigating the existence of solutions to fractional multi-point boundary-value problems at resonance in a Hilbert space. More precisely, the dimension of the kernel of the fractional differential operator with the boundary conditions be any positive integer. We point out that the problem is new even when the system under consideration is reduced to a second-order ordinary differential system with resonant boundary conditions. We show that the considered system admits at least a solution by applying coincidence degree theory first introduced by Mawhin. An example is presented to illustrate our results.

1. Introduction

In this article, we are concerned with the existence of solutions to the following fractional multi-point boundary value problems(BVPs) at resonance

$$\begin{gathered} D_{0+}^{\alpha}x(t) = f(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad 1 < \alpha \le 2,\ t \in (0,1), \\ I_{0+}^{2-\alpha}x(t)|_{t=0} = \theta, \quad x(1) = Ax(\xi), \end{gathered} \tag{1.1}$$

where D_{0+}^{α} and I_{0+}^{α} are the Riemann-Liouville differentiation and integration, respectively; θ is the zero vector in $l^2 := \{x = (x_1, x_2, \dots, .) : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$; $A : l^2 \to l^2$ is a bounded linear operator satisfying $1 \le \dim\ker(I - A\xi^{\alpha-1}) < \infty$; $\xi \in (0,1)$ is a fixed constant; $f : [0,1] \times l^2 \times l^2 \to l^2$ is a Carathéodory function; that is,

(i) for each $(u, v) \in l^2 \times l^2$, $t \mapsto f(t, u, v)$ is measurable on $[0, 1]$;
(ii) for a.e. $t \in [0, 1]$, $(u, v) \mapsto f(t, u, v)$ is continuous on $l^2 \times l^2$;
(iii) for every bounded set $\Omega \subseteq l^2 \times l^2$, the set $\{f(t, u, v) : (u, v) \in \Omega\}$ is a relatively compact set in l^2. Moreover, the function
$$\varphi_{\Omega}(t) = \sup\{\|f(t, u, v)\|_{l^2} : (u, v) \in \Omega\} \in L^1[0, 1],$$
where $\|x\|_{l^2} = \sqrt{\sum_{i=1}^{\infty} |x_i|^2}$ is the norm of $x = (x_1, x_2, \dots, \cdot)^{\top}$ in l^2.

System (1.1) is said to be at resonance in l^2 if $\dim\ker(I - A\xi^{\alpha-1}) \ge 1$, otherwise, it is said to be non-resonant. In the past three decades, the existence of solutions for the fractional differential equations with the boundary value conditions have been

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given considerable attention by many mathematical researchers. The attempts on $\dim\ker(I - A\xi^{\alpha-1}) = 0$, non-resonance case, for fractional differential equations are available in [1, 2, 10, 11, 17, 21, 22, 23], and the attempts on $1 \le \dim\ker(I - A\xi^{\alpha-1}) \le 2$, resonance case, can be found in [3, 4, 8, 9, 13, 14, 18, 20]. However, to the best of our knowledge, all results derived in these papers are for one equation with $\dim\ker L = 0$ or 1 and for two equations with $\dim\ker L = 2$. Recently, the authors in [16] investigated the following second differential system

$$\begin{gathered} u''(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \\ u'(0) = \theta, u(1) = Au(\eta) \end{gathered} \tag{1.2}$$

where $f : [0,1] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and the square matrix A satisfies certain condition. Moreover, fractional order $\alpha \in (1,2]$ case was investigated in [7], where the results for second order ordinary differential equation in [16] was generalized to fractional order case. However, these considered problems were investigated in finite dimensional space. Therefore, it is more natural to ask whether it exists a solution when such kind of boundary value problem considered in a infinite dimensional space. Recently, in [24], the author discussed the existence of solution for fractional boundary value problem with non-resonant conditions in an arbitrary Banach space which, of course, can be in the infinite dimensional space. However, it is still open for the equation in infinite dimensional space with resonance conditions. It deservers to point out that the problem new even when $\alpha = 2$ the system (1.1) becomes second order ordinary differential system with resonant boundary conditions. In this paper, we investigate the existence of solution for fractional differential equation in l^2. There is remarkable difference that any bounded closed set is compact in finite dimensional space, while bounded closed set may be not compact in the infinite dimensional, for instance, $\{x \in l^2 : \|x\| \le 1\} \subset l^2$ is non-compact in l^2. Therefore, compactness criterion of the infinite dimensional space is more complicated, the problem we considered is in the infinite dimensional setting.

To apply the coincidence degree theory of Mawhin [15], we suppose additionally that A satisfies $1 \le \dim\ker(I - A\xi^{\alpha-1}) < \infty$ and one of the following conditions

-](A1)] $A\xi^{\alpha-1}$ is idempotent, that is, $A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}$, or;
-](A2)] $A^2\xi^{2\alpha-2} = I$, where I stands for the identity operator from l^2 to l^2.

The requirement $1 \le \dim\ker(I - A\xi^{\alpha-1})$ is to make the problem to be resonant and the requirement $\dim\ker(I - A\xi^{\alpha-1}) < \infty$ is to make the kernel operator to be a Fredholm operator which is a basic requirement in applying the Mawhin theorem.

It is also obvious that $\dim\ker(I - A\xi^{\alpha-1})$ can take any integer $n \in \mathbb{N}$ for suitable A, which can be regards as a generalization of the previous efforts [3, 4, 8, 9, 13, 14, 18, 20]. However, we point out that without the above assumptions (A1) or (A2), it will be difficult to construct the projector Q as (3.1) below. Actually, the assumptions (A1) or (A2) play a key role in the process of the proof. This is the reason why we only choose the two special cases of A. Without such an assumption, i.e., the general A satisfying $\dim\ker(I - A\xi^{\alpha-1}) < \infty$, (1.1) may be a challenge problem, which we will study in the future.

In particular, when $A = \xi^{1-\alpha}I$, it is clear that A satisfies (A2) but with $\dim\ker(I - A\xi^{\alpha-1}) = \infty$, which leads to the kernel operator not to be Fredholm operator. Thus, such operator is excluded. Unlike the case in $\mathbb{R}^n$, the operator A is allowed to be identity operator $\xi^{\alpha-1}I$. Let $\mathbb{A} = \operatorname{diag}(\xi^{1-\alpha}I_k, B)$ with

$\dim\ker(I - B\xi^{\alpha-1}) = 0$ and B satisfying (A1) or (A2), where I_k is the identity matrix in $\mathbb{R}^k$. It is seen that $\dim\ker(I - \mathbb{A}\xi^{\alpha-1}) = k$,

$$\ker L = \{(c_1, c_2, \dots, c_k, 0, 0, \dots,)^\top t^{\alpha-1} : c_i \in \mathbb{R}, i = 1, 2, \dots, k\}$$

and $\dim\ker L = k$, where L is defined by (2.2) below. So under this boundary condition, the system (1.1) is at resonance. The goal of this paper is to study the existence of solutions for the fractional differential equations with boundary value conditions at resonance in Hilbert space l^2.

We proceed as follows: In Section 2, we give some necessary background and some preparations for our consideration. The proof for the main results is presented in Section 3 by applying the coincidence degree theory of Mawhin. In Section 4, an example is given to illustrate the main result.

2. Preliminaries

In this section, we introduce some necessary definitions and lemmas which will be used later. For more details, we refer the reader to [5, 12, 15] and the references therein.

Definition 2.1 ([12]). The fractional integral of order $\alpha > 0$ of a function $x : (0, \infty) \to \mathbb{R}$ is given by

$$I^\alpha_{0^+} x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds,$$

provided that the right-hand side is pointwise defined on $(0, \infty)$.

Remark 2.2. The notation $I^\alpha_{0^+} x(t)|_{t=0}$ means that the limit is taken at almost all points of the right-sided neighborhood $(0, \varepsilon)(\varepsilon > 0)$ of 0 as follows:

$$I^\alpha_{0^+} x(t)|_{t=0} = \lim_{t\to 0^+} I^\alpha_{0^+} x(t).$$

Generally, $I^\alpha_{0^+} x(t)|_{t=0}$ is not necessarily to be zero. For instance, let $\alpha \in (0, 1)$, $x(t) = t^{-\alpha}$. Then

$$I^\alpha_{0^+} t^{-\alpha}|_{t=0} = \lim_{t\to 0^+} \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = \lim_{t\to 0^+} \Gamma(1-\alpha) = \Gamma(1-\alpha).$$

Definition 2.3 ([12]). The fractional derivative of order $\alpha > 0$ of a function $x : (0, \infty) \to \mathbb{R}$ is given by

$$D^\alpha_{0^+} x(t) = \frac{1}{\Gamma(n-\alpha)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{x(s)}{(t-s)^{\alpha-n+1}} ds,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0, \infty)$.

Lemma 2.4 ([12]). *Assume that $x \in C(0, +\infty) \cap L_{\rm loc}(0, +\infty)$ with a fractional derivative of order $\alpha > 0$ belonging to $C(0, +\infty) \cap L_{\rm loc}(0, +\infty)$. Then*

$$I^\alpha_{0^+} D^\alpha_{0^+} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n},$$

for some $c_i \in \mathbb{R}, i = 1, \dots, n$, where $n = [\alpha] + 1$.

For any $x(t) = (x_1(t), x_2(t), \dots)^\top \in l^2$, the fractional derivative of order $\alpha > 0$ of x is defined by

$$D^\alpha_{0^+} x(t) = (D^\alpha_{0^+} x_1(t), D^\alpha_{0^+} x_2(t), \dots)^\top \in l^2.$$

The following definitions and the coincidence degree theory are fundamental in the proof of our main result. We refer the reader to [5, 15].

Definition 2.5. Let X and Y be normed spaces. A linear operator $L : \operatorname{dom}(L) \subset X \to Y$ is said to be a Fredholm operator of index zero provided that

(i) $\operatorname{im} L$ is a closed subset of Y, and
(ii) $\dim \ker L = \operatorname{codim} \operatorname{im} L < +\infty$.

It follows from definition 2.5 that there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{im} P = \ker L, \quad \ker Q = \operatorname{im} L, \quad X = \ker L \oplus \ker P, \quad Y = \operatorname{im} L \oplus \operatorname{im} Q$$

and the mapping $L|_{\operatorname{dom} L \cap \ker P} : \operatorname{dom} L \cap \ker P \to \operatorname{im} L$ is invertible. We denote the inverse of $L|_{\operatorname{dom} L\cap\ker P}$ by $K_P : \operatorname{im} L \to \operatorname{dom} L \cap \ker P$. The generalized inverse of L denoted by $K_{P,Q} : Y \to \operatorname{dom} L \cap \ker P$ is defined by $K_{P,Q} = K_P(I - Q)$. Furthermore, for every isomorphism $J : \operatorname{im} Q \to \ker L$, we can obtain that the mapping $K_{P,Q} + JQ : Y \to \operatorname{dom} L$ is also an isomorphism and for all $x \in \operatorname{dom} L$, we know that

$$(K_{P,Q} + JQ)^{-1}x = (L + J^{-1}P)x. \tag{2.1}$$

Definition 2.6. Let L be a Fredholm operator of index zero, let $\Omega \subseteq X$ be a bounded subset and $\operatorname{dom} L \cap \Omega \neq \emptyset$. Then the operator $N : \overline{\Omega} \to Y$ is called to be L-compact in $\overline{\Omega}$ if

(i) the mapping $QN : \overline{\Omega} \to Y$ is continuous and $QN(\overline{\Omega}) \subseteq Y$ is bounded, and
(ii) the mapping $K_{P,Q}N : \overline{\Omega} \to X$ is completely continuous.

Assume that L is defined by Definition 2.6 and $N : \overline{\Omega} \to Y$ is L-compact. For any $x \in \overline{\Omega}$, by (2.1), we shall see that

$$\begin{aligned} Lx &= (K_{P,Q} + JQ)^{-1}x - J^{-1}Px \\ &= (K_{P,Q} + JQ)^{-1}\left[Ix - K_{P,Q}J^{-1}Px - JQJ^{-1}Px\right] \\ &= (K_{P,Q} + JQ)^{-1}(Ix - Px). \end{aligned}$$

Then we can equivalently transform the existence problem of the equation $Lx = Nx, x \in \overline{\Omega}$ into a fixed point problem of the operator $P + (K_{P,Q} + JQ)N$ in $\overline{\Omega}$.

This can be guaranteed by the following lemma, which is also the main tool in this paper.

Lemma 2.7 ([15])**.** *Let $\Omega \subset X$ be bounded, L be a Fredholm mapping of index zero and N be L-compact on $\overline{\Omega}$. Suppose that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in ((\operatorname{dom} L \backslash \ker L) \cap \partial\Omega) \times (0, 1)$;*
(ii) *$Nx \notin \operatorname{im} L$ for every $x \in \ker L \cap \partial\Omega$;*
(iii) *$\deg(JQN|_{\ker L\cap\partial\Omega}, \Omega\cap\ker L, 0) \neq 0$, with $Q : Y \to Y$ a continuous projector such that $\ker Q = \operatorname{im} L$ and $J : \operatorname{im} Q \to \ker L$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $\operatorname{dom} L \cap \overline{\Omega}$.

In this paper, we use spaces $\mathbb{X}$, $\mathbb{Y}$ introduced as

$$\mathbb{X} = \left\{x(t) \in l^2 : x(t) = I_{0+}^{\alpha-1}u(t), u \in C([0, 1]; l^2), t \in [0, 1]\right\}$$

with the norm $\|x\|_{\mathbb{X}} = \max\{\|x\|_{C([0,1];l^2)}, \|D_{0+}^{\alpha-1}x\|_{C([0,1];l^2)}\}$ and $\mathbb{Y} = L^1([0, 1]; l^2)$ with the norm $\|y\|_{L^1([0,1];l^2)} = \int_0^1 \|y(s)\|_{l^2} ds$, respectively, where $\|x\|_{C([0,1];l^2)} = \sup_{t\in[0,1]} \|x(t)\|_{l^2}$.

We have the following compactness criterion on subset F of $\mathbb{X}$ which is a slight modification of [19, Lemma 2.2] (see also the Ascoli-Arzela theorem [6, Theorem 1.2.5, p. 15]).

Lemma 2.8. *$F \subset \mathbb{X}$ is a sequentially compact set if and only if $F(t)$ is a relatively compact set and equicontinuous which are understood in the following sense:*

(1) *for any $t \in [0,1]$, $F(t) := \{x(t) | x \in F\}$ is a relatively compact set in l^2;*
(2) *for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that*
$$\|x(t_1) - x(t_2)\|_{l^2} < \varepsilon,\ \|D_{0+}^{\alpha-1}x(t_1) - D_{0+}^{\alpha-1}x(t_2)\|_{l^2} < \varepsilon,$$
for $t_1, t_2 \in [0,1]$, $|t_1 - t_2| < \delta$, for all $x \in F$.

Now we define the linear operator $L : \operatorname{dom} L \subseteq \mathbb{X} \to \mathbb{Y}$ by
$$Lx := D_{0+}^{\alpha}x, \tag{2.2}$$
where $\operatorname{dom} L = \{x \in X : D_{0+}^{\alpha}x \in Y, x(0) = \theta, x(1) = Ax(\xi)\}$. Define $N : X \to Y$ by
$$Nx(t) := f(t, x(t), D_{0+}^{\alpha-1}x(t)), \quad t \in [0,1]. \tag{2.3}$$
Then the problem can be equivalently rewritten as $Lx = Nx$.

The next lemma plays a vital role in estimating the boundedness of some sets.

Lemma 2.9. *Let $z_1, z_2 \geq 0$, $\gamma_1, \gamma_2 \in [0,1)$ and $\lambda_i, \mu_i \geq 0, i = 1,2,3$, and the following two inequalities hold,*
$$\begin{aligned} z_1 &\leq \lambda_1 z_1^{\gamma_1} + \lambda_2 z_2 + \lambda_3, \\ z_2 &\leq \mu_1 z_1 + \mu_2 z_2^{\gamma_2} + \mu_3 \end{aligned} \tag{2.4}$$
Then z_1, z_2 is bounded if $\lambda_2\mu_1 < 1$.

Proof. From (2.4), we have
$$\begin{aligned} z_1 &\leq \frac{\lambda_1 z_1^{\gamma_1} + \lambda_2\mu_2 z_2^{\gamma_2} + \lambda_2\mu_3 + \lambda_3}{1 - \lambda_2\mu_1}, \\ z_2 &\leq \frac{\lambda_1\mu_1 z_1^{\gamma_1} + \mu_2 z_2^{\gamma_2} + \lambda_3\mu_1 + \mu_3}{1 - \lambda_2\mu_1}. \end{aligned} \tag{2.5}$$
Let $z = \max\{z_1, z_2\}$, $\kappa_1 = \max\{\lambda_1, \lambda_1\mu_1\}$ and $\kappa_2 = \max\{\lambda_2\mu_2, \mu_2\}$. It follows from (2.5) that
$$z \leq \frac{\kappa_1 z^{\gamma_1} + \kappa_2 z^{\gamma_2} + \lambda_2\mu_3 + \lambda_3\mu_1 + \lambda_3 + \mu_3}{1 - \lambda_2\mu_1}.$$
This, together with $\gamma_1, \gamma_2 \in [0,1)$, yields that z is bounded. $\square$

Lemma 2.10. *The operator L, defined by (2.2), is a Fredholm operator of index zero.*

Proof. For any $x \in \operatorname{dom} L$, by Lemma 2.4 and $x(0) = \theta$, we obtain
$$x(t) = I_{0+}^{\alpha}Lx(t) + ct^{\alpha-1}, \quad c \in l^2,\ t \in [0,1], \tag{2.6}$$
which, together with $x(1) = Ax(\xi)$, yields
$$\begin{aligned} \ker L &= \{x \in \mathbb{X} : x(t) = ct^{\alpha-1}, t \in [0,1], c \in \ker(I - A\xi^{\alpha-1})\} \\ &\backsimeq \ker(I - A\xi^{\alpha-1})t^{\alpha-1}. \end{aligned} \tag{2.7}$$
Now we claim that
$$\operatorname{im} L = \{y \in Y : h(y) \in \operatorname{im}(I - A\xi^{\alpha-1})\}, \tag{2.8}$$

where $h : \mathbb{Y} \to l^2$ is a continuous linear operator defined by

$$h(y) := \frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) ds. \tag{2.9}$$

Actually, for any $y \in \operatorname{im} L$, there exists a function $x \in \operatorname{dom} L$ such that $y = Lx$. It follows from (2.6) that $x(t) = I^{\alpha}_{0+} y(t) + ct^{\alpha-1}$. From this equality and $x(1) = Ax(\xi)$, we obtain

$$\frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) ds = (I - A\xi^{\alpha-1})c, \quad c \in l^2,$$

which means that $h(y) \in \operatorname{im}(I - A\xi^{\alpha-1})$.

On the other hand, for any $y \in \mathbb{Y}$ satisfying $h(y) \in \operatorname{im}(I - A\xi^{\alpha-1})$, there exists a constant c^* such that $h(y) = (I - A\xi^{\alpha-1})c^*$. Let $x^*(t) = I^{\alpha}_{0+} y(t) + c^* t^{\alpha-1}$. A straightforward computation shows that $x^*(0) = \theta$ and $x^*(1) = Ax^*(\xi)$. Hence, $x^* \in \operatorname{dom} L$ and $y(t) = D^{\alpha}_{0+} x^*(t)$, which implies that $y \in \operatorname{im} L$.

Next, put $\rho_A = \kappa(I - A\xi^{\alpha-1})$, where

$$\kappa = \begin{cases} 1, & \text{if (A1) holds, i.e., } A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}; \\ \frac{1}{2}, & \text{if (A2) holds, i.e., } A^2\xi^{2\alpha-2} = I. \end{cases} \tag{2.10}$$

For $A^2\xi^{2\alpha-2} = A\xi^{\alpha-1}$, we have

$$\begin{aligned} \rho_A^2 &= (I - A\xi^{\alpha-1})^2 = I - 2A\xi^{\alpha-1} + A^2\xi^{2\alpha-2} = I - A\xi^{\alpha-1} = \rho_A, \\ (I - \rho_A)(\xi^{2\alpha-1}A - I) &= A\xi^{\alpha-1}(\xi^{2\alpha-1}A - I) = \xi^{3\alpha-2}A^2 - A\xi^{\alpha-1} \\ &= (\xi^{\alpha} - 1)A\xi^{\alpha-1} = (\xi^{\alpha} - 1)(I - \rho_A). \end{aligned} \tag{2.11}$$

For $A^2\xi^{2\alpha-2} = I$, we have

$$\begin{aligned} \rho_A^2 &= \frac{1}{4}(I - A\xi^{\alpha-1})^2 = \frac{1}{4}(I - 2A\xi^{\alpha-1} + A^2\xi^{2\alpha-2}) = \frac{1}{2}(I - A\xi^{\alpha-1}) = \rho_A, \\ &(I - \rho_A)(\xi^{2\alpha-1}A - I) \\ &= \frac{1}{2}(I + A\xi^{\alpha-1})(\xi^{2\alpha-1}A - I) \\ &= \frac{1}{2}[\xi^{2\alpha-1}A - I + \xi^{3\alpha-2}A^2 - A\xi^{\alpha-1}] = \frac{1}{2}(\xi^{\alpha} - 1)(I + A\xi^{\alpha-1}) \\ &= (\xi^{\alpha} - 1)(I - \rho_A). \end{aligned} \tag{2.12}$$

It follows from (2.11) and (2.12) that ρ_A satisfies the following properties

$$\rho_A^2 = \rho_A, \quad (I - \rho_A)(\xi^{2\alpha-1}A - I) = (\xi^{\alpha} - 1)(I - \rho_A). \tag{2.13}$$

Furthermore, we note that if $y = ct^{\alpha-1}$, $c \in l^2$, then

$$\begin{aligned} h(y) &= \frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} y(s) ds - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} y(s) ds \\ &= \frac{(\xi^{2\alpha-1}A - I)c}{\Gamma(\alpha)\Gamma(2\alpha)}. \end{aligned} \tag{2.14}$$

Define the continuous linear mapping $Q : \mathbb{Y} \to \mathbb{Y}$ by

$$Qy(t) := \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^{\alpha} - 1}(I - \rho_A)h(y)t^{\alpha-1}, \quad t \in [0, 1],\ y \in \mathbb{Y}. \tag{2.15}$$

By the first identity in (2.13), we obtain $(I-\rho_A)^2=(I-\rho_A)$, which together with (2.13) implies

$$\begin{aligned}
Q^2y(t) &= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Qy(t))t^{\alpha-1}\\
&= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)\frac{(\xi^{2\alpha-1}A-I)}{\Gamma(\alpha)\Gamma(2\alpha)}\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(y)t^{\alpha-1}\\
&= \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)^2h(y)t^{\alpha-1}=Qy(t);
\end{aligned}$$

that is, Q is a projection operator. The equality $\ker Q=\operatorname{im}L$ follows from the trivial fact that

$$\begin{aligned}
&y\in\ker Q\Leftrightarrow h(y)\in\ker(I-\rho_A)\Leftrightarrow h(y)\in\operatorname{im}\rho_A\\
&\Leftrightarrow h(y)\in\operatorname{im}(I-A\xi^{\alpha-1})\Leftrightarrow y\in\operatorname{im}L.
\end{aligned}$$

Therefore, we get $\mathbb{Y}=\ker Q\oplus\operatorname{im}Q=\operatorname{im}L\oplus\operatorname{im}Q$.

Finally, we shall prove that $\operatorname{im}Q=\ker L$. Indeed, for any $z\in\operatorname{im}Q$, let $z=Qy$, $y\in\mathbb{Y}$. By (2.13), we have

$$k(I-A\xi^{\alpha-1})z(t)=\rho_Az(t)=\rho_AQy(t)=\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}\rho_A(I-\rho_A)g(y)t^{\alpha-1}=\theta,$$

which implies $z\in\ker L$. Conversely, for each $z\in\ker L$, there exists a constant $c^*\in\ker(I-A\xi^{\alpha-1})$ such that $z=c^*t^{\alpha-1}$ for $t\in[0,1]$. By (2.13) and (2.14), we derive

$$Qz(t)=\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(c^*t^{\alpha-1})t^{\alpha-1}=c^*t^{\alpha-1}=z(t),\quad t\in[0,1],$$

which implies that $z\in\operatorname{im}Q$. Hence we know that $\operatorname{im}Q=\ker L$, i.e., the operator L is a Fredholm operator of index zero. The proof is complete. □

Define the operator $P:\mathbb{X}\to\mathbb{X}$ as follows

$$Px(t)=\frac{1}{\Gamma(\alpha)}(I-\rho_A)D_{0+}^{\alpha-1}x(0)t^{\alpha-1}. \tag{2.16}$$

Lemma 2.11. *The mapping $P:\mathbb{X}\to\mathbb{X}$, defined by* (2.16)*, is a continuous projector such that*

$$\operatorname{im}P=\ker L,\quad \mathbb{X}=\ker L\oplus\ker P$$

and the linear operator $K_P:\operatorname{im}L\to\operatorname{dom}L\cap\ker P$ can be written as

$$K_Py(t)=I_{0+}^\alpha y(t),$$

also

$$K_P=(L|_{\operatorname{dom}L\cap\ker P})^{-1},\quad \|K_Py\|_{\mathbb{X}}\le 1/\Gamma(\alpha)\|y\|_{L^1([0,1];l^2)}.$$

Proof. By (2.16), one can see that P is a continuous operator. From the first identity of (2.13), we have $(I-\rho_A)^2=(I-\rho_A)$, which implies that the mapping P is a projector. Moreover, if $v\in\operatorname{im}P$, there exists a $x\in\mathbb{X}$ such that $v=Px$. By the first identity of (2.13) again, we see that

$$\frac{1}{\Gamma(\alpha)}(I-A\xi^{\alpha-1})(I-\rho_A)D_{0+}^{\alpha-1}x(0)=\frac{1}{k\Gamma(\alpha)}\rho_A(I-\rho_A)D_{0+}^{\alpha-1}x(0)=0,$$

which gives us $v \in \ker L$. Conversely, if $v \in \ker L$, then $v(t) = c_* t^{\alpha-1}$ for some $c_* \in \ker(I - A\xi^{\alpha-1})$, and we deduce that

$$Pv(t) = \frac{1}{\Gamma(\alpha)}(I-\rho_A)D_{0+}^{\alpha-1}v(0)t^{\alpha-1} = (I-\rho_A)c_* t^{\alpha-1} = c_* t^{\alpha-1} = v(t), \quad t \in [0,1],$$

which gives us $v \in \operatorname{im} P$. Thus, we get that $\ker L = \operatorname{im} P$ and consequently $\mathbb{X} = \ker L \oplus \ker P$.

Moreover, let $y \in \operatorname{im} L$. There exists $x \in \operatorname{dom} L$ such that $y = Lx$, and we obtain

$$K_P y(t) = x(t) + ct^{\alpha-1}$$

where $c \in l^2$ satisfies $c = \xi^{\alpha-1}Ac$. It is easy to see that $K_P y \in \operatorname{dom} L$ and $K_P y \in \ker P$. Therefore, K_P is well defined. Further, for $y \in \operatorname{im} L$, we have

$$L(K_P y(t)) = D_{0+}^{\alpha}(K_P y(t)) = y(t)$$

and for $x \in \operatorname{dom} L \cap \ker P$, we obtain that

$$K_P(Lx(t)) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-1},$$

for some $c_1, c_2 \in l^2$. In view of $x \in \operatorname{dom} L \cap \ker P$, we know that $c_1 = c_2 = \theta$. Therefore, $(K_P L)x(t) = x(t)$. This shows that $K_P = (L|_{\operatorname{dom} L \cap \ker P})^{-1}$. Finally, by the definition of K_P, we derive

$$\|D_{0+}^{\alpha-1}K_P y\|_{C([0,1];l^2)} = \Big\| \int_0^{\cdot} y(s)ds \Big\|_{C([0,1];l^2)} \le \|y\|_{L^1([0,1];l^2)} \tag{2.17}$$

and

$$\|K_P y\|_{C([0,1];l^2)} = \Big\| \frac{1}{\Gamma(\alpha)} \int_0^{\cdot} (\cdot - s)^{\alpha-1} y(s)ds \Big\|_{C([0,1];l^2)} \le \frac{1}{\Gamma(\alpha)} \|y\|_{L^1([0,1];l^2)}. \tag{2.18}$$

It follows from (2.17) and (2.18) that

$$\begin{aligned} \|K_P y\|_{\mathbb{X}} &= \max\{\|D_{0+}^{\alpha-1}K_P y\|_{C([0,1];l^2)}, \|K_P y\|_{C([0,1];l^2)}\} \\ &\le \max\Big\{\|y\|_{L^1([0,1];l^2)}, \frac{1}{\Gamma(\alpha)}\|y\|_{L^1([0,1];l^2)}\Big\} \\ &= \frac{1}{\Gamma(\alpha)}\|y\|_{L^1([0,1];l^2)}. \end{aligned} \tag{2.19}$$

This completes of the proof. □

Lemma 2.12. *Let f be a Carathéodory function. Then N, defined by* (2.3)*), is L-compact.*

Proof. Let Ω be a bounded subset in $\mathbb{X}$. By hypothesis (iii) on the function f, there exists a function $\varphi_\Omega(t) \in L^1[0,1]$ such that for all $x \in \Omega$,

$$\|f(t, x(t), D_{0+}^{\alpha-1}x(t))\|_{l^2} \le \varphi_\Omega(t), \quad \text{a.e. } t \in [0,1], \tag{2.20}$$

which, along with (2.9) implies

$$\begin{aligned} \|h(Nx(t))\|_{l^2} &= \Big\| \frac{A}{\Gamma(\alpha)} \int_0^{\xi} (\xi - s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1}x(s))ds \\ &\quad - \frac{I}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), D_{0+}^{\alpha-1}x(s))ds \Big\|_{l^2} \\ &\le \frac{\|A\|+1}{\Gamma(\alpha)} \|\varphi_\Omega\|_{L^1[0,1]}. \end{aligned} \tag{2.21}$$

Thus, from (2.15) and (2.21) it follows that

$$\begin{aligned} \|QNx\|_{L^1([0,1];l^2)} &= \Big\|\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1}(I-\rho_A)h(Nx)\Big\|_{l^2}\int_0^1 s^{\alpha-1}ds \\ &\le \frac{\Gamma(2\alpha)(\|A\|+1)\|I-\rho_A\|}{|1-\xi^\alpha|}\|\varphi_\Omega\|_{L^1[0,1]} < \infty. \end{aligned} \tag{2.22}$$

This shows that $QN(\overline{\Omega}) \subseteq \mathbb{Y}$ is bounded. The continuity of QN follows from the hypothesis on f and the Lebesgue dominated convergence theorem.

Next, we shall show that $K_{P,Q}N$ is completely continuous. First, for any $x \in \Omega$, we have

$$\begin{aligned} K_{P,Q}Nx(t) &= K_P(I-Q)Nx(t) = K_PNx(t) - K_PQNx(t) \\ &= I_{0+}^\alpha Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))I_{0+}^\alpha t^{\alpha-1}. \end{aligned} \tag{2.23}$$

and

$$D_{0^+}^{\alpha-1}K_{P,Q}Nx(t) = I_{0+}^1 Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))I_{0+}^1 t^{\alpha-1}. \tag{2.24}$$

By the hypothesis on f and the Lebesgue dominated convergence theorem, it is easy to see that $K_{P,Q}N$ is continuous. Since f is a Carathéodory function, for every bounded set $\Omega_0 \subseteq l^2 \times l^2$, the set $\{f(t,u,v) : (u,v) \in \Omega_0\}$ is relatively compact set in l^2. Therefore, for almost all $t \in [0,1]$, $\{K_{P,Q}Nx(t) : x \in \Omega\}$ and $\{D_{0^+}^{\alpha-1}K_{P,Q}Nx(t) : x \in \Omega\}$ are relatively compact in l^2.

From (2.21), (2.23) and (2.24), we derive that

$$\begin{aligned} &\|K_{P,Q}Nx\|_{C([0,1];l^2)} \\ &= \Big\|I_{0+}^\alpha Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))I_{0+}^\alpha t^{\alpha-1}\Big\|_{C([0,1];l^2)} \\ &\le \frac{1}{\Gamma(\alpha)}\|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I-\rho_A\|}{|\xi^\alpha-1|}\|h(Nx(t))\|_{l^2} \\ &\le \frac{1}{\Gamma(\alpha)}\|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I-\rho_A\|(\|A\|+1)}{\Gamma(\alpha)|\xi^\alpha-1|}\|\varphi_\Omega\|_{L^1(0,1)} < \infty, \end{aligned}$$

and

$$\begin{aligned} &\|D_{0^+}^{\alpha-1}K_{P,Q}Nx\|_{C([0,1];l^2)} \\ &= \Big\|I_{0+}^1 Nx(t) - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))I_{0+}^1 t^{\alpha-1}\Big\|_{C([0,1];l^2)} \\ &\le \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I-\rho_A\|}{|\xi^\alpha-1|}\|h(Nx(t))\|_{l^2} \\ &\le \|\varphi_\Omega\|_{L^1(0,1)} + \frac{\Gamma(2\alpha)\|I-\rho_A\|(\|A\|+1)}{\Gamma(\alpha)|\xi^\alpha-1|}\|\varphi_\Omega\|_{L^1(0,1)} < \infty, \end{aligned}$$

which shows that $K_{P,Q}N\overline{\Omega}$ is uniformly bounded in $\mathbb{X}$. Noting that

$$b^p - a^p \le (b-a)^p \quad \text{for any } b \ge a > 0, 0 < p \le 1. \tag{2.25}$$

for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, we shall see that

$$\|K_{P,Q}Nx(t_2) - K_{P,Q}Nx(t_1)\|_{l^2}$$

$$
\begin{aligned}
&= \frac{1}{\Gamma(\alpha)}\Big\| \int_0^{t_1} [(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]Nx(s)ds + \int_{t_1}^{t_2} (t_2-s)^{\alpha-1}Nx(s)ds \\
&\quad - \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))[I_{0+}^\alpha t_2^{\alpha-1} - I_{0+}^\alpha t_1^{\alpha-1}]\Big\|_{l^2} \\
&\le \frac{1}{\Gamma(\alpha)}\int_0^{t_1} (t_2-t_1)^{\alpha-1}\varphi_\Omega(s)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \varphi_\Omega(s)ds \\
&\quad + \frac{\Gamma^2(\alpha)\|I-\rho_A\|(\|A\|+1)}{|\xi^\alpha-1|}\|\varphi_\Omega\|_{L^1(0,1)}|t_2^{2\alpha-1}-t_1^{2\alpha-1}| \to 0 \quad \text{as } t_2\to t_1
\end{aligned}
$$

and

$$
\begin{aligned}
&\|D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_2) - D_{0^+}^{\alpha-1}K_{P,Q}Nx(t_1)\|_{l^2} \\
&= \Big\|\int_{t_1}^{t_2} Nx(s)ds\Big\|_{l^2} + \Big\|\frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha-1}(I-\rho_A)h(Nx(t))\int_{t_1}^{t_2} s^{\alpha-1}ds\Big\|_{l^2} \\
&\le \int_{t_1}^{t_2}\varphi_\Omega(s)ds + \frac{\Gamma(2\alpha)\|I-\rho_A\|(\|A\|+1)}{|\xi^\alpha-1|}\|\varphi_\Omega\|_{L^1(0,1)}|t_2^\alpha - t_1^\alpha| \to 0 \quad \text{as } t_2\to t_1.
\end{aligned}
$$

Then $K_{P,Q}N\overline{\Omega}$ is equicontinuous in $\mathbb{X}$. By Lemma 2.8, $K_{P,Q}N\overline{\Omega}\subseteq\mathbb{X}$ is relatively compact. Thus we can conclude that the operator N is L-compact in $\overline{\Omega}$. The proof is complete. □

3. Main results

Theorem 3.1. *Let f be a Carathéodory function and the following conditions hold:*

(H1) *There exist five nonnegative functions $a_1,a_2,b_1,b_2,c\in L^1[0,1]$ and constants $\gamma_1,\gamma_2\in[0,1)$ such that for all $t\in[0,1]$, $u,v\in l^2$,*

$$
\|f(t,u,v)\|_{l^2} \le a_1(t)\|u\|_{l^2} + b_1(t)\|v\|_{l^2} + a_2(t)\|u\|_{l^2}^{\gamma_1} + b_2(t)\|v\|_{l^2}^{\gamma_2} + c(t).
$$

(H2) *There exists a constant $A_1>0$ such that for $x\in\operatorname{dom}L$, if $\|D_{0^+}^{\alpha-1}x(t)\|_{l^2} > A_1$ for all $t\in[0,1]$, then*

$$
\begin{aligned}
&\frac{A}{\Gamma(\alpha)}\int_0^\xi (\xi-s)^{\alpha-1}f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds \\
&- \frac{I}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}f(s,x(s),D_{0^+}^{\alpha-1}x(s))ds \notin \operatorname{im}(I-A\xi^{\alpha-1}).
\end{aligned}
$$

(H3) *There exists a constant $A_2>0$ such that for any $e=\{(e_i)\}\in l^2$ satisfying $e=\xi^{\alpha-1}Ae$ and $\|e\|_{l^2}>A_2$, either*

$$
\langle e,QNe\rangle_{l^2}\le 0 \quad \text{or} \quad \langle e,QNe\rangle_{l^2}\ge 0,
$$

where $\langle\cdot,\cdot\rangle_{l^2}$ is the inner product in l^2.

Then (1.1) has at least one solution in space X provided that

$$
\begin{gathered}
\Gamma(\alpha) > \max\big\{(\|I-\rho_A\|+1)\|a_1\|_{L^1(0,1)}, (\|I-\rho_A\|+1)\|b_1\|_{L^1(0,1)}\big\}, \\
\frac{(\|I-\rho_A\|+1)^2\|a_1\|_{L^1(0,1)}\|b_1\|_{L^1(0,1)}}{(\Gamma(\alpha)-(\|I-\rho_A\|+1)\|a_1\|_{L^1(0,1)})(\Gamma(\alpha)-(\|I-\rho_A\|+1)\|b_1\|_{L^1(0,1)})} < 1.
\end{gathered} \tag{3.1}
$$

Proof. We shall construct an open bounded subset Ω in X satisfying all assumption of Lemma 2.7. Let

$$
\Omega_1 = \big\{x\in\operatorname{dom}L\setminus\ker L : Lx=\lambda Nx \text{ for some } \lambda\in[0,1]\big\}. \tag{3.2}
$$

For any $x \in \Omega_1$, $x \notin \ker L$, we have $\lambda \neq 0$. Since $Nx \in \operatorname{im} L = \ker Q$, by (2.8), we have $h(Nx) \in \operatorname{im}(I - A\xi^{\alpha-1})$, where

$$\begin{aligned} h(Nx) = \frac{A}{\Gamma(\alpha)} \int_0^\xi (\xi - s)^{\alpha-1} f(s, x(s), D_{0^+}^{\alpha-1} x(s)) ds \\ - \frac{I}{\Gamma(\alpha)} \int_0^1 (1 - s)^{\alpha-1} f(s, x(s), D_{0^+}^{\alpha-1} x(s)) ds. \end{aligned} \tag{3.3}$$

From (H2) there exists $t_0 \in [0, 1]$ such that $|D_{0^+}^{\alpha-1} x(t_0)|_{l^2} \le A_1$. Then from the equality $D_{0^+}^{\alpha-1} x(0) = D_{0^+}^{\alpha-1} x(t_0) - \int_0^t D_{0+}^\alpha x(s) ds$, we deduce that

$$\|D_{0^+}^{\alpha-1} x(0)\|_{l^2} \le A_1 + \|D_{0+}^\alpha x\|_{L^1(0,1;l^2)} = A_1 + \|Lx\|_1 \le A_1 + \|Nx\|_{L^1(0,1;l^2)},$$

which implies

$$\|Px\|_{\mathbb{X}} = \|\frac{1}{\Gamma(\alpha)} (I - \rho_A) D_{0^+}^{\alpha-1} x(0) t^{\alpha-1}\|_{\mathbb{X}} \le \frac{\|I - \rho_A\|}{\Gamma(\alpha)} (A_1 + \|Nx\|_{L^1(0,1;l^2)}). \tag{3.4}$$

Further, for $x \in \Omega_1$, since $\operatorname{im} P = \ker L, X = \ker L \oplus \ker P$, we have $(I - P)x \in \operatorname{dom} L \cap \ker P$ and $LPx = \theta$. Then

$$\begin{aligned} \|(I - P)x\|_{\mathbb{X}} &= \|K_P L(I - P)x\|_{\mathbb{X}} \le \|K_P Lx\|_{\mathbb{X}} \\ &\le \frac{1}{\Gamma(\alpha)} \|Lx\|_{L^1(0,1;l^2)} \le \frac{1}{\Gamma(\alpha)} \|Nx\|_{L^1(0,1;l^2)}. \end{aligned} \tag{3.5}$$

From (3.4) and (3.5), we conclude that

$$\begin{aligned} \|x\|_{\mathbb{X}} &= \|Px + (I - P)x\|_{\mathbb{X}} \le \|Px\|_{\mathbb{X}} + \|(I - P)x\|_{\mathbb{X}} \\ &\le \frac{\|I - \rho_A\|}{\Gamma(\alpha)} A_1 + \frac{\|I - \rho_A\| + 1}{\Gamma(\alpha)} \|Nx\|_{L^1(0,1;l^2)}. \end{aligned} \tag{3.6}$$

Moreover, by the definition of N and (H1), one has

$$\begin{aligned} &\|Nx\|_{L^1(0,1;l^2)} \\ &= \int_0^1 \|f(s, x(s), D_{0^+}^{\alpha-1} x(s))\|_{l^2} dt \\ &\le \|a_1\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)} + \|b_1\|_{L^1(0,1)} \|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)} \\ &\quad + \|a_2\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)}^{\gamma_1} + \|b_2\|_{L^1(0,1)} \|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)}^{\gamma_2} + \|c\|_{L^1(0,1)}. \end{aligned} \tag{3.7}$$

Thus,

$$\begin{aligned} \|x\|_{\mathbb{X}} &\le \frac{\|I - \rho_A\|}{\Gamma(\alpha)} A_1 + \frac{\|I - \rho_A\| + 1}{\Gamma(\alpha)} \Big(\|a_1\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)} \\ &\quad + \|b_1\|_{L^1(0,1)} \|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)} \Big) + \frac{\|I - \rho_A\| + 1}{\Gamma(\alpha)} \\ &\quad \times \Big(\|a_2\|_{L^1(0,1)} \|x\|_{C([0,1];l^2)}^{\gamma_1} + \|b_2\|_{L^1(0,1)} \|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)}^{\gamma_2} \\ &\quad + \|c\|_{L^1(0,1)} \Big). \end{aligned} \tag{3.8}$$

It follows from (3.1), (3.8), $\|x\|_{C([0,1];l^2)} \le \|x\|_{\mathbb{X}}$, $\|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)} \le \|x\|_{\mathbb{X}}$ and Lemma 2.9 that there exists $M > 0$ such that

$$\max\{\|x\|_{C([0,1];l^2)}, \|D_{0^+}^{\alpha-1} x\|_{C([0,1];l^2)}\} \le M,$$

that is to say Ω_1 is bounded.

Let

$$\Omega_2 = \{x \in \ker L : Nx \in \operatorname{im} L\}. \tag{3.9}$$

For any $x \in \Omega_2$, it follows from $x \in \ker L$ that $x = et^{\alpha-1}$ for some $e \in \ker(I - A\xi^{\alpha-1}) \subset l^2$, and it follows from $Nx \in \operatorname{im} L$ that $h(Nx) \in \operatorname{im}(I - A\xi^{\alpha-1})$, where $h(Nx)$ is defined by (3.3). By hypothesis (H_2), we arrive at $\|D_{0^+}^{\alpha-1}x(t_0)\|_{l^\infty} = \|e\|_{l^2}\Gamma(\alpha) \le A_1$. Thus we obtain

$$\|x\| \le \|e\|_{l^\infty}\Gamma(\alpha) \le A_1.$$

That is, Ω_2 is bounded in X. If the first part of (H_3) holds, denote

$$\Omega_3 = \{x \in \ker L : -\lambda x + (1-\lambda)QNx = \theta,\ \lambda \in [0,1]\},$$

then for any $x \in \Omega_3$, we know that

$$x = et^{\alpha-1} \quad \text{with } e \in \ker(I - A\xi^{\alpha-1}) \text{ and } \lambda x = (1-\lambda)QNx.$$

If $\lambda = 0$, we have $Nx \in \ker Q = \operatorname{im} L$, then $x \in \Omega_2$, by the argument above, we get that $\|x\| \le A_1$. Moreover, if $\lambda \in (0,1]$ and if $\|e\|_{l^2} > A_2$, by (H3), we deduce that

$$0 < \lambda\|e\|_{l^2}^2 = \lambda\langle e, e\rangle_{l^2} = (1-\lambda)\langle e, QNe\rangle_{l^2} \le 0,$$

which is a contradiction. Then $\|x\|_{\mathbb{X}} = \|et^{\alpha-1}\|_{\mathbb{X}} \le \max\{\|e\|_{l^2}, \Gamma(\alpha)\|e\|_{l^2}\}$. That is to say, Ω_3 is bounded. If the other part of (H3) holds, we take

$$\Omega_3 = \{x \in \ker L : \lambda x + (1-\lambda)QNx = \theta,\ \lambda \in [0,1]\}.$$

By using the same arguments as above, we can conclude that Ω_3 is also bounded.

Next, we show that all conditions of Lemma 2.7 are satisfied. Assume that Ω is a bounded open subset of $\mathbb{X}$ such that $\cup_{i=1}^3 \overline{\Omega}_i \subseteq \Omega$. It follows from Lemmas 2.10 and 2.12 that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. By the definition of Ω and the argument above, to complete the theorem, we only need to prove that condition (iii) of Lemma 2.7 is satisfied. For this purpose, let

$$H(x,\lambda) = \pm\lambda x + (1-\lambda)QNx, \tag{3.10}$$

where we let the isomorphism the $J : \operatorname{im} Q \to \ker L$ be the identical operator. Since $\Omega_3 \subseteq \Omega$, $H(x,\lambda) \ne 0$ for $(x,\lambda) \in \ker L \cap \partial\Omega \times [0,1]$, then by homotopy property of degree, we obtain

$$\begin{aligned}
\deg\left(JQN|_{\ker L\cap\partial\Omega}, \Omega \cap \ker L, 0\right) &= \deg\left(H(\cdot,0), \Omega \cap \ker L, 0\right) \\
&= \deg\left(H(\cdot,1), \Omega \cap \ker L, 0\right) \\
&= \deg\left(\pm Id, \Omega \cap \ker L, 0\right) = \pm 1 \ne 0.
\end{aligned}$$

Thus (H3) of Lemma 2.7 is fulfilled and Theorem 3.1 is proved. □

4. Example

In this section, we shall present an example to illustrate our main result in l^2 with $\dim\ker L = 2k$, which surely generalize the previous results [3, 4, 8, 9, 13, 14, 18, 20], where the dimension of $\dim\ker L$ is only 1 or 2.

Consider the following system with $\dim\ker L = 2k,\ k = 1, 2, 3, \dots$ in l^2.

$$
\begin{gathered}
D_{0^+}^{3/2}\begin{pmatrix} x_1(t)\\ x_2(t)\\ x_3(t)\\ x_4(t)\\ \vdots \end{pmatrix}
= \frac{1}{10}\begin{pmatrix}
\begin{cases} 1, & \text{if } \|D_{0^+}^{1/2}x(t)\|_{l^2} < 1\\ D_{0^+}^{1/2}x_1(t) + [D_{0^+}^{1/2}x_1(t)]^{-1} - 1, & \text{if } \|D_{0^+}^{1/2}x(t)\|_{l^2} \geq 1 \end{cases}\\
\big(x_2(t) + D_{0^+}^{1/2}x_3(t)\big)/2\\
\big(x_3(t) + D_{0^+}^{1/2}x_3(t)\big)/2^2\\
\big(x_4(t) + D_{0^+}^{1/2}x_4(t)\big)/2^3\\
\vdots
\end{pmatrix} \\
x_i(0) = 0, \quad i = 1, 2, \dots \\
x(1) = Ax(1/9).
\end{gathered} \tag{4.1}
$$

Let $\alpha = 3/2, \xi = 1/9$. For all $t \in [0, 1]$, let $u = (x_1, x_2, x_3, \dots)$, $v = (y_1, y_2, y_3, \dots) \in l^2$ and $f = (f_1, f_2, \dots)^T$ with

$$
f_1(t, u, v) = \begin{cases} 1/10, & \text{if } \|v\|_{l^2} < 1,\\ (y_1 + y_1^{-1} - 1)/10, & \text{if } \|v\|_{l^2} \geq 1, \end{cases}
$$

$f_2(t, u, v) = (x_2 + y_3)/20$ and $f_i(t, u, v) = \frac{1}{5}\frac{x_i + y_i}{2^i}$, $i = 3, 4, \dots$. Moreover,

$$
A = \begin{bmatrix}
B_1 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & B_2 & 0 & 0 & 0 & 0 & \dots\\
\vdots & & \ddots & & & & \vdots\\
0 & 0 & 0 & B_k & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots\\
0 & 0 & 0 & 0 & 0 & 0 & \dots\\
\vdots & & & & & & \ddots
\end{bmatrix} \quad \text{with} \quad B_i = \begin{bmatrix} 3 & 0 & 0\\ 0 & -3 & 6\\ 0 & 0 & 3 \end{bmatrix}, \tag{4.2}
$$

$i = 1, 2, \dots, k$, $k \in \mathbf{N}$. Obviously, we see that $B_i^2 = 9I_3$ and $\dim\ker(I_3 - \xi^{\alpha-1}B_i) = \dim\ker(I_3 - B_i/3) = 2$, $i = 1, 2, \dots,$ where I_3 is the 3×3 identity matrix. Then $A^2\xi^{2\alpha-2} = I$, $\dim\ker(I - A\xi^{\alpha-1}) = 2k$, $k \in \mathbf{N}$ and the problem (4.1), with A and f defined above, has one solution if and only if problem (1.1) admits one solution.

Checking (H1) of Theorem 3.1: For some $r \in \mathbb{R}$, $\Omega = \{(u, v) \in l^2 \times l^2 : \|u\|_{l^2} \leq r, \|v\|_{l^2} \leq r\}$, let $\varphi_\Omega(t) = \frac{1}{10}[(r + 1/r + 1)^2 + \frac{4r^2}{3}]^{1/2} \in L^1[0, 1]$. Since $\|A\|_{l^2} \leq 9\sqrt{k}$, letting

$$
a_1(t) = b_1(t) = \frac{1}{5\sqrt{3}}, \quad a_2(t) = b_2(t) = 0, \quad c(t) = \frac{r + 1/r + 1}{10}. \tag{4.3}
$$

condition (H1) is satisfied.

Checking (H2) of Theorem 3.1: From the definition of f it follows that $f_1 > 1/10 > 0$ when $\|D_{0^+}^{1/2}x(t)\|_{l^2} > 1$. This,

$$(B_1\xi^\alpha - I)\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix} = \begin{bmatrix} -8/9 & 0 & 0\\ 0 & -10/9 & 2/9\\ 0 & 0 & -8/9\end{bmatrix}\begin{pmatrix} f_1\\ f_2\\ f_3\end{pmatrix} = \begin{pmatrix} \frac{-8f_1}{9}\\ *\\ *\end{pmatrix}$$

and $\operatorname{im}(I - A\xi^{\alpha-1}) = \{(0,0,\tau_3,0,0,\tau_6,\dots,0,0,\tau_{3i},\dots) : \tau_{3i} \in \mathbb{R}, i = 1,2,\dots\}$ implies that condition (H2) is satisfied.

Checking (H3) of Theorem 3.1: Since $\dim\ker(I - A\xi^{\alpha-1}) = \dim\ker(I - A/3) = 2k$, $k \in \mathbf{N}$, for any $e \in l^2$ satisfying $e = Ae$, e can be expressed as $e = e_1 + e_2 + \dots + e_k$, with

$$e_i = \sigma_{i1}\varepsilon_{3i-2} + \sigma_{i2}(\varepsilon_{3i-1} + \varepsilon_{3i}), \quad \sigma_{ij} \in \mathbb{R}, \quad i = 1,2,\dots,k, j = 1,2,$$

where $\varepsilon_j = \big(0,0,\dots 0, 1_{\mathrm{j}-th}, 0,0,\dots\big) \in l^2$ is a vector with all elements equaling to 0 except the j-th equaling to 1, $j = 1,2,\dots$. In addition, for any $y \in \mathbb{Y}$, by (2.15) and $\rho_A = \frac{1}{2}(I - A/3)$, we have

$$Qy(t) = \frac{\Gamma(\alpha)\Gamma(2\alpha)}{\xi^\alpha - 1}(I - \rho_A)h(y)t^{\alpha-1} = \frac{-27\sqrt{\pi}}{52}(I + A/3)h(y)t^{\alpha-1}, \tag{4.4}$$

where

$$h(y) = \frac{A}{\Gamma(\alpha)}\int_0^\xi (\xi - s)^{\alpha-1}y(s)ds - \frac{I}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}y(s)ds. \tag{4.5}$$

By (2.3), let $d = t^{1/2} + \frac{\sqrt{\pi}}{2}$, we have

$$N(et^{1/2}) = \frac{1}{10}\begin{cases} \Big(1, \frac{d\sigma_{12}}{2}, \frac{d\sigma_{12}}{2^2}, \frac{d\sigma_{21}}{2^3}, \frac{d\sigma_{22}}{2^4}, \frac{d\sigma_{22}}{2^5}, \dots, \\ \frac{d\sigma_{i1}}{2^{3i-3}}, \frac{d\sigma_{i2}}{2^{3i-2}}, \frac{d\sigma_{i2}}{2^{3i-1}}, \dots\Big)^\top_, \\ \quad \text{if } |\sigma_{11}| < 1,\ 2 \le i \le k; \\ \Big(\sigma_{11} + \frac{1}{\sigma_{11}} - 1, \frac{d\sigma_{12}}{2}, \frac{d\sigma_{12}}{2^2}, \frac{d\sigma_{21}}{2^3}, \frac{d\sigma_{22}}{2^4}, \frac{d\sigma_{22}}{2^5}, \dots, \\ \frac{d\sigma_{i1}}{2^{3i-3}}, \frac{d\sigma_{i2}}{2^{3i-2}}, \frac{d\sigma_{i2}}{2^{3i-1}}, \dots\Big)^\top_, \\ \quad \text{if } |\sigma_{11}| \ge 1,\ 2 \le i \le k. \end{cases} \tag{4.6}$$

Suppose that $|\sigma_{11}| > 1$, $\sigma_{12} \neq 0$, and let $A_2 = 1$, $\widetilde{d} = \frac{-27\pi - 208\sqrt{\pi}}{648} < 0$. From (4.4) and (4.6) it follows that

$$\begin{aligned} Q(Net^{1/2}) &= \frac{-27\sqrt{\pi}}{520}(I + A/3)h(Net^{1/2})t^{1/2}\\ &= \frac{-27\sqrt{\pi}t^{1/2}}{520}\Big(\frac{-64}{27\sqrt{\pi}}(\sigma_{11} + \frac{1}{\sigma_{11}} - 1), \frac{\widetilde{d}\sigma_{12}}{2^2}, \frac{\widetilde{d}\sigma_{12}}{2^2}, \frac{\widetilde{d}\sigma_{21}}{2^3}, \frac{\widetilde{d}\sigma_{22}}{2^5}, \frac{\widetilde{d}\sigma_{22}}{2^5},\\ &\qquad \dots, \frac{\widetilde{d}\sigma_{i1}}{2^{3i-3}}, \frac{\widetilde{d}\sigma_{i2}}{2^{3i-1}}, \frac{\widetilde{d}\sigma_{i2}}{2^{3i-1}}, \dots\Big)^T \end{aligned}$$

and

$$\begin{aligned} \langle e, QNet^{1/2}\rangle &= \frac{-27\sqrt{\pi}t^{1/2}}{520}\Big[\frac{-64}{27\sqrt{\pi}}((\sigma_{11} - 1/2)^2 + 3/4)\\ &\quad + \widetilde{d}\Big(\frac{2\sigma_{12}^2}{2^2} + \frac{\sigma_{21}^2}{2^3} + \frac{2\sigma_{22}^2}{2^5} + \dots + \frac{\sigma_{i1}^2}{2^{3i-3}} + \frac{2\sigma_{i2}^2}{2^{3i-1}} + \dots\Big)\Big] > 0. \end{aligned}$$

Therefore, (4.1) admits at least one solution.

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