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QUASI-SPECTRAL DECOMPOSITION OF THE HEAT POTENTIAL

TYNYSBEK SH. KAL'MENOV, GAUKHAR D. AREPOVA

ABSTRACT. In this article, by multiplying of the unitary operator

$$(Pf)(x,t) = f(x, T-t), \quad 0 \le t \le T,$$

the heat potential turns into a self-adjoint operator. From the spectral decomposition of this completely continuous self-adjoint operator we obtain a quasi-spectral decomposition of the heat potential operator.

1. Introduction

In the works of Gohberg and Krein [2], it is proven that for any linear completely-continuous operator A, in a Hilbert space H, has a triangular representation $A = U(A^*A)^{1/2}$, where A^* is an adjoint operator to A, and U a unitary operator. When the operator A is a completely-continuous Volterra operator generated by a mixed solution of the Cauchy problem for parabolic and hyperbolic equations proposes, it is of great interest. In this article we give a new analogue of a triangular representation of multi-dimensional heat potential and its quasi-spectral expansion.

2. Main results

Let $\Omega \subset R^n$ be a finite domain with a smooth boundary $\partial\Omega \in C^1$, and $D = \Omega \times (0,T)$. In the domain D we define the heat potential (see e.g. [1, 11]) by the formula

$$u = \Diamond^{-1} f \equiv \int_0^t d\tau \int_\Omega \varepsilon_n(x-\xi, t-\tau) f(\xi,\tau) d\xi \tag{2.1}$$

where

$$\varepsilon_n(x,t) = \frac{\theta(t)}{(2\sqrt{\pi t})^n} e^{-\frac{|x|^2}{4t}} \tag{2.2}$$

is the fundamental solution of the heat equation

$$\Diamond \varepsilon_n(x,t) \equiv (\frac{\partial}{\partial t} - \Delta_x)\varepsilon_n(x,t) = \delta(x,t), \tag{2.3}$$

$$\varepsilon_n(x,t)|_{t=0} = 0. \tag{2.4}$$

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For $f \in L_2(\Omega)$ it is easy to verify that

$$\lozenge u = \lozenge\lozenge^{-1} f = \lozenge \int_0^t d\tau \int_\Omega \varepsilon_n(x-\xi, t-\tau) f(\xi,\tau) d\xi = f(x,t), \quad u|_{t=0} = 0. \tag{2.5}$$

In the work by Kalmenov, Tokmagambetov [7] (see also [3, 4, 5, 6, 9]), it is shown that the heat potential $u = \lozenge^{-1} f$ at any $f \in L_2(\Omega)$ satisfies the following boundary conditions

$$\begin{aligned} &\frac{u(x,t)}{2} - \int_0^t d\tau \int_{\partial\Omega} \Big(\frac{\partial \varepsilon_n}{\partial n_\xi}(x-\xi, t-\tau) u(\xi,\tau) \\ &- \varepsilon_n(x-\xi, \tau - t) \frac{\partial u}{\partial n_\xi}(\xi,\tau) \Big) d\xi = 0, \quad x \in \partial\Omega, \ t \in [0,T]. \end{aligned} \tag{2.6}$$

Conversely, for any $f \in L_2(D)$, solution of (2.5) defines the heat potential by formula (2.1). Here, $\frac{\partial}{\partial n_\xi}$ is unit normal derivative at $\partial\Omega$.

Note that the operator $\lozenge^{-1}$ is completely-continuous on L_2 for any $f \in L_2(\Omega)$, $u = \lozenge^{-1} f \in W_2^{2,1}(D)$. The operator $\lozenge^{-1}$ is a Volterra operator, i.e. it has no nontrivial eigenvectors.

Let us define the operator P by

$$(Pf)(x,t) = f(x, T-t), \quad 0 \le t \le T. \tag{2.7}$$

It is clear that P is a bounded self-adjoint operator satisfying

$$P = P^*, \quad P^2 = I. \tag{2.8}$$

Lemma 2.1. *The operator $P\lozenge^{-1}$ is a completely-continuous self-adjoint operator.*

Proof. Let us rewrite the operator $P\lozenge^{-1}$ in the form

$$\begin{aligned} P\lozenge^{-1} f &= P\Big(\int_0^T \theta(t-\tau) d\tau \int_\Omega \varepsilon_n(x-\xi, t-\tau) f(\xi,\tau) d\xi \Big) \\ &= \int_0^T \theta(T-t-\tau) d\tau \int_\Omega \varepsilon_n(x-\xi, T-t-\tau) f(\xi,\tau) d\xi. \end{aligned} \tag{2.9}$$

By using a direct computation for any $f, g \in L_2(D)$ it can be shown that

$$\begin{aligned} &(P\lozenge^{-1} f, g)_{L_2(D)} \\ &= \int_0^T dt \int_\Omega (P\lozenge^{-1} f)(x,t) g(x,t) dx \\ &= \int_0^T dt \int_\Omega \int_0^T \theta(T-t-\tau) \int_\Omega \varepsilon_n(x-\xi, T-t-\tau) f(\xi,\tau) d\xi g(x,t) dx \\ &= \int_0^T \int_\Omega f(\xi,t) dx \int_0^T \theta(T-t-\tau) \int_\Omega \varepsilon_n(x-\xi, T-t-\tau) g(x,t) dx d\xi \\ &= \int_0^T d\tau \int_\Omega f(\xi,\tau) P\Big(\int_0^T \theta(\tau - t) dt \int_\Omega \varepsilon_n(x-\xi, \tau-t) g(x,t) dx \Big) d\xi \\ &= (f, P\lozenge^{-1} g)_{L_2(D)}. \end{aligned} \tag{2.10}$$

On the other hand,

$$(P\lozenge^{-1} f, g)_{L_2(D)} = (f, (P\lozenge^{-1})^* g)_{L_2(D)}. \tag{2.11}$$

Because of the arbitrariness of $f, g \in L_2(D)$ we obtain

$$(P\lozenge^{-1})^* = P\lozenge^{-1}.$$

This completes the proof. □

According to the theory of regular extensions of the linear operator (Otelbaev [8] and Vishik [10]) self-adjoint differential operators are generated only by boundary conditions.

Lemma 2.2. *For $f \in L_2(D)$ the function $u = P\lozenge^{-1}f \in W_2^{1,2}(D) \cap W_2^1(\partial D)$ satisfies the equation*

$$\lozenge Pu = f, \tag{2.12}$$

the initial condition

$$u|_{t=T} = 0, \tag{2.13}$$

and the lateral boundary condition

$$\begin{aligned} &-\frac{(Pu)(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} (\frac{\partial \varepsilon_n}{\partial n_\xi}(x-\xi,\tau-t)Pu(\xi,\tau)d\xi) \\ &-\int_0^t d\tau \int_{\Omega} (\varepsilon_n(x-\xi,\tau-t)P\frac{\partial u}{\partial n_\xi}(\xi,\tau)d\tau) = 0, \quad x \in \partial\Omega, t \in [0,T]. \end{aligned} \tag{2.14}$$

Conversely, if $u \in W_2^{1,2}(D) \cap W_2^1(\partial D)$ satisfies (2.12)*, the initial condition* (2.13) *and the lateral boundary condition* (2.14)*, then $u = P\lozenge^{-1}f$.*

Proof. In view of $\lozenge Pu = f$, where $u \in W_2^{1,2}(D) \cap W_2^1(\partial D)$ satisfies the initial condition (2.13) and the lateral boundary condition (2.14), it is easy to prove (see [7]) that $v = Pu = \lozenge^{-}1f$, where

$$v = \lozenge^{-1}\lozenge\vartheta = \int_0^t d\tau \int_\Omega \varepsilon_n(x-\xi,\tau-t)(\frac{\partial}{\partial\tau} - \Delta_\xi)\vartheta(\xi,\tau)d\xi. \tag{2.15}$$

It is easy to check as in [7] that

$$\begin{aligned} &-\frac{\vartheta(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} (\frac{\partial \varepsilon_n}{\partial n_\xi}(x-\xi,t-\tau)\vartheta(\xi,\tau) \\ &-\varepsilon_n(x-\xi,\tau-t)\frac{\partial u}{\partial n_\xi}(\xi,\tau))d\xi = 0, \quad x \in \partial\Omega,\ t \in [0,T]. \end{aligned} \tag{2.16}$$

$$v\big|_{t=0} = 0 \tag{2.17}$$

By taking into account $v = Pu$ we will rewrite (2.16)–(2.17) in the form

$$\begin{aligned} &-\frac{(Pu)(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} (\frac{\partial \varepsilon_n}{\partial n_\xi}(x-\xi,t-\tau)(Pu)(\xi,\tau) \\ &-\varepsilon_n(x-\xi,\tau-t)\frac{\partial Pu}{\partial n_\xi}(\xi,\tau))d\xi = 0, \quad x \in \partial\Omega,\ t \in [0,T]. \end{aligned} \tag{2.18}$$

$$u\big|_{t=T} = 0 \tag{2.19}$$

This completes the proof. □

Since the operator $P\lozenge^{-1}$ is completely-continuous and self-adjoint throughout $L_2(\Omega)$, then it has a complete orthonormal system of eigenvectors $e_k(x,t)$ associated with real eigenvalues λ_k,

$$\lambda_k(P\lozenge^{-1})e_k = e_k. \tag{2.20}$$

Then

$$\begin{aligned} P\Diamond^{-1}f &= \sum_k (P\Diamond^{-1}f, e_k)_0 e_k = \sum_k (f, (P\Diamond^{-1})e_k)_0 e_k \\ &= \sum_k (f, \frac{e_k}{\lambda_k})e_k = \sum_k \frac{1}{\lambda_k}(f, e_k)e_k. \end{aligned} \tag{2.21}$$

Applying the operator P to both sides of (2.21), we obtain

$$\Diamond^{-1}f = \sum_k \frac{1}{\lambda_k}(f, e_k)Pe_k. \tag{2.22}$$

The decomposition of $\Diamond^{-1}f$ through orthonormal system Pe_k is called a quasi-spectral expansion of the heat potential $\Diamond^{-1}$. This proves the following theorem.

Theorem 2.3. *Let e_k be a complete orthonormal system of eigenvectors of the self-adjoint operator $\lambda_k(P\Diamond^{-1})e_k = e_k$. Then, for any $f \in L_2(D)$, $\Diamond^{-1}f$ has quasi-spectral expansion in the form*

$$\Diamond^{-1}f = \sum_k \frac{1}{\lambda_k}(f, e_k)Pe_k. \tag{2.23}$$

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Tynysbek Sh. Kal'menov
Institute of Mathematics and Mathematical Modeling, 125 Pushkin str., 050010 Almaty, Kazakhstan
E-mail address: kalmenov@math.kz

Gaukhar D. Arepova
Institute of Mathematics and Mathematical Modeling, 125 Pushkin str., 050010 Almaty, Kazakhstan and Al-Farabi Kazakh national university, 71 Al-Farabi ave., 050040 Almaty, Kazakhstan
E-mail address: arepova@math.kz