Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 87, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

STABILITY OF THE BASIS PROPERTY OF EIGENVALUE SYSTEMS OF STURM-LIOUVILLE OPERATORS WITH INTEGRAL BOUNDARY CONDITION

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Abstract. We study a question on stability and instability of the basis property of a system of eigenfunctions of the Sturm - Liouville operator, with an integral perturbation of anti-periodic type on the boundary conditions.

1. Introduction

Spectral theory of non-self-adjoint boundary value problems for ordinary differential equations on a finite interval goes back to the classical works of Birkhoff [3] and Tamarkin [29]. They introduced the concept of regular boundary conditions and investigated asymptotic behavior of eigenvalues and eigenfunctions of such problems. In their works Malamud and Oridoroga [18, 19] proved completeness of eigenfunctions and associated functions for a wide class of boundary value problems which includes regular boundary conditions. In space $L^2(0,1)$ we consider an operator L_0, generated by the following ordinary differential expression:

$$L_0(u) \equiv -u''(x) + q(x)u(x), \quad q(x) \in C[0,1],\ 0 < x < 1 \tag{1.1}$$

and the boundary value conditions of the form

$$U_j(u) = a_{j1}u'(0) + a_{j2}u'(1) + a_{j3}u(0) + a_{j4}u(1) = 0, \quad j = 1, 2. \tag{1.2}$$

When the boundary conditions (1.2) are strongly regular, the results by Dunford [6, 7], Mikhailov [20] and Kesel'man [15] provide the Riesz basis property in $L^2(0,1)$ of the eigenfunctions and associated functions (E and AF) system of the problem. In the case when the boundary conditions are regular but not strongly regular, the question on basis property of E and AF system is not yet completely resolved. We introduce the matrix of coefficients of the boundary conditions (1.2):

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{pmatrix}.$$

By $A(ij)$ we denote the matrix composed of the i-th and j-th columns of the matrix A, $A_{ij} = \det A(ij)$. Let the boundary conditions (1.2) be regular but not

2010 *Mathematics Subject Classification.* 34L20, 34L40.
Key words and phrases. Sturm-Liouville operator basis property; eigenfunction; eigenvalue; non self-adjoint problem; anti-periodic boundary condition; perturbation; stability.

Submitted August 27, 2015. Published March 30, 2016.

strongly regular. According to [26, p. 73], if the following conditions hold:

$$A_{12} = 0, \quad A_{14} + A_{23} \neq 0, \quad A_{14} + A_{23} = \mp(A_{13} + A_{24}), \tag{1.3}$$

then the boundary conditions (1.2) are regular, but not strongly regular boundary conditions.

Makin [21] suggested dividing all regular, but not strongly regular, boundary conditions into four types:

I $A_{14} = A_{23}$, $A_{34} = 0$;
II $A_{14} = A_{23}$, $A_{34} \neq 0$;
III $A_{14} \neq A_{23}$, $A_{34} = 0$;
IV $A_{14} \neq A_{23}$, $A_{34} \neq 0$.

For example, periodical or antiperiodical boundary conditions form the type I, and can be determined in the following form:

$$A_{14} = A_{23}, \quad A_{34} = 0,$$

That is, $a_{11} = -a_{12}, a_{13} = a_{14} = a_{21} = a_{22} = 0$ and $a_{23} = -a_{24}$.

These conditions will be equivalent to matrix A, where the following two options are possible:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

are periodical or

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

are antiperiodical, and the same boundary conditions with "the lowest coefficients" form the type II. The boundary value conditions defined as $A_{14} \neq A_{23}, \quad A_{34} = 0$ form the type III. These conditions are always equivalent to boundary conditions given by the matrix A:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

This case will be the aim of our research in this paper. Moreover, Makin [21] allocated the one type of non-strongly regular boundary value conditions, when E and AF systems of the spectral problem

$$L_0(u) \equiv -u''(x) + q(x)u(x) = \lambda u(x), \quad q(x) \in C[0,1],\ 0 < x < 1, \tag{1.4}$$

with boundary conditions of the form (1.2) forms Riesz basis for any potentials $q(x)$. When $q(x) \equiv 0$, the problem about basis property of E and AF system of the problem with general regular boundary conditions has been completely resolved in [17]. In [4, 5] questions on convergence of eigenfunctions expansion of the Dirac operator in vector - matrix form and the Hill operator, forming Riesz basis in $L^2(0,1)$, with regular, but not strongly regular, boundary value conditions have been considered. For Dirac operators Mityagin [23] proved that periodic (or anti-periodic) boundary conditions give a rise to a Riesz system of 2D projections.

2. Statement of the problem

The spectral problem (1.4)-(1.2) with boundary conditions of the type I when $q(x) \equiv 0$ is self-adjoint; the system of its eigenfunctions is the usual trigonometric system, forming an orthonormal basis in $L^2(0,1)$. For the case of non-self-adjoint initial operator the question about preservation of the basis properties with some (weak in a certain sense) perturbation was shown in the type of several examples in [8].

Riesz basis property of eigenfunctions and associated functions of periodic and antiperiodic Sturm-Liouville problems was considered in [30]. In [12, 27] questions on stability of basis properties of the periodic problem for (1.4) were investigated with integral perturbation of the boundary conditions (1.2), when $j = 2$, of the type I; that is, $A_{14} = A_{23}$, $A_{34} = 0$. Moreover, in [22] similar issues at $q(x) \equiv 0$ have been studied. In the present paper we consider a spectral problem close to research of [22] when $q(x) \equiv 0$, with integral perturbation of the boundary conditions (1.2) when $j = 2$, which belong to type I:

$$L_1(u) \equiv -u''(x) = \lambda u(x), \quad 0 < x < 1, \tag{2.1}$$

$$U_1(u) \equiv u(0) + u(1) = \int_0^1 \overline{p(x)} u(x) dx, \quad p(x) \in L^1(0,1), \tag{2.2}$$

$$U_2(u) \equiv u'(0) + u'(1) = 0. \tag{2.3}$$

From [28] it follows that the E and AF system of the problem (2.1)-(2.3) is complete and minimal in $L^2(0,1)$. Moreover, the E and AF system for any $p(x)$ forms Riesz basis with brackets. Our aim is to show that the basis property in $L^2(0,1)$ of the E and AF system of problem (2.1)-(2.3) is unstable with respect to small changes of kernel $p(x)$ of integral perturbation. In [13] the method of constructing the characteristic determinant of the spectral problem with integral perturbation of the boundary conditions has been suggested.

The basis properties in $L^p(-1,1)$ of root functions of the nonlocal problem for the equations with involution have been studied in [16]. Instability of basis properties of root functions of the Schrodinger operator with nonlocal perturbation of the boundary condition has been investigated in [10]. In [9] they studied the question of stability and instability of basis property of system of eigenfunctions and associated functions of the double differentiation operator with an integral perturbation of Samarskii - Ionkin type boundary conditions. In [24] they considered the eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point. Boundary value problems with transmission conditions were investigated extensively in the recent years (see, for example, [1, 2, 14, 25]).

3. Characteristic determinant of a spectral problem

In this section we use the method in [13] to construct the characteristic determinant of the problem with integral perturbation of the boundary condition. Applying integration by parts, for smooth enough complex-valued functions $u(x)$ and $v(x)$ we obtain the Lagrange formula:

$$\begin{aligned} &\int_0^1 L_0(u)\overline{v(x)}\, dx - \int_0^1 u(x)\overline{L_0^*(v)} dx \\ &= [u'(0) + u'(1)]\overline{v(0)} + u'(1)[\overline{v(0)} + \overline{v(1)}] \\ &\quad - [u(0) + u(1)]\overline{v'(0)} - u(1)[\overline{v'(0)} + \overline{v'(1)}]. \end{aligned} \tag{3.1}$$

Here $L_0^*(v)$ is the adjoint differential expression

$$L_0^* v = -v''(x) + \overline{q(x)} v(x), \quad 0 < x < 1. \tag{3.2}$$

Consequently the operator L_0^* corresponding to the operator L_0 is given by differential expression (3.2) and the boundary conditions

$$V_1(\upsilon) = \upsilon(0) + \upsilon(1) = 0, \quad V_2(\upsilon) = \upsilon'(0) + \upsilon'(1) = 0. \tag{3.3}$$

Also the operator L_1^* corresponding to the operator L_1 is given by the loaded differential expression

$$L_1^*(\upsilon) = -\upsilon''(x) + \overline{q(x)}\upsilon(x) + p(x)\upsilon'(0), \quad 0 < x < 1, \tag{3.4}$$

and antiperiodic boundary conditions (3.3). One of the aspects of this problem is the fact that the adjoint problem to (2.1)-(2.3) is the spectral problem for the loaded differential equation

$$\begin{gathered} L_1^*(\upsilon) = -\upsilon''(x) + p(x)\upsilon'(0) = \overline{\lambda}\upsilon(x), \\ V_1(\upsilon) = \upsilon(0) + \upsilon(1) = 0, \\ V_2(\upsilon) = \upsilon'(0) + \upsilon'(1) = 0. \end{gathered} \tag{3.5}$$

First, we construct the characteristic determinant of the spectral problem (2.1)-(2.3). Presenting the general solution of the equation (2.1) by the formula

$$u(x, \lambda) = C_1 \cos\sqrt{\lambda}x + C_2 \sin\sqrt{\lambda}x,$$

and with respect to the boundary conditions (2.2)-(2.3), we obtain the following linear system for the coefficients C_k:

$$\begin{gathered} C_1\Big[1 + \cos\sqrt{\lambda} - \int_0^1 \overline{p(x)} \cos\sqrt{\lambda}x\, dx\Big] + C_2\Big[\sin\sqrt{\lambda} - \int_0^1 \overline{p(x)} \sin\sqrt{\lambda}x\, dx\Big] = 0, \\ C_1\Big[-\sin\sqrt{\lambda}\Big] + C_2\Big[1 + \cos\sqrt{\lambda}\Big] = 0. \end{gathered}$$

Its determinant will be the characteristic determinant of the spectral problem (2.1)-(2.3):

$$\Delta_1(\lambda) = \begin{vmatrix} 1 + \cos\sqrt{\lambda} - \int_0^1 \overline{p(x)} \cos\sqrt{\lambda}x\, dx & \sin\sqrt{\lambda} - \int_0^1 \overline{p(x)} \sin\sqrt{\lambda}x\, dx \\ -\sin\sqrt{\lambda} & 1 + \cos\sqrt{\lambda} \end{vmatrix}. \tag{3.6}$$

When $p(x) = 0$ we obtain the characteristic determinant of the unperturbed problem (2.1)-(2.3). It is de denote by $\Delta_0(\lambda) = 2(1 + \cos\sqrt{\lambda})$. The number $\lambda_k^0 = ((2k-1)\pi)^2$ is the eigenvalue of the unperturbed antiperiodic problem, and $u_{k0}^0 = \sqrt{2}\cos((2k-1)\pi x)$, $u_{k1}^0 = \sqrt{2}\sin((2k-1)\pi x)$ are eigenfunctions. We represent the function $p(x)$ in the Fourier series form by the trigonometric system

$$p(x) = \sum_{k=1}^{\infty} [a_k \cos((2k-1)\pi x) + b_k \sin((2k-1)\pi x)]. \tag{3.7}$$

Using (3.7), we find more convenient representation for the determinant $\Delta_1(\lambda)$. To do this, first, we calculate integrals in (3.6). Simple calculations show that

$$\begin{aligned} \int_0^1 \overline{p(x)} \cos(\sqrt{\lambda}x) dx &= \sum_{k=1}^{\infty} \frac{[\bar{a}_k\sqrt{\lambda}\sin\sqrt{\lambda} + \bar{b}_k((2k-1)\pi)(\cos\sqrt{\lambda} + 1)]}{\lambda - ((2k-1)\pi)^2}, \\ \int_0^1 \overline{p(x)} \sin(\sqrt{\lambda}x) dx &= \sum_{k=1}^{\infty} \frac{[\bar{a}_k\sqrt{\lambda}(1 + \cos\sqrt{\lambda}) + \bar{b}_k((2k-1)\pi)\sin\sqrt{\lambda}]}{\lambda - ((2k-1)\pi)^2}. \end{aligned}$$

Using these results and standard transformations, the determinant (3.6) is reduced to the form

$$\Delta_1(\lambda) = \Delta_0(\lambda) \cdot A(\lambda),$$

where

$$A(\lambda) = \Big[1 + \sum_{k=1}^{\infty} \bar{b}_k \frac{(2k-1)\pi}{\lambda - ((2k-1)\pi)^2}\Big]. \tag{3.8}$$

Hence, the following theorem is proved.

Theorem 3.1. *The characteristic determinant of the spectral problem* (2.1)-(2.3) *with the perturbed boundary value conditions can be represented in the form* (3.8)*, where* $\Delta_0(\lambda)$ *is the characteristic determinant of the unperturbed antiperiodic spectral problem,* b_k *are coefficients of the expansion* (3.7) *of the function* $p(x)$ *into trigonometric Fourier series.*

The function $A(\lambda)$ in (3.8) has a first-order pole at the points $\lambda = \lambda_k^0$, and the function $\Delta_0(\lambda)$ has a second order zero at these points. Therefore, the function $\Delta_1(\lambda)$, represented by the formula (3.8), is an entire analytic function of the variable λ. The characteristic determinant, which is an entire analytical function, related with the problem on eigenvalues of differential operator of the third order with nonlocal boundary conditions has been studied in [11].

4. Particular cases of the characteristic determinant

The characteristic determinant (3.8) looks simpler when

$$p(x) = \sum_{k=1}^{N} \big[a_k \cos((2k-1)\pi x) + b_k \sin((2k-1)\pi x)\big].$$

That is, there exists such a number N such that $a_k = b_k = 0$ for all $k > N$. In this case, formula (3.8) takes the form

$$\Delta_1(\lambda) = \Delta_0(\lambda)\Big[1 + \sum_{k=1}^{N} \bar{b}_k \frac{(2k-1)\pi}{\lambda - ((2k-1)\pi)^2}\Big]. \tag{4.1}$$

From this particular case of formula (3.8), the we have the following corollary.

Corollary 4.1. *For any preassigned numbers - a complex* $\hat{\lambda}$ *and a positive integer* $\hat{m}$ *there always exists a function* $p(x)$ *such that* $\hat{\lambda}$ *will be an eigenvalue of the problem* (2.1)-(2.3) *of multiplicity* $\hat{m}$*.*

From the analysis of formula (4.1) it is easy to see that $\Delta_1(\lambda_k^0) = 0$ for all $k > N$. That is, all eigenvalues λ_k^0, $k > N$, of the unperturbed periodic problem are the eigenvalues of the perturbed spectral problem (2.1)-(2.3). It is also not difficult to show that the multiplicity of the eigenvalues λ_k^0, $k > N$ is also preserved. Moreover, from the condition of orthogonality of the trigonometric system it follows that in this case:

$$\int_0^1 \overline{p(x)} u_{kj}^0(x) dx = 0, \quad j = \overline{0,1},\ k > N.$$

Thus, the eigenfunctions $u_{kj}^0(x)$ of the antiperiodic problem when $k > N$ satisfy the boundary value conditions (2.2)-(2.3) and, therefore, they are eigenfunctions of the perturbed problem (2.1)-(2.3). Hence, in this case the system of eigenfunctions of (2.1)-(2.3) and the system of eigenfunctions of the periodic problem (an orthonormal basis) differ from each other only in a finite number of the first members. Consequently, the system of eigenfunctions of (2.1)-(2.3) also forms a Riesz basis in $L^2(0,1)$. The set of functions $p(x)$, that can be represented as a finite series (3.7), is dense in $L^1(0,1)$. Thus, we have proved the following result.

Theorem 4.2. *Let $A_{14} = A_{23}$, $A_{34} = 0$; that is, the boundary conditions* (2.2)-(2.3) *belong to type I with integral perturbation. Then the set of functions $p(x) \in L^1(0,1)$, such that the system of eigenfunctions of the perturbed problem* (2.1)-(2.3) *forms Riesz basis in $L^2(0,1)$, is dense in $L^1(0,1)$.*

5. Instability of the basis property

Now we show that basis properties of eigenfunctions system of the perturbed problem (2.1)-(2.3) is unstable for an arbitrarily small integral perturbation of the boundary-value condition (2.2).

Theorem 5.1. *Suppose that $A_{14} = A_{23}$, $A_{34} = 0$; that is, the boundary-value conditions* (2.2)-(2.3) *belong to type I. Then the set of functions $p(x) \in L^1(0,1)$, such that the system of eigenfunctions of the perturbed problem* (2.1)-(2.3) *does not form even a normal basis in $L^2(0,1)$, is dense in $L^1(0,1)$.*

Proof. Let in (3.7) the coefficients $b_k \neq 0$ for all sufficiently large k. Then from (3.8) we note that $\lambda = \lambda_k^0$ is a simple eigenvalue of problem (2.1)-(2.3). By direct calculation we get that $u_k^1 = b_k \cos((2k-1)\pi)x - a_k \sin((2k-1)\pi)x$ are eigenfunctions of (2.1)-(2.3), corresponding to $\lambda_k^0 = ((2k-1)\pi)^2$. Moreover, the eigenfunction of the dual problem (3.5), corresponding to the eigenvalue λ_k^0, is $v_k^1(x) = c_k \cos((2k-1)\pi)x$.

Since the eigenfunctions of the dual problems form biorthogonal system, then we have the equality of the scalar product $(u_k^1, v_k^1) = 1$. Hence, it is easy to obtain $b_k\bar{c}_k = 2$. Therefore,

$$\|u_k^1\| \cdot \|v_k^1\| = \sqrt{1 + |\frac{a_k}{b_k}|^2}. \tag{5.1}$$

Denote by $\sigma_N(x)$ partial sum of the Fourier series (3.7). It is obvious, that the set of functions, which can be represented as the infinite series

$$\tilde{p}(x) = \sigma_N(x) + \sum_{k=N+1}^{\infty} [\tilde{a}_k \cos((2k-1)\pi x) + \tilde{b}_k \sin((2k-1)\pi x)],$$

where $\tilde{a}_k = 2^{-k}$, $\tilde{b}_k = 2^{-k}/k$, $k > N$, is dense in $L^1(0,1)$. However, from (5.1) it follows that for such kind of functions $\tilde{p}(x)$ for the corresponding eigenfunctions systems of the direct and conjugate problems there holds: $\lim_{k\to\infty} \|u_k^1\|\|v_k^1\| = \infty$.

That is, the condition of uniform minimal property (see [8] and references in it) of the system does not hold, and therefore, it does not form even a basis in $L^2(0,1)$. ∎

Since adjoint operators possess the Riesz basis property of the root functions, we obtain the corollary.

Corollary 5.2. *Suppose that $A_{14} = A_{23}$, $A_{34} = 0$, that is boundary value conditions* (2.2)-(2.3) *belongs to type I. Then the set P of functions $p(x) \in L^1(0,1)$, such that the system of eigenfunctions of* (3.5) *for the loaded differential equations forms Riesz basis in $L^2(0,1)$, is everywhere dense in $L^1(0,1)$. The set $L^1(0,1)\backslash P$ is also everywhere dense in $L^1(0,1)$.*

The results of this paper, in contrast to [28], demonstrate instability of basis properties of the root functions of the problem with an integral perturbation of the boundary value conditions of type I, which are regular, but not strongly regular.

Acknowledgements. This research was financially supported by a grant from the Ministry of Science and Education of the Republic of Kazakhstan (Grant No. 0825/GF4).

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