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INFINITELY MANY SOLUTIONS FOR FRACTIONAL SCHRÖDINGER EQUATIONS IN $\mathbb{R}^N$

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Abstract. Using variational methods we prove the existence of infinitely many solutions to the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,$$

where $N \geq 2, s \in (0,1)$. $(-\Delta)^s$ stands for the fractional Laplacian. The potential function satisfies $V(x) \geq V_0 > 0$. The nonlinearity $f(x,u)$ is superlinear, has subcritical growth in u, and may or may not satisfy the (AR) condition.

1. Introduction and statement of main results

In this article, we investigate the existence of infinitely many solutions to the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 2$, $s \in (0,1)$. $(-\Delta)^s$ stands for the fractional Laplacian. The function $f(x,u)$ is odd, sublinear or suplinear and subcritical in u, $V(x)$ is positive and bounded below in $\mathbb{R}^N$.

Equation (1.1) arises in the study of the fractional Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + (-\Delta)^s \psi + V(x)\psi = f(x,\psi), \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.2}$$

when looking for standing waves, that is, solutions with the form $\psi(x,t) = e^{i\omega t}u(x)$, where ω is a constant. This equation was introduced by Laskin [14, 15] and comes from an expansion of the Feynman path integral and from Brownian-like to Lévy-like quantum mechanical paths.

This equation is of particular interest in fractional quantum mechanics for the study of particles on stochastic fields modelled by Lévy processes, which occur widely in physics, chemistry and biology. The stable Lévy processes that gives rise to equations with the fractional Laplacian have recently attracted much research interest. For more details, we can see [5].

Nonlinear equations like (1.1) have recently been studied by Cabré and Roquejoffre [3], Cabré and Tan [4], Sire and Valdinoci [22], Iannizzotto et al. [13], Hua

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and Yu [12]. A one-dimensional version of (1.1) has been studied in the context of solitary waves by Weinstein [27].

Equations of the form (1.1) in the whole space $\mathbb{R}^N$ were studied by a number of authors; see for instance [6, 9, 19, 20, 21] and the references therein. Felmer et al. [9] considered the existence and regularity of positive solution of (1.1) with $V(x) = 1$ and $s \in (0, 1)$ when f has subcritical growth and satisfies the Ambrosetti-Rabinowitz ((AR) for short) condition. Secchi [19] obtained the existence of ground state solutions of (1.1) for $s \in (0, 1)$ when $V(x) \to \infty$ as $|x| \to \infty$ and (AR) condition holds. In [8], the authors proved the existence of infinitely many weak solutions for (1.1) by variant fountain theorem under the assumption

$$0 < \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to\infty} V(x) = V_\infty < \infty. \tag{1.3}$$

Tang [25] studied (1.1) with a potential $V(x)$ satisfying

$$0 < \inf_{x\in\mathbb{R}^N} V(x), \quad \operatorname{meas}(\{x \in \mathbb{R}^N | V(x) \le d\}) < \infty, \quad \forall d > 0. \tag{1.4}$$

Similar assumptions can be found in [10, 21, 23, 28]. Each of these conditions ensures that the embedding $W^{s,2}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact for $2 \le q < 2_s^* = \frac{2N}{N-2s}$. On the other hand, Gou and Sun [11], Chang and Wang [7] investigated the existence of radial solutions for (1.1).

In this article, we are interest in the existence of infinitely many solutions for (1.1) under the assumptions (A3)–(A7) below. Our assumptions on $f(x, u)$ are different from that in the above papers. The weighted functions $h_1(x), h_2(x)$ and $h_3(x)$ depend on the potential function $V(x)$ and the nonlinear function $f(x, u)$ either satisfies (AR) condition or does not. Moreover, two cases that $f(x, u)$ is bounded and unbounded in $x \in \mathbb{R}^N$ are considered. We note that, in [17, 23, 26], $f(x, u)$ is assumed to bounded in $x \in \mathbb{R}^N$

To state our main results, we recall some fractional Sobolev spaces and norms [16]. Let $V(x)$ satisfy (A1) below and

$$E = \Big\{u \in W^{s,2}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 d\xi + \int_{\mathbb{R}^N} V(x)|u|^2 dx < \infty\Big\} \tag{1.5}$$

endowed with the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} |\xi|^{2s}|\hat{u}|^2 d\xi + \|u\|_{2,V}^2\Big)^{1/2}, \tag{1.6}$$

where and in the sequel, $\|u\|_{2,V}^2 = \int_{\mathbb{R}^N} V(x)|u|^2 dx$ and $\hat{\omega} = \hat{\omega}(\xi)$ is the Fourier transform of $\omega(x)$; that is,

$$\begin{gathered}\hat{\omega} = \mathcal{F}[\omega(x)] = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \omega(x)e^{-i\xi\cdot x} dx,\\ \omega(x) = \mathcal{F}^{-1}[\hat{\omega}] = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} \hat{\omega}(\xi)e^{i\xi\cdot x} d\xi.\end{gathered} \tag{1.7}$$

In [16], the author shows that

$$((-\Delta)^s u)(x) = \mathcal{F}^{-1}[|\xi|^2 \hat{u}], \quad \forall x \in \mathbb{R}^N, \tag{1.8}$$

$$[u]_E^2 = \frac{2}{C(N, s)} \int_{\mathbb{R}^N} |\xi|^2 |\hat{u}|^2 d\xi, \tag{1.9}$$

where
$$[u]_E = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2} \tag{1.10}$$
is called the Gagliardo norm, and the constant $C(N,s)$ depends only on the space dimensional N and the order s, and it is explicitly given by the integral
$$\frac{1}{C(N,s)} = \int_{\mathbb{R}^N} \frac{1-\cos(\zeta_1)}{|\zeta|^{N+2s}} d\zeta, \quad \zeta = (\zeta_1, \zeta_2, \dots, \zeta_N) \in \mathbb{R}^N. \tag{1.11}$$
Moreover, by the Plancherel formula in Fourier analysis, we have
$$[u]_E^2 = \frac{2}{C(N,s)} \|(-\Delta)^{s/2} u\|_2^2. \tag{1.12}$$
Then, from (1.8)-(1.12), we obtain that the norm $\|\cdot\|_E$ is equivalent to the norms
$$\begin{gathered} \|u\|_1 = \Big(\int_{\mathbb{R}^N} V(x)|u|^2 dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy\Big)^{1/2}, \\ \|u\|_2 = \Big(\int_{\mathbb{R}^N} V(x)|u|^2 dx + \|(-\Delta)^{s/2} u\|_2^2\Big)^{1/2}. \end{gathered} \tag{1.13}$$

In general, we define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)(0<s<1<p, sp<N)$ as follows
$$W^{s,p}(\mathbb{R}^N) = \Big\{ u \in L^p(\mathbb{R}^N) : \frac{|u(x)-u(y)|}{|x-y|^{\frac{N}{p}+s}} \in L^p(\mathbb{R}^{2N})\Big\}. \tag{1.14}$$
This space is endowed with the natural norm
$$\|u\|_{W^{s,p}} = \Big(\int_{\mathbb{R}^N} |u|^p dx + \iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}, \tag{1.15}$$
while
$$[u]_{W^{s,p}} = \Big(\iint_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p} \tag{1.16}$$
is called the Gagliardo norm. For the reader's convenience, we recall the main embedding results for $W^{s,p}(\mathbb{R}^N)$.

Lemma 1.1 ([16]). *Let $s \in (0,1)$ and $p \geq 1$ such that $sp < N$. Then there exists a positive constant $S_0 = S_0(N,p,s)$ such that, for any measurable and compactly supported function $u : \mathbb{R}^N \to \mathbb{R}$, we have*
$$\|u\|_{p_s^*} \leq S_0 [u]_{W^{s,p}}, \tag{1.17}$$
where $p_s^* = pN/(N-ps)$ is the fractional critical exponent. Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$. Moreover, the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is locally compact whenever $1 < q < p_s^*$.

Remark 1.2. By the density of the compactly supported functions in $W^{s,p}(\mathbb{R}^N)$, we know that (1.17) holds for any $u \in W^{s,p}(\mathbb{R}^N)$.

From the Hölder inequality and Lemma 1.1, we obtain the following lemma.

Lemma 1.3. *Let $s \in (0,1)$, $sp < N$ and $p \leq q \leq p_s^*$. Then for any $u \in X = W^{s,p}(\mathbb{R}^N)$,*
$$\|u\|_q \leq S_q \|u\|_X \tag{1.18}$$
where S_q is a constant depending on s, q, p, N. In particular, we denote $S_{p_s^*}$ by S_0. The inequality (1.18) shows that the embedding $X \hookrightarrow L^q(\mathbb{R}^N)$ is continuous.

Proof. When $q = p$, inequality (1.18) is obvious. For $q = p_s^*$, (1.18) can be obtained from (1.17). Let $p < q < p_s^*$. Then there exists $t \in (0,1)$ such that $q = pt + p_s^*(1-t)$. It follows from the Hölder inequality and (1.17) that

$$\begin{aligned}\int_{\mathbb{R}^N} |u|^q dx &= \int_{\mathbb{R}^N} |u|^{p_s^*(1-t)} |u|^{pt} dx \leq \Big(\int_{\mathbb{R}^N} |u|^{p_s^*} dx\Big)^{(1-t)} \Big(\int_{\mathbb{R}^N} |u|^p dx\Big)^t \\ &\leq S_0^{1-t} [u]_X^{p_s^*(1-t)} \|u\|_p^{pt} \leq S_q^q \|u\|_X^{p_s^*(1-t)} \|u\|_X^{pt} = S_q^q \|u\|_X^q,\end{aligned} \tag{1.19}$$

where $S_q^q = S_0^{1-t}$. This implies (1.18). ☐

Similarly, for the Sobolev space E defined by (1.5), we have the following result.

Lemma 1.4. *Let $s \in (0,1), 2s < N$ and $2 \leq q \leq 2_s^*$. Assume $V(x) \geq V_0 > 0$ in $\mathbb{R}^N$. Then, for any $u \in E$,*

$$\|u\|_q \leq S_q \|u\|_E \tag{1.20}$$

where S_q is a constant depending on s, q, p, N and V_0. In particular, we denote $S_{2_s^*}$ by S_0.

Definition 1.5. A function $u \in E$ is said to be a (weak) solution of (1.1) if for any $\varphi \in E$, we have

$$\int_{\mathbb{R}^N} |\xi|^{2s} \hat{u}\hat{\varphi} d\xi + \int_{\mathbb{R}^N} V(x) u\varphi dx = \int_{\mathbb{R}^N} f(x,u)\varphi dx. \tag{1.21}$$

Let $J(u) : E \to \mathbb{R}$ be the energy functional associated with (1.1) defined by

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}|^2 d\xi + \frac{1}{2}\int_{\mathbb{R}^N} V(x)|u|^2 dx - \int_{\mathbb{R}^N} F(x,u) dx, \tag{1.22}$$

where $F(x,u) = \int_0^u f(x,t)dt$.

Using (1.18) and assumptions (A3)–(A7) below, we see that the functional J is well defined and $J \in C^1(E, \mathbb{R})$ with

$$J'(u)\varphi = \int_{\mathbb{R}^N} |\xi|^2 \hat{u}\hat{\varphi} d\xi + \int_{\mathbb{R}^N} V(x) u\varphi dx - \int_{\mathbb{R}^N} f(x,u)\varphi dx, \quad \forall \varphi \in E. \tag{1.23}$$

Throughout this article, the function $f(x,u) \in C(\mathbb{R}^N \times \mathbb{R})$ is odd in u. In addition, we use the following assumptions.

(A1) The function $V(x) \in C(\mathbb{R}^N)$ satisfies $\inf_{x\in\mathbb{R}^N} V(x) \geq V_0 > 0$, where V_0 is a constant.

(A2) There exists $a > 0$ such that $\lim_{|y|\to\infty} \mathrm{meas}(\{x \in B_a(y) : V(x) \leq d\}) = 0$ for any $d > 0$, where "meas" denotes the Lebesgue measure on $\mathbb{R}^N$ and $B_r(x)$ denotes any open ball of $\mathbb{R}^N$ centered at x and of radius $r > 0$, while we simply write B_r when $x = 0$.

(A3) There exist $2 < \alpha < \beta < 2_s^*$ such that

$$|f(x,u)| \leq h_1(x)|u|^{\alpha-1} + h_2(x)|u|^{\beta-1}, \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.24}$$

where $h_1(x), h_2(x) \in C(\mathbb{R}^N)$ and

$$\lim_{r\to\infty} \sup_{x\in B_r^c} \frac{h_1(x)}{V^{t_1}(x)} = 0, \quad \lim_{r\to\infty} \sup_{x\in B_r^c} \frac{h_2(x)}{V^{t_2}(x)} = 0 \tag{1.25}$$

with $t_1 = (2_s^* - \alpha)/(2_s^* - 2)$, $t_2 = (2_s^* - \beta)/(2_s^* - 2)$ and $B_r^c = \mathbb{R}^N \setminus \overline{B}_r = \{x \in \mathbb{R}^N : |x| > r\}$.

(A4) There exists $\mu > 2$ such that

$$tf(x,t) - \mu F(x,t) \geq 0, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.26}$$

(A5) $\lim_{|t|\to\infty}(F(x,t)t^{-2}) = \infty$ for any $x \in \mathbb{R}^N$.

(A6) There exist $k > \frac{N}{2s}$ and $2 < \alpha < \beta \leq \frac{2k}{k-1}$ such that (1.24) and (1.25) hold. Furthermore, there exist $b, c_1 \geq 1$ such that for $x \in \mathbb{R}^N$ and $|u| \geq b$,

$$\begin{gathered} F(x,u) \geq 0, \quad G(x,u) = \frac{1}{2}uf(x,u) - F(x,u) \geq 0, \\ |F(x,u)|^k \leq c_1^k |u|^{2k} |h_3(x)|^{2k} G(x,u), \end{gathered} \tag{1.27}$$

where $h_3(x) \in C(\mathbb{R}^N)$ satisfies

$$\lim_{r\to\infty} \sup_{x\in B_r^c} \frac{|h_3(x)|^{2k'}}{V^{t_3}(x)} = 0, \quad \text{with } k' = \frac{k}{k-1},\ t_3 = \frac{2_s^* - 2k'}{2_s^* - 2}. \tag{1.28}$$

(A7) There exist a constant $C_0 > 0$ and $2 < \alpha < \beta < 2_s^*$, such that

$$|f(x,u)| \leq C_0(|u|^{\alpha-1} + |u|^{\beta-1}), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}. \tag{1.29}$$

Remark 1.6. Condition (A2), which is weaker than the coercivity assumption $V(x) \to \infty$ as $|x| \to \infty$, was originally introduced by Bartsch and Wang in [1] to overcome the lack of compactness. Clearly, if $V(x) \to \infty$ as $|x| \to \infty$, it is possible that the functions $h_1(x), h_2(x)$ and $h_3(x)$ in $(A3)$ and (A6) are unbounded on $\mathbb{R}^N$. So, it is necessary to consider the condition (A7).

Our main results in this paper are as follows.

Theorem 1.7. *Let $s \in (0,1), 2s < N$. Assume* (A1), (A2) *and* (A5) *hold. In addition, suppose that either* (A3), (A4) *or (A6) are satisfied. Then* (1.1) *admits infinitely many solutions $u_n \in E$ such that $J(u_n) \to \infty$ as $n \to \infty$.*

Theorem 1.8. *Let $s \in (0,1), 2s < N$. Assume* (A1)–(A3) *and* (A7) *hold. In addition, suppose that either (A4) or* (1.27) *is satisfied with $h_3(x) \equiv 1$. Then* (1.1) *admits infinitely many solutions $u_n \in E$ such that $J(u_n) \to \infty$ as $n \to \infty$.*

Remark 1.9. Assumption (1.25) implies that the functions (h_1V^{-1}), (h_2V^{-1}), $(h_1V^{-t_1})$, $(h_2V^{-t_2})$ belong to $L^\infty(\mathbb{R}^N)$ and

$$\lim_{r\to\infty} \sup_{x\in B_r^c} [h_1(x)V^{-1}(x)] = \lim_{r\to\infty} \sup_{x\in B_r^c} [h_2(x)V^{-1}(x)] = 0.$$

Moreover, the condition $k > \frac{N}{2s}$ in (A7) implies that $\frac{2k}{k-1} < 2_s^*$.

Remark 1.10. Assumption (A4) is called the (AR) condition. Obviously, the power functions in u like $f(x,u) = \sum_{i=1}^n h_i(x)|u|^{\beta_i-2}u$ with $2 < \beta_i < 2_s^*$ satisfy (A3) and (A4) for appropriate functions $h_i \in C(\mathbb{R}^N)$. The functions like $f(x,u) = h(x)u\log(1+|u|)$ fails to satisfy condition (A4), but it satisfies (A6).

Teng [26] considered problem (1.1) under assumption (A4) with $h_1(x), h_2(x) \in L^\infty(\mathbb{R}^N)$. Obviously, our assumptions on h, h_1 and h_2 are weaker than that in [26]. Without loss of generality, we let $V_0 = 1$ in (A1).

2. Proof of main results

To prove the main results, we recall some useful concepts and results.

Definition 2.1. Let E be a real Banach space and the functional $J \in C^1(E, \mathbb{R})$. We say that J satisfies the $(C)_c$ condition if any $(C)_c$ sequence $\{u_n\} \subset E$:

$$J(u_n) \to c, \quad (1 + \|u_n\|_E)\|J'(u_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty \tag{2.1}$$

has a convergent subsequence in E.

Lemma 2.2 ([18, 24])**.** *Let E be an infinite dimensional real Banach space, the functional $J \in C^1(E, \mathbb{R})$ be even and satisfy the $(C)_c$ condition for all $c > 0$ and $J(0) = 0$. In addition, assume $E = Y \oplus Z$, in which Y is finite dimensional, and J satisfies*

- (A8) *there exist constants $\rho, \alpha_0 > 0$ such that $J(z) \geq \alpha_0$ on $\partial B_\rho \cap Z$;*
- (A9) *for each finite dimensional subspace $E_0 \subset E$, there is an $R = R(E_0)$ such that $J(z) \leq 0$ on $E_0 \setminus \overline{B}_R$, where $B_R = \{z \in E : \|z\|_E < R\}, \partial B_R = \{z \in E : \|z\|_E = R\}$.*

Then, J possesses an unbounded sequence of critical values, i.e. there exists a sequence $\{u_n\} \subset E$ such that $J'(u_n) = 0$ and $J(u_n) \to \infty$ as $n \to \infty$.

In the proof of our results, we use the following lemma.

Lemma 2.3 ([17])**.** *Let $s \in (0, 1), 2s < N$ and $2 \leq q < 2_s^* = \frac{2N}{N-2s}$. Assume* (A1) *and* (A2)*. Then the embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact.*

For the prove Theorems 1.7 and 1.8, we need the following lemmas.

Lemma 2.4. *Assume* (A1) *and* (A2)*. If* (A4) *is satisfied, then any $(C)_c$ sequence $\{u_n\}$ is bounded in E.*

Proof. Let the sequence $\{u_n\}$ satisfy (2.1) and $\mu > 2$. Then for large n, we have

$$\begin{aligned} c + 1 + \|u_n\|_E &\geq J(u_n) - \frac{1}{\mu} J'(u_n)u_n \\ &= (\frac{1}{2} - \frac{1}{\mu})\|u_n\|_E^2 + \frac{1}{\mu} \int_{\mathbb{R}^N} f(x, u_n)u_n - \mu F(x, u_n))dx. \end{aligned} \tag{2.2}$$

Then (A4) implies that $\{u_n\}$ is bounded in E. The proof is complete. □

Lemma 2.5. *Assume* (A1), (A2), (A5), (A6) *hold. Then any $(C)_c$ sequence $\{u_n\}$ is bounded in E.*

Proof. To prove the boundedness of $\{u_n\}$, arguing contradiction, we suppose that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n(x) = \frac{u_n(x)}{\|u_n\|_E}$. Then $\|v_n\|_E = 1$ for all $n \geq 1$. By Lemma 2.3, there exists a subsequence of $\{v_n\}$, still denoted by $\{v_n\}$, and $v \in E$ such that $\|v\|_E \leq 1$ and

$$\begin{gathered} v_n \rightharpoonup v \text{ weakly in } E; \quad v_n \to v \text{ in } L^q(\mathbb{R}^N) \ (2 \leq q < 2_s^*); \\ v_n(x) \to v(x) \text{ a.e. in } \mathbb{R}^N. \end{gathered} \tag{2.3}$$

Clearly, it follows from (2.3) that there exists $\omega(x) \in L^q(\mathbb{R}^N)(2 \leq q < 2_s^*)$ such that $|v_n(x)| \leq \omega(x)$ a.e. in $\mathbb{R}^N$ for all $n \geq 1$.

From (1.22), (1.23) and (2.1), it follows that for, n large,

$$c+1 \geq J(u_n) - \frac{1}{2}J'(u_n)u_n = \int_{\mathbb{R}^N} G(x,u_n)dx, \tag{2.4}$$

where $G(x,u) = \frac{1}{2}uf(x,u) - F(x,u)$, and

$$\begin{aligned} \frac{1}{2} &\leq \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{|F(x,u_n)|}{\|u_n\|_E^2}dx \\ &\leq \limsup_{n\to\infty} \int_{B_r} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx + \limsup_{n\to\infty} \int_{B_r^c} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx, \end{aligned} \tag{2.5}$$

for any $r > 0$. By (A6), we obtain, for any $\varepsilon > 0$, there exists $\delta > 0$ such that $\frac{|F(x,t)|}{|t|^2} \leq \varepsilon(h_1(x) + h_2(x))$ for all $0 < |t| \leq \delta$ and all $x \in \mathbb{R}^N$. Denote $X_n = \{x \in \mathbb{R}^N : |u_n(x)| \leq \delta\}$, $Y_n = \{x \in \mathbb{R}^N : \delta < |u_n(x)| \leq b\}$, $Z_n = \{x \in \mathbb{R}^N : |u_n(x)| \geq b\}$, where the constant b is given in (A6). Obviously, $\mathbb{R}^N = X_n \cup Y_n \cup Z_n$ and $B_r^c = B_r^c \cap (X_n \cup Y_n \cup Z_n)$. Then

$$\begin{aligned} \int_{B_r^c\cap X_n} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx &\leq \varepsilon \int_{B_r^c\cap X_n} (h_1(x)+h_2(x))|v_n|^2dx \\ &\leq \varepsilon(\|h_1V^{-1}\|_\infty + \|h_2V^{-1}\|_\infty) \int_{\mathbb{R}^N} V|v_n|^2dx \\ &\leq 2\varepsilon M_1\|v_n\|_E^2 = 2\varepsilon M_1, \end{aligned} \tag{2.6}$$

where $M_1 = \max\{\|h_1V^{-1}\|_\infty, \|h_2V^{-1}\|_\infty\}$. Furthermore, by (1.24) and (1.25), one sees that

$$\begin{aligned} &\int_{B_r^c\cap Y_n} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2\,dx \\ &\leq b^{\beta-2}\int_{B_r^c\cap Y_n} (h_1(x)+h_2(x))|v_n(x)|^2\,dx \\ &\leq b^{\beta-2}\Big(\sup_{x\in B_r^c}\frac{h_1(x)}{V(x)} + \sup_{x\in B_r^c}\frac{h_2(x)}{V(x)}\Big)\int_{B_r^c} V(x)|v_n|^2dx \\ &\leq b^{\beta-2}\Big(\sup_{x\in B_r^c}\frac{h_1(x)}{V(x)} + \sup_{x\in B_r^c}\frac{h_2(x)}{V(x)}\Big) \to 0 \quad \text{as } r=|x|\to\infty. \end{aligned} \tag{2.7}$$

On the other hand, from (1.27) and (2.4) it follows that

$$\begin{aligned} &\int_{B_r^c\cap Z_n} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx \\ &\leq \Big(\int_{B_r^c\cap Z_n}\Big(\frac{|F(x,u_n)|}{h_3^2|u_n|^2}\Big)^k dx\Big)^{1/k}\Big(\int_{B_r^c\cap Z_n}|h_3v_n|^{2k'}dx\Big)^{1/k'} \\ &\leq c_1\Big(\int_{\mathbb{R}^N} G(x,u_n)dx\Big)^{1/k}\Big(\int_{B_r^c\cap Z_n}|h_3v_n|^{2k'}dx\Big)^{1/k'} \\ &\leq c_1(c+1)^{1/k}\Big(\int_{B_r^c}|h_3v_n|^{2k'}dx\Big)^{1/k'}. \end{aligned} \tag{2.8}$$

Note that $2 < 2k' < 2_s^*$. Let $t_3 = (2_s^* - 2k')/(2_s^* - 2)$. By the Hölder inequality and (1.20), we obtain

$$\begin{aligned} \int_{B_r^c} |h_3 v_n|^{2k'} dx &\le \sup_{x\in B_r^c} \frac{|h_3(x)|^{2k'}}{V^{t_3}(x)} \Big(\int_{B_r^c} V v_n^2 dx\Big)^{t_3} \Big(\int_{B_r^c} |v_n|^{2_s^*} dx\Big)^{1-t_3} \\ &\le S_0 \|v_n\|_E^{2k'} \sup_{x\in B_r^c} \frac{|h_3(x)|^{2k'}}{V^{t_3}(x)} \le S_0 \sup_{x\in B_r^c} \frac{|h_3(x)|^{2k'}}{V^{t_3}(x)} \to 0 \end{aligned} \tag{2.9}$$

as $r = |x| \to \infty$, where $S_0 = S_{2_s^*}$ is the constant in (1.20). Then, and application of (2.6)-(2.9) implies that for any $\varepsilon > 0$, there exist $n_0, r_0 \ge 1$, such that $n \ge n_0$, $r \ge r_0$,

$$\int_{B_r^c} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 dx \le \varepsilon(2M_1 + 1). \tag{2.10}$$

Set $T_n = X_n \cup Y_n$. Notice that for all $x \in B_{r_0} \cap T_n$,

$$\begin{aligned} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 &\le b^{\beta-2}(h_1(x) + h_2(x))|v_n(x)|^2 \\ &\le b^{\beta-2} M_2 |\omega(x)|^2 \equiv d(x) \in L^1(B_{r_0}), \end{aligned} \tag{2.11}$$

where

$$M_2 = \sup_{x\in B_{r_0}} (h_1(x) + h_2(x)). \tag{2.12}$$

If $v(x) = 0$ in B_{r_0}, it follows from Fatou's lemma that

$$\begin{aligned} \limsup_{n\to\infty} \int_{B_{r_0}\cap T_n} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 dx &\le M_2 b^{\beta-2} \int_{B_{r_0}} \limsup_{n\to\infty} |v_n|^2 dx \\ &= M_2 b^{\beta-2} \int_{B_{r_0}} |v|^2 dx = 0. \end{aligned} \tag{2.13}$$

Arguing as in (2.8) and (2.9), we obtain

$$\int_{B_{r_0}\cap Z_n} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 dx \le C_1 \sup_{x\in B_{r_0}} |h_3(x)|^2 \Big(\int_{B_{r_0}} |v_n|^{2k'} dx\Big)^{1/k'} \tag{2.14}$$

with $C_1 = c_1(c+1)^{1/k}$. Similarly, since $|v_n(x)|^{2k'} \le |\omega(x)|^{2k'}$ a.e. in $\mathbb{R}^N$ and $|\omega(x)|^{2k'} \in L^1(\mathbb{R}^N)$, we obtain

$$\limsup_{n\to\infty} \int_{B_{r_0}} |v_n|^{2k'} dx \le \int_{B_{r_0}} \limsup_{n\to\infty} |v_n|^{2k'} dx = \int_{B_{r_0}} |v|^{2k'} dx = 0.$$

Hence,

$$\limsup_{n\to\infty} \int_{B_{r_0}} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 dx = 0. \tag{2.15}$$

So, an application of (2.10) and (2.15) contradicts (2.5) and then meas$(A) > 0$, where $A = \{x \in \mathbb{R}^N : v(x) \neq 0\}$. Obviously, for a.e. $x \in A$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$. Hence, $A \subset Z_n$ for large n. Moreover, one sees that

$$\begin{aligned} &\int_{T_n} \frac{|F(x,u_n)|}{|u_n|^2} |v_n|^2 dx \le b^{\beta-2} \int_{\mathbb{R}^N} (h_1(x) + h_2(x))|v_n|^2 dx \\ &\le b^{\beta-2}(\|h_1 V^{-1}\|_\infty + \|h_2 V^{-1}\|_\infty) \int_{\mathbb{R}^N} V|v_n|^2 dx \\ &\le 2b^{\beta-2} M_1 \|v_n\|_E^2 = 2b^{\beta-2} M_1. \end{aligned} \tag{2.16}$$

Moreover, using assumption (A5) and Fatou's lemma, it follows from $J(u_n) \to c$ that

$$\begin{aligned} 0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|u_n\|_E^2} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n\|_E^2} \le \lim_{n\to\infty} \Big[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2 dx\Big] \\ &\le \frac{1}{2} + 2b^{\beta-2}M_1 - \liminf_{n\to\infty} \int_{Z_n} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2 dx \\ &\le \frac{1}{2} + 2b^{\beta-2}M_1 - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^2}\chi_{Z_n}(x)|v_n|^2 dx = -\infty, \end{aligned} \tag{2.17}$$

where χ_I denotes the characteristic function associated to the mensurable subset $I \subset \mathbb{R}^N$. Clearly, (2.17) is impossible. Thus $\{u_n\}$ is bounded in E and the proof of Lemma 2.5 is finished. □

Lemma 2.6. *Assume* (A1), (A2), (A5), (A7). *In addition, suppose that* (1.27) *is satisfied with* $h_3(x) \equiv 1$. *Then any* $(C)_c$ *sequence* $\{u_n\}$ *is bounded in* E.

Proof. Arguing as the proof of Lemma 2.5, we suppose that $\|u_n\|_E \to \infty$ as $n \to \infty$. Let $v_n(x) = \frac{u_n(x)}{\|u_n\|_E}$. Then $\{v_n\}$ satisfies (2.3). For any $\varepsilon > 0$, we choose $r_1 > 0$ such that $\int_{B_r^c} |v(x)|^2 dx < \varepsilon$ when $r \ge r_1$. Since $v_n(x) \to v(x)$ in $L^2(\mathbb{R}^N)$, we obtain

$$\limsup_{n\to\infty} \int_{B_r^c} |v_n(x)|^2 dx \le \int_{B_r^c} \limsup_{n\to\infty} |v_n(x)|^2 dx \le \int_{B_r^c} |v(x)|^2 dx < \varepsilon. \tag{2.18}$$

Then

$$\begin{aligned} \limsup_{n\to\infty} \int_{B_r^c\cap T_n} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx &\le 2b^{\beta-2}C_0 \limsup_{n\to\infty} \int_{B_r^c} |v_n|^2 dx \\ &\le C_2 \int_{B_r^c} |\omega(x)|^2 dx \le C_2\varepsilon, \end{aligned} \tag{2.19}$$

where $C_2 = 2b^{\beta-2}C_0$, b is the constant in (A6) and C_0 is given in (1.29).

On the other hand, from (1.27) with $h_3(x) = 1$ and (2.4) it follows that

$$\begin{aligned} &\int_{B_r^c\cap Z_n} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx \\ &\le \Big(\int_{B_r^c\cap Z_n} \Big(\frac{|F(x,u_n)|}{|u_n|^2}\Big)^k dx\Big)^{1/k} \Big(\int_{B_r^c\cap Z_n} |v_n|^{2k'} dx\Big)^{1/k'} \\ &\le c_1\Big(\int_{\mathbb{R}^N} G(x,u_n) dx\Big)^{1/k} \Big(\int_{B_r^c\cap Z_n} |v_n|^{2k'} dx\Big)^{1/k'} \\ &\le c_1(c+1)^{1/k}\Big(\int_{B_r^c} |v_n|^{2k'} dx\Big)^{1/k'}. \end{aligned} \tag{2.20}$$

From (2.9) and (2.18), for large n, we obtain

$$\int_{B_r^c} |v_n|^{2k'} dx \le \|v_n\|_{L^2(B_r^c)}^{2t_3} \|v_n\|_{L^{2_s^*}(B_r^c)}^{(1-t_3)2_s^*} \le S_0\|v_n\|_{L^2(B_r^c)}^{2t_3} \le S_0\varepsilon. \tag{2.21}$$

Then, an application of (2.19)-(2.21) gives that for any $\varepsilon > 0$ there exist $n_0, r_0 \ge 1$ such that $n \ge n_0, r \ge r_0$,

$$\int_{B_r^c} \frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 dx \le \varepsilon(C_2 + S_0C_1). \tag{2.22}$$

Similar to (2.11), for a.e. $x \in B_{r_0} \cap T_n$ and $n \geq 1$, we obtain

$$\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 \leq C_2|v_n(x)|^2 \leq C_2|\omega(x)|^2 \equiv d(x) \in L^1(B_{r_0}). \tag{2.23}$$

By Fatou's lemma,

$$\begin{aligned}\limsup_{n\to\infty}\int_{B_{r_0}\cap T_n}\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 &\leq C_2\int_{B_{r_0}}\limsup_{n\to\infty}|v_n(x)|^2\\ &= C_2\int_{B_{r_0}}|v(x)|^2dx.\end{aligned}\tag{2.24}$$

Similar to (2.20), we derive

$$\limsup_{n\to\infty}\int_{B_{r_0}\cap Z_n}\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 \leq C_1\Big(\int_{B_{r_0}}|v(x)|^{2k'}dx\Big)^{1/k'}. \tag{2.25}$$

If $v(x) = 0$ in B_{r_0}, an application of (2.24) and (2.25) gives that

$$\limsup_{n\to\infty}\int_{B_{r_0}}\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2 = 0. \tag{2.26}$$

Combining (2.22) with (2.26) contradicts (2.5). So, $\operatorname{meas}(A) > 0$, where $A = \{x \in \mathbb{R}^N : v(x) \neq 0\}$ and for a.e. $x \in A$, we have $\lim_{n\to\infty}|u_n(x)| = \infty$. Hence, $A \subset Z_n$ for large n. Moreover, one sees that

$$\int_{T_n}\frac{|F(x,u_n)|}{|u_n|^2}|v_n|^2dx \leq C_1\int_{\mathbb{R}^N}|v_n|^2dx \leq C_1\int_{\mathbb{R}^N}V|v_n|^2dx \leq C_1\|v_n\|_E^2 = C_1. \tag{2.27}$$

Moreover, using assumption (A5) and Fatou's lemma, from $J(u_n) \to c$ it follows that

$$\begin{aligned}0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|_E^2} = \lim_{n\to\infty}\frac{J(u_n)}{\|u_n\|_E^2} \leq \lim_{n\to\infty}\Big[\frac{1}{2} - \int_{\mathbb{R}^N}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2dx\Big]\\ &\leq \frac{1}{2} + C_1 - \liminf_{n\to\infty}\int_{Z_n}\frac{F(x,u_n)}{|u_n|^2}|v_n|^2dx\\ &\leq \frac{1}{2} + C_1 - \int_{\mathbb{R}^N}\liminf_{n\to\infty}\frac{F(x,u_n)}{|u_n|^2}\chi_{Z_n}(x)|v_n|^2dx = -\infty.\end{aligned}\tag{2.28}$$

Clearly, the limit (2.28) is impossible. Thus $\{u_n\}$ is bounded in E and the proof is complete. $\square$

From Lemmas 2.4–2.6, we know that any $(PS)_c$ sequence and $(C)_c$ sequence $\{u_n\}$ of the functional J are bounded in E. Therefore, by Lemma 2.3, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, and $u \in E$ such that $\|u_n\|_E + \|u\|_E \leq M(\forall n \geq 1)$ and

$$\begin{gathered}u_n \rightharpoonup u \text{ weakly in } E, \quad u_n \to u \text{ in } L^q(\mathbb{R}^N)\ (2 \leq q < 2_s^*),\\ u_n(x) \to u(x) \text{ a.e. in } \mathbb{R}^N\end{gathered}\tag{2.29}$$

with some constant $M > 0$.

Lemma 2.7. *Assume* (A1)–(A6) *hold. If the sequence* $\{u_n\}$ *satisfies* (2.29)*, then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}h_1(|u_n|^\alpha - |u|^\alpha)dx = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^N}h_2(|u_n|^\beta - |u|^\beta)dx = 0, \tag{2.30}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)dx=0,\quad \lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u)(u_n-u)dx=0, \tag{2.31}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} F(x,u_n)dx=\int_{\mathbb{R}^N} F(x,u)dx. \tag{2.32}$$

Proof. First, we assume (A3), (A4). From (2.29), we obtain

$$\lim_{n\to\infty}\int_{B_r} h_1(|u_n|^\alpha-|u|^\alpha)dx=0,\quad \lim_{n\to\infty}\int_{B_r} h_2(|u_n|^\beta-|u|^\beta)dx=0 \tag{2.33}$$

for any $r>0$. On the other hand, we see from the Hölder inequality and (A3) that

$$\begin{aligned}\int_{B_r^c} h_1|u_n|^\alpha dx &\le \sup_{x\in B_r^c}\frac{h_1(x)}{V^{t_1}(x)}\Big(\int_{B_r^c} V|u_n|^2dx\Big)^{t_1}\Big(\int_{B_r^c}|u_n|^{2_s^*}dx\Big)^{1-t_1}\\ &\le S_0\sup_{x\in B_r^c}\frac{h_1(x)}{V^{t_1}(x)}\|u_n\|_E^{2t_1}\|u_n\|_E^{(1-t_1)2_s^*}\\ &\le S_0M^\alpha\sup_{x\in B_r^c}\frac{h_1(x)}{V^{t_1}(x)}\to 0,\quad \text{as } r\to\infty,\end{aligned} \tag{2.34}$$

where $t_1=(2_s^*-\alpha)/(2_s^*-2), S_0=S_{2_s^*}$. Similarly, as $r\to\infty$,

$$\begin{aligned}\int_{B_r^c} h_2|u_n|^\beta dx &\le S_0\sup_{x\in B_r^c}\frac{h_2(x)}{V^{t_2}(x)}\|u_n\|_E^{2t_2}\|u_n\|_E^{(1-t_2)2_s^*}\\ &\le S_0M^\beta\sup_{x\in B_r^c}\frac{h_2(x)}{V^{t_2}(x)}\to 0,\end{aligned} \tag{2.35}$$

where $t_2=(2_s^*-\beta)/(2_s^*-2)$. Then an application of (2.33), (2.34) and (2.35) gives (2.30). Moreover, the limit (2.30) and Brezis-Lieb lemma [2] give that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} h_1|u_n-u|^\alpha dx=0,\quad \lim_{n\to\infty}\int_{\mathbb{R}^N} h_2|u_n-u|^\beta dx=0. \tag{2.36}$$

Thus, from (2.36), it follows that

$$\begin{aligned}&\int_{\mathbb{R}^N} h_1|u_n|^{\alpha-1}|u_n-u|dx \le \Big(\int_{\mathbb{R}^N} h_1|u_n|^\alpha dx\Big)^{(\alpha-1)/\alpha}\Big(\int_{\mathbb{R}^N} h_1|u_n-u|^\alpha dx\Big)^{1/\alpha}\\ &\le (S_0M^\alpha\|h_1V^{-t_1}\|_\infty)^{(\alpha-1)/\alpha}\Big(\int_{\mathbb{R}^N} h_1|u_n-u|^\alpha dx\Big)^{1/\alpha}\to 0,\quad \text{as } n\to\infty\end{aligned} \tag{2.37}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^N} h_2|u_n|^{\beta-1}|u_n-u|dx\\ &\le \Big(\int_{\mathbb{R}^N} h_2|u_n|^\beta dx\Big)^{1-1/\beta}\Big(\int_{\mathbb{R}^N} h_2|u_n-u|^\beta dx\Big)^{1/\beta}\\ &\le (S_0M^\beta\|h_2V^{-t_2}\|_\infty)^{1-1/\beta}\Big(\int_{\mathbb{R}^N} h_2|u_n-u|^\beta dx\Big)^{1/\beta}\to 0,\quad \text{as } n\to\infty.\end{aligned} \tag{2.38}$$

Hence,

$$\int_{\mathbb{R}^N}|f(x,u_n)(u_n-u)|dx\le\int_{\mathbb{R}^N}(h_1|u_n|^{\alpha-1}+h_2|u_n|^{\beta-1})|u_n-u|dx\to 0, \tag{2.39}$$

as $n\to\infty$. This proves the first limit of (2.31). The second limit of (2.31) can be obtained in a similar way.

To prove the limit (2.32), we use (1.24) and derive that

$$\begin{aligned}&|F(x,u_n)-F(x,v)|\\&\le C\Big[h_1(x)(|u_n|^{\alpha-1}+|u|^{\alpha-1})+h_2(x)(|u_n|^{\beta-1}+|u|^{\beta-1})\Big]|u_n-u|.\end{aligned} \tag{2.40}$$

Then an application of (2.37) and (2.38) yields that the limit (2.32). The proof is complete. □

Lemma 2.8. *Let the assumptions in Theorem 1.8 hold. If the sequence* $\{u_n\}$ *satisfies* (2.29)*, then the limits* (2.31) *and* (2.32) *hold.*

Proof. Choose $\psi \in C_0^\infty(\mathbb{R})$ such that $supp\psi \subset [-2,2]$ and $\psi(t)=1$ on $[-1,1]$. Denote $g(x,t)=\psi(t)f(x,t)$, $H(x,t)=(1-\psi(t))f(x,t)$. Then $f(x,t)=g(x,t)+H(x,t)$. Furthermore, from (1.29), there exist the constants $a_1,b_1>0$ such that

$$|g(x,t)|\le a_1|t|^{\alpha-1},\quad |H(x,t)|\le b_1|t|^{\beta-1},\quad \forall (x,t)\in\mathbb{R}^N\times\mathbb{R}. \tag{2.41}$$

Denote

$$A_n=\int_{\mathbb{R}^N}|g(x,u_n)-g(x,u)|^{\alpha'}dx,\quad D_n=\int_{\mathbb{R}^N}|H(x,u_n)-H(x,u)|^{\beta'}dx, \tag{2.42}$$

where $t'=t/(t-1)$. By (2.29), there exist $\omega_1(x)\in L^\alpha(\mathbb{R}^N)$ and $\omega_2(x)\in L^\beta(\mathbb{R}^N)$ such that $|u_n(x)|\le\omega_1(x)$ and $|u_n(x)|\le\omega_2(x)$ a.e. in $\mathbb{R}^N$ for all $n\ge 1$. Note that

$$\begin{aligned}|g(x,u_n)-g(x,u)|^{\alpha'}&\le C_3(|u_n(x)|^\alpha+|u(x)|^\alpha)\\&\le C_3(|\omega_1(x)|^\alpha+|u(x)|^\alpha)\equiv d_1(x)\in L^1(\mathbb{R}^N)\end{aligned} \tag{2.43}$$

and

$$\begin{aligned}|H(x,u_n)-H(x,u)|^{\beta'}&\le C_3(|u_n(x)|^\beta+|u(x)|^\beta)\\&\le C_3(|\omega_2(x)|^\beta+|u(x)|^\beta)\equiv d_2(x)\in L^1(\mathbb{R}^N),\end{aligned} \tag{2.44}$$

where C_3 is a constant independent of n. By the Lebesgue dominated convergence theorem and (2.29), we have

$$\begin{aligned}\lim_{n\to\infty}A_n&=\int_{\mathbb{R}^N}\lim_{n\to\infty}|g(x,u_n)-g(x,u)|^{\alpha'}dx=0,\\ \lim_{n\to\infty}D_n&=\int_{\mathbb{R}^N}\lim_{n\to\infty}|H(x,u_n)-H(x,u)|^{\beta'}dx=0.\end{aligned} \tag{2.45}$$

Therefore, by the Hölder inequality,

$$\begin{aligned}&\int_{\mathbb{R}^N}|f(x,u_n)-f(x,u)||u_n-u|dx\\&\le\int_{\mathbb{R}^N}(|g(x,u_n)-g(x,u)|+|H(x,u_n)-H(x,u)|)|u_n-u|dx\\&\le A_n^{1/\alpha'}\|u_n-u\|_\alpha+D_n^{1/\beta'}\|u_n-u\|_\beta\\&\le(A_n^{1/\alpha'}+D_n^{1/\beta'})\|u_n-u\|_E\le M(A_n^{1/\alpha'}+D_n^{1/\beta'}).\end{aligned} \tag{2.46}$$

An application of (2.45) and (2.46) gives

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)dx=\lim_{n\to\infty}\int_{\mathbb{R}^N}f(x,u)(u_n-u)dx. \tag{2.47}$$

Similarly, from (2.41), (2.43) and (2.44), we can derive that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} f(x,u)(u_n-u)dx=\int_{\mathbb{R}^N}\lim_{n\to\infty} f(x,u)(u_n-u)dx=0. \tag{2.48}$$

Consequently, the limit (2.31) is given. Similarly, (2.32) can be proved and the proof is complete. □

Lemma 2.9. *Let the assumptions in Theorems 1.7 and 1.8 hold. Let $\{u_n\}$ be the sequence in Lemmas 2.4–2.6 satisfying* (2.29)*. Then u is a critical point of the functional J and $u_n\to u$ in E.*

Proof. First, we show that $J'(u)=0$ in E^*. By Lemmas 2.4–2.6, the sequence $\{u_n\}$ is bounded in E. So, there exists a subsequence, still denoted by $\{u_n\}$, such that $\{u_n\}$ satisfies (2.29). Moreover, one sees that for all $\varphi\in C_0^\infty(\mathbb{R}^N)$,

$$\lim_{n\to\infty}\Big(\int_{\mathbb{R}^N}|\xi|^2(\hat{u_n}-\hat{u})\hat{\varphi}dx+\int_{\mathbb{R}^N}V(x)(u_n-u)\varphi dx\Big)=0. \tag{2.49}$$

Under assumptions (A3)–(A7), we obtain

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}(f(x,u_n)-f(x,u))\varphi dx=0. \tag{2.50}$$

Furthermore, from (2.49), (2.50) and the assumption $J'(u_n)\to 0$ in E^*, we have

$$0=\lim_{n\to\infty}J'(u_n)\varphi=J'(u)\varphi,\quad \forall\varphi\in C_0^\infty(\mathbb{R}^N). \tag{2.51}$$

By the denseness of $C_0^\infty(\mathbb{R}^N)$ in E, it follows that $J'(u)\varphi=0,\forall\varphi\in E$. Hence, u is a critical point of J in E. On the other hand, from (2.29) it follows that

$$R_n=\int_{\mathbb{R}^N}|\xi|^2\hat{u}(\hat{u}_n-\hat{u})d\xi+\int_{\mathbb{R}^N}V(x)u(u_n-u)dx\to 0,\ \text{ as }\quad n\to\infty. \tag{2.52}$$

Set

$$W_n:=\int_{\mathbb{R}^N}f(x,u_n)(u_n-u)dx,\quad S_n:=\int_{\mathbb{R}^N}f(x,u)(u_n-u)dx,\quad \forall n\in\mathbb{N}. \tag{2.53}$$

From (2.31), it follows that $W_n,S_n\to 0$ as $n\to\infty$ and so

$$J'(u)(u_n-u)=R_n-S_n\to 0. \tag{2.54}$$

Similarly, we set

$$Q_n:=(J'(u_n)-J'(u))(u_n-u)=\|u_n-u\|_E^2-W_n+S_n,\quad \forall n\in\mathbb{N}. \tag{2.55}$$

Obviously, relation (2.55) can be reduced to the form

$$\|u_n-u\|_E^2=W_n+Q_n-S_n,\quad \forall n\in\mathbb{N}. \tag{2.56}$$

From (2.53), (2.54) and $J'(u_n)\to 0$, we find $Q_n\to 0$ and $\|u_n-u\|_E\to 0$ as $n\to\infty$. Thus $u_n\to u$ in E as $n\to\infty$ under assumptions (A3)–(A7). Therefore, J satisfies the $(C)_c$ condition in E and the proof is complete. □

Proof of Theorem 1.7. Clearly, the functional J defined by (1.22) is even. By Lemma 2.9, the functional J satisfies the $(C)_c$ condition. Next, we prove that J satisfies (A8) and (A9) in Lemma 2.2. From (A3), we have

$$J(u)=\frac{1}{2}\|u\|_E^2-\int_{\mathbb{R}^N}F(x,u)dx\ge\frac{1}{2}\|u\|_E^2-\int_{\mathbb{R}^N}(h_1|u|^\alpha+h_2|u|^\beta)dx. \tag{2.57}$$

Arguing as in the proof of (2.34) and (2.35), we obtain

$$J(u) \geq \frac{1}{2}\|u\|_E^2 - S_0 M_3(\|u\|_E^\alpha + \|u\|_E^\beta) \geq \rho^2(\frac{1}{2} - 2S_0 M_3 \rho^{\alpha-2}) \geq \frac{\rho^2}{4} \equiv \alpha_0 > 0, \tag{2.58}$$

where $M_3 = \max\{\|h_1 V^{-t_1}\|_\infty, \|h_2 V^{-t_2}\|_\infty\}$ and $\|u\|_E = \rho = \min\{1, (8S_0M_3)^{\frac{1}{2-\alpha}}\}$. Thus, by (2.58), condition (A8) is satisfied. We now satisfy condition (A9). For any finite dimensional subspace $E_0 \subset E$, we assert that there holds $J(u_n) \to -\infty$ when $u_n \in E_0$ and $\|u_n\|_E \to \infty$. Arguing by contradiction, suppose that for some sequence $\{u_n\} \subset E_0$ with $\|u_n\|_E \to \infty$, there is $M_4 > 0$ such that $J(u_n) \geq -M_4$, for all $n \geq 1$. Set $v_n(x) = \frac{u_n(x)}{\|u_n\|_E}$, then $\|v_n\|_E = 1$. Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n(x) \to v(x)$ a.e on $\mathbb{R}^N$. Since E_0 is finite dimensional, then $v_n \to v$ in E_0 and so $v \neq 0$ a.e.in $\mathbb{R}^N$. Set $\Omega = \{x \in \mathbb{R}^N : v(x) \neq 0\}$, then $\mathrm{meas}(\Omega) > 0$. For $x \in \Omega$, we have $\lim_{n\to\infty} |u_n(x)| = \infty$.

Then, from (A5) it follows that

$$\begin{aligned} 0 &= \limsup_{n\to\infty} \frac{-M}{\|u_n\|_E^2} \leq \limsup_{n\to\infty} \frac{J(u_n)}{\|u_n\|_E^2} = \limsup_{n\to\infty} \Big[\frac{1}{2} - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|u_n|^2}|v_n|^2 dx\Big] \\ &\leq \frac{1}{2} - \int_{\mathbb{R}^N} \liminf_{n\to\infty} \frac{F(x,u_n)}{|u_n|^2} \chi_\Omega(x)|v_n|^2 dx = -\infty, \end{aligned} \tag{2.59}$$

and we have a contradiction. So, there exists $R = R(E_0) > 0$ such that $J(u) < 0$ for $u \in E_0$ and $\|u\|_E \geq R$. Therefore, condition (A9) is satisfied. Then an application of Lemma 2.2 shows that (1.1) admits infinitely many solutions $u_n \in E$ with $J(u_n) \to \infty$ as $n \to \infty$. This completes the proof. $\square$

Proof of Theorem 1.8. Clearly, the functional J defined by (1.22) is even. By Lemma 2.9, the functional J satisfies the $(C)_c$ condition. Next, we prove that J satisfies (A8) and (A9) in Lemma 2.2. From (A7), we have

$$J(u) = \frac{1}{2}\|u\|_E^2 - \int_{\mathbb{R}^N} F(x,u)dx \geq \frac{1}{2}\|u\|_E^2 - C_0 \int_{\mathbb{R}^N} (|u|^\alpha + |u|^\beta)dx. \tag{2.60}$$

Furthermore, from (1.20) it follows that

$$J(u) \geq \frac{1}{2}\|u\|_E^2 - C_4(\|u\|_E^\alpha + \|u\|_E^\beta) \geq \rho^2(\frac{1}{2} - C_4\rho^{\alpha-2}) \geq \frac{\rho^2}{4} \equiv \alpha_1 > 0, \tag{2.61}$$

with $\|u\|_E = \rho = \min\{1, (4C_4)^{\frac{1}{2-\alpha}}\}$ and $C_4 = \max\{S_\alpha^\alpha, S_\beta^\beta\}$. Thus, by (2.61), condition (A8) is satisfied. Similarly, we can derive (2.59) and the verification of condition (A9) is finished. Again, using Lemma 2.2, we complete the proof of Theorem 1.8. $\square$

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