MULTIPLE SOLUTIONS FOR CRITICAL ELLIPTIC PROBLEMS
WITH FRACTIONAL LAPLACIAN

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Abstract. This article is devoted to the study of the nonlocal fractional equation involving critical nonlinearities

\((-\Delta)^{\alpha/2} u = \lambda u + |u|^{2^*_\alpha} - 2u \quad \text{in } \Omega,\)

\(u = 0 \quad \text{on } \partial \Omega,\)

where \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^N\), \(N \geq 2\alpha, \alpha \in (0, 2)\), \(\lambda \in (0, \lambda_1)\) and \(2^*_\alpha = \frac{2N}{N-\alpha}\) is critical exponent. We show the existence of at least \(\text{cat} \Omega (\Omega)\) nontrivial solutions for this problem.

1. Introduction

This article concerns the critical elliptic problem with the fractional Laplacian

\((-\Delta)^{\alpha/2} u = \lambda u + |u|^{2^*_\alpha} - 2u \quad \text{in } \Omega,\)

\(u = 0 \quad \text{on } \partial \Omega,\) \hspace{1cm} (1.1)

where \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^N\) with \(N > \alpha, \alpha \in (0, 2)\) is fixed and \(2^*_\alpha = \frac{2N}{N-\alpha}\) is the critical Sobolev exponent.

In a bounded domain \(\Omega \subset \mathbb{R}^N\), the operator \((-\Delta)^{\alpha/2}\) can be defined as in [3, 6] as follows. Let \(\{ (\lambda_k, \varphi_k) \}_{k=1}^\infty \) be the eigenvalues and corresponding eigenfunctions of the Laplacian \(-\Delta\) in \(\Omega\) with zero Dirichlet boundary values on \(\partial \Omega\) normalized by \(\| \varphi_k \|_{L^2(\Omega)} = 1\), i.e.

\(-\Delta \varphi_k = \lambda_k \varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \quad \text{on } \partial \Omega.\)

We define the space \(H_0^{\alpha/2}(\Omega)\) by

\(H_0^{\alpha/2}(\Omega) = \{ u = \sum_{k=1}^\infty u_k \varphi_k \in L^2(\Omega) : \sum_{k=1}^\infty u_k^2 \lambda_k^{\frac{\alpha}{2}} < \infty \},\)

which is equipped with the norm

\(\| u \|_{H_0^{\alpha/2}(\Omega)} = \left( \sum_{k=1}^\infty u_k^2 \lambda_k^{\frac{\alpha}{2}} \right)^{\frac{1}{2}}.\)
For $u \in H_0^{\alpha/2}(\Omega)$, the fractional Laplacian $(-\Delta)^{\alpha/2}$ is defined by

$$(-\Delta)^{\alpha/2}u = \sum_{k=1}^{\infty} u_k \lambda_k^{\alpha/2} \varphi_k.$$  

Problem (1.1) is the Brézis-Nirenberg type problem with the fractional Laplacian. Brézis and Nirenberg [4] considered the existence of positive solutions for problem (1.1) with $\alpha = 2$. Such a problem involves the critical Sobolev exponent $2^* = \frac{2N}{N-2}$ for $N \geq 3$, and it is well known that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is not compact even if $\Omega$ is bounded. Hence, the associated functional of problem (1.1) does not satisfy the Palais-Smale condition, and critical point theory cannot be applied directly to find solutions of the problem. However, it is found in [4] that the functional satisfies the $(PS)_c$ condition for $c \in (0, \frac{1}{N} S^{N/2}_N)$, where $S$ is the best Sobolev constant and $\frac{1}{N} S^{N/2}_N$ is the least level at which the Palais-Smale condition fails. So a positive solution can be found if the mountain pass value corresponding to problem (1.1) is strictly less than $\frac{1}{N} S^{N/2}_N$.

Problems with the fractional Laplacian have been extensively studied, see for example [2, 3, 5, 6, 7, 9, 10, 12, 13] and the references therein. In particular, the Brézis-Nirenberg type problem was discussed in [12] for the special case $\alpha = \frac{1}{2}$, and in [2] for the general case, $0 < \alpha < 2$, where existence of one positive solution was proved. To use the idea in [4] to prove the existence of one positive solution for the fractional Laplacian, the authors in [2, 12] used the following results in [10] (see also [3]): for any $u \in H_0^\alpha(\Omega)$, the solution $v \in H^1_{0,\text{loc}}(\mathcal{C}_\Omega)$ of the problem

$$-\text{div}(y^{1-\alpha} \nabla v) = 0, \quad \text{in } \mathcal{C}_\Omega = \Omega \times (0, \infty),$$

$$v = 0, \quad \text{on } \partial_L \mathcal{C}_\Omega = \partial \Omega \times (0, \infty),$$

$$v = u, \quad \text{on } \Omega \times \{0\},$$

(1.2)
satisfies

$$-\lim_{y \to 0^+} k_\alpha y^{1-\alpha} \frac{\partial v}{\partial y} = (-\Delta)^\alpha u,$$

where we use $(x, y) = (x_1, \ldots, x_N, y) \in \mathbb{R}^{N+1}$, and

$$H^1_{0,\text{loc}}(\mathcal{C}_\Omega) = \{ w \in L^2(\mathcal{C}_\Omega) : w = 0 \text{ on } \partial_L \mathcal{C}_\Omega, \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w|^2 \, dx \, dy < \infty \}. \quad (1.3)$$

Therefore, the nonlocal problem (1.1) can be reformulated as the local problem

$$-\text{div}(y^{1-\alpha} \nabla w) = 0, \quad \text{in } \mathcal{C}_\Omega,$$

$$v = 0, \quad \text{on } \partial_L \mathcal{C}_\Omega,$$

$$\lim_{y \to 0^+} y^{1-\alpha} \frac{\partial w}{\partial y} = |w(x, 0)|^{2^* - 2} w(x, 0) + \lambda w(x, 0), \quad \text{on } \Omega \times \{0\},$$

(1.4)

where $\frac{\partial}{\partial \nu}$ is the outward normal derivative of $\partial \mathcal{C}_\Omega$. Hence, critical points of the functional

$$J(w) = \frac{1}{2} \int_{\mathcal{C}_\Omega} y^{1-\alpha} |\nabla w|^2 \, dx \, dy - \frac{1}{2^*} \int_{\Omega \times \{0\}} |w(x, 0)|^{2^*} \, dx$$

$$- \frac{\lambda}{2} \int_{\Omega \times \{0\}} |w(x, 0)|^2 \, dx$$

(1.5)
defined on \( H^{1}_{0,L}(C_{\Omega}) \) correspond to solutions of \( (1.4) \), and the trace \( u = tr w \) of \( w \) is a solution of \( (1.1) \). A critical point of the functional \( J(u) \) at the mountain pass level was found in \([2, 12]\). On the other hand, it can be shown by using the Pohozaev type identity that the problem

\[
(-\Delta)^{\alpha/2} u = |u|^{p-1} u \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{on } \partial \Omega
\]

has no nontrivial solution if \( p + 1 \geq \frac{2N}{N-\alpha} \) and \( \Omega \) is star-shaped, see for example \([3]\) and \([12]\).

It is well-known that if \( \Omega \) has a rich topology, \( (1.1) \) with \( \alpha = 2, \lambda = 0 \) has a solution, see \([1, 8, 14]\) etc. In this paper, we assume \( 0 < \lambda < \lambda_{1} \), where \( \lambda_{1} \) is the first eigenvalue of the fractional Laplacian \( (-\Delta)^{\alpha/2} \). We investigate the existence of multiple solutions of problem \( (1.1) \). Let \( A \) be a closed subset of a topology space \( X \). The category of \( A \) is the least integer \( n \) such that there exist \( n \) closed subsets \( A_{1}, \ldots, A_{n} \) of \( X \) satisfying \( A = \bigcup_{j=1}^{n} A_{j} \) and \( A_{1}, \ldots, A_{n} \) are contractible in \( X \). Our main result is as follows.

**Theorem 1.1.** If \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^{N} \), \( N \geq 4 \), \( 0 < \alpha < 2 \) and \( 0 < \lambda < \lambda_{1} \), \( (1.1) \) has at least \( \text{cat}_{\Omega}(\Omega) \) nontrivial solutions. Equivalently, \( (1.1) \) possesses at least \( \text{cat}_{\Omega}(\Omega) \) positive solutions.

We say that \( w \in H^{1}_{0,L}(C_{\Omega}) \) is a solution to \( (1.4) \) if for every function \( \varphi \in H^{1}_{0,L}(C_{\Omega}) \), we have

\[
k_{\alpha} \int_{C_{\Omega}} y^{1-\alpha} \langle \nabla w, \nabla \varphi \rangle \, dx \, dy = \int_{\Omega} (\lambda w + w^{\frac{N+\alpha}{N-\alpha}}) \varphi \, dx.
\]

We will find solutions of \( J \) at energy levels below a value related to the best Sobolev constant \( S_{\alpha,N} \), where

\[
S_{\alpha,N} = \inf_{w \in H^{1}_{0,L}(C_{\Omega}), \nu \not= 0} \frac{k_{\alpha} \int_{C_{\Omega}} y^{1-\alpha} |\nabla w|^{2} \, dx \, dy}{(\int_{\Omega} |w(x,0)|^{2} \, dx)^{\frac{2}{N-\alpha}}},
\]

which is not achieved in any bounded domain and is indeed achieved in the case \( \Omega = \mathbb{R}^{N+1} \). We know from \([2]\) that the trace \( u_{\epsilon}(x) = w_{\epsilon}(x,0) \) of the family of minimizers \( w_{\epsilon} \) of \( S_{\alpha,N} \) takes the form

\[
u(x) = u_{\epsilon}(x) = \frac{\epsilon^{\frac{N-\alpha}{2}}}{(|x|^{2} + \epsilon^{2})^{\frac{N-\alpha}{2}}},
\]

with \( \epsilon > 0 \). Using this property, we are able to find critical values of \( J \) in a right range.

In section 2, we prove the \( (PS)_{c} \) condition and the main result is shown in section 3.

2. Palais-Smale condition

In this section, we show that the functional \( J(w) \) satisfies \( (PS)_{c} \) condition for \( c \) in certain interval. By a \( (PS)_{c} \) condition for the functional \( J(w) \) we mean that a sequence \( \{w_{n}\} \subset H^{1}_{0,L}(C_{\Omega}) \) such that \( J(w_{n}) \to c, J'(w_{n}) \to 0 \) contains a convergent subsequence.
Define on the space $H^1_{0,L}(C_\Omega)$ the functionals
\[ \psi(w) = \int_\Omega (w^+(x,0))^2 \, dx, \]
\[ \varphi_\lambda(w) = k_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla w|^2 \, dx dy - \lambda \int_\Omega |w(x,0)|^2 \, dx. \]
We may verify as in [14] that on the manifold $V = \{ w \in H^1_{0,L}(C_\Omega) : \psi(w) = 1 \}$,
\[ \psi'(w) \neq 0 \text{ for every } w \in V. \]
Hence, the tangent space of $V$ at $v$ is given by
\[ T_v V := \{ w \in H^1_{0,L}(C_\Omega) : \langle \psi'(v), w \rangle = 0 \}, \]
and the norm of the derivative of $\varphi_\lambda(v)$ at $v$ restricted to $V$ is defined by
\[ \| \varphi_\lambda'(v) \|_* = \sup_{w \in T_v V, \|w\|=1} |\langle \varphi_\lambda'(v), w \rangle|. \]
It is well known that
\[ \| \varphi_\lambda'(v) \|_* = \min_{\mu \in \mathbb{R}} \| \varphi_\lambda'(v) - \mu \psi'(w) \|. \]
A critical point $v \in V$ of $\varphi_\lambda$ is a point such that $\| \varphi_\lambda'(v) \|_* = 0$.
Since $\lambda_1$ is the first eigenvalue of the fractional Laplacian $(-\Delta)^{\alpha/2}$, it can be characterized as
\[ \lambda_1 = \inf_{w \in H^1_{0,L}(C_\Omega), w \neq 0} \frac{k_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla w|^2 \, dx dy}{\int_\Omega |w(x,0)|^2 \, dx}. \]
If $0 < \lambda < \lambda_1$, we see that
\[ \| w \|_1 := \left( k_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla w|^2 \, dx dy - \lambda \int_\Omega |w(x,0)|^2 \, dx \right)^{1/2} \]
is an equivalent norm on $H^1_{0,L}(C_\Omega)$.

**Lemma 2.1.** Any sequence $\{v_n\} \subset H^1_{0,L}(C_\Omega)$ such that
\[ d := \sup_n J(v_n) < C^* := \frac{\alpha}{2N} S_\alpha N, \quad J'(v_n) \to 0 \quad \text{in } H^{-1}_{0,L}(C_\Omega) \]
contains a convergent subsequence.

**Proof.** It is easy to show from the assumptions that
\[ d + 1 + \|v_n\|_1 \geq J(v_n) - \frac{1}{2^*} \langle J'(v_n), v_n \rangle \]
\[ = \left( \frac{1}{2} - \frac{1}{2^*} \right) \left( \int_{C_\Omega} k_\alpha y^{1-\alpha} |\nabla v_n|^2 \, dx dy - \lambda \int_\Omega |v_n|^2 \, dx \right) \]
\[ = \left( \frac{1}{2} - \frac{1}{2^*} \right) \|v_n\|_1^2; \]
that is, $\|v_n\|_1$ is bounded. We may assume that
\[ v_n(x,y) \to v(x,y) \quad \text{in } H^1_{0,L}(C_\Omega), \]
\[ v_n(x,0) \to v(x,0) \quad \text{in } L^2(\Omega), \]
\[ v_n(x,0) \to v(x,0) \quad \text{a.e. in } \Omega. \]
Therefore, for every $\varphi \in H^1_{0,L}(\Omega)$,
$$
\langle J'(v_n), \varphi \rangle \to \langle J'(v), \varphi \rangle = 0
$$
as $n \to +\infty$. We also have that $J(v) \geq 0$. By Brézis-Lieb’s lemma,
$$
J(v) + \frac{1}{2} \|v_n - v\|_1^2 - \frac{1}{2\alpha} \int_{\Omega} (v_n - v)^{2^*_+} dx = J(v_n) + o(1) = C + o(1).
$$
Since $\langle J'(v_n), v_n \rangle \to 0$, we obtain
$$
\|v_n - v\|_1^2 - 2\alpha \int_{\Omega} (v_n - v)^{2^*_+} dx = -\|v\|_1^2 + 2\alpha \int_{\Omega} v_+^{2^*_+} dx = -\langle J'(v), v \rangle = 0.
$$
Hence, there exist a constant $b$ such that
$$
\|v_n - v\|_1^2 \to b, \quad 2\alpha \int_{C\Omega} (v_n)^{2^*_+} dx \to b, \quad \text{as } n \to +\infty.
$$
It follows by $v_n \to v$ in $L^2(\Omega)$ that
$$
\int_{C\Omega} k_\alpha y^{1-\alpha} |\nabla (v_n - v)|^2 dx dy \to b.
$$
The trace inequality
$$
\int_{C\Omega} k_\alpha y^{1-\alpha} |\nabla (v_n - v)|^2 dx dy \geq S_{\alpha,N} \|v_n - v(x,0)\|_{L^{2^*_+}(\Omega)}^2
$$
implies $b \geq S_{\alpha,N} b^{\frac{2}{2^*_+}}$. Hence, either $b = 0$ or $b \geq S_{\alpha,N}^{\frac{2}{2^*_+}}$.

If $b = 0$, then $v_n \to v$ in $H^1_{0,L}(\Omega)$, and the proof is complete. If $b \geq S_{\alpha,N}^{\frac{2}{2^*_+}}$, we deduce that
$$
C^* = \frac{\alpha}{2N} S_{\alpha,N}^{\alpha/N} \leq \left( \frac{1}{2} - \frac{1}{2^*_+} \right) b
$$
$$
= \left( \frac{1}{2} - \frac{1}{2\alpha} \right) \|v_n - v\|_1^2 + o(1)
$$
$$
\leq J(v) + \frac{1}{2} \|v_n - v\|_1^2 - \frac{1}{2\alpha} \|v_n - v\|_1^2 + o(1)
$$
$$
= J(v) + \frac{1}{2} \|v_n - v\|_1^2 - \frac{1}{2\alpha} \int_{\Omega} (v_n - v)^{2^*_+} + o(1)
$$
$$
= C \leq d < C^*,
$$
which is a contradiction. \qed

Alternatively, we have the following result.

**Lemma 2.2.** Every sequence $\{w_n\} \subset V$ satisfying $\varphi_\lambda(w_n) \to c < S_{\alpha,N}$ and $\|\varphi_\lambda'(w_n)\| \to 0$, as $n \to +\infty$, contains a convergent subsequence.
Proof. Since
\[ \| \varphi'_{\lambda}(w_n) \|_s = \min_{\mu \in \mathbb{R}} \| \varphi'_{\lambda}(w_n) - \mu \psi'(w_n) \|, \]
there exists a sequence \( \{ \alpha_n \} \subset \mathbb{R} \) such that \( \| \varphi'_{\lambda}(w_n) - \alpha_n \psi'(w_n) \| \to 0 \). It follows that for every \( h \in H_{0,L}^1(\Omega) \),
\[ k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w_n| dy - \lambda \int_{\Omega} w_n h dx - \mu_n \int_{\Omega} (w_n^+)^{2\alpha-1} h dx \to 0, \quad (2.1) \]
where \( \mu_n = \frac{\alpha_n 2^\alpha}{2} \). Choosing \( h = w_n \) in (2.1) and using the fact \( \psi(w_n^+) = 1 \), we obtain
\[ \varphi_{\lambda}(w_n) - \mu_n = k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w_n| dy - \lambda \int_{\Omega} |w_n|^2 - \mu_n \int_{\Omega} (w_n^+)^{2\alpha} \to 0. \]
Whence \( \varphi_{\lambda}(w_n) \to c, \mu_n \to c \) as \( n \to +\infty \). Setting \( v_n := \mu_n \frac{\alpha_n}{\alpha_n 2^\alpha} w_n \), we obtain
\[ J(v_n) = \frac{1}{2} \mu_n^\frac{\alpha}{\alpha_n 2^\alpha} \left( \int_{\Omega} k_\alpha y^{1-\alpha} |\nabla w_n|^2 dy - \lambda \int_{\Omega} |w_n|^2 dx \right) - \frac{1}{\alpha} \mu_n \int_{\Omega} (w_n^+)^{2\alpha} \]
\[ = \frac{1}{2} \mu_n^\frac{\alpha}{\alpha_n 2^\alpha} \varphi_{\lambda}(w_n) - \frac{N - \alpha}{2N} \mu_n^\frac{\alpha}{\alpha_n 2^\alpha}. \]
Thus,
\[ J(v_n) \to \frac{\alpha}{2N} c_0^\frac{\alpha}{\alpha_n 2^\alpha} < \frac{\alpha}{2N} S^\frac{\alpha}{\alpha_n 2^\alpha}. \]
In the same way, for every \( h \in H_{0,L}^1(\Omega) \), by (2.1),
\[ \langle J'(v_n), h \rangle \]
\[ = \frac{\alpha_n 2^\alpha}{\alpha} k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w_n| dy - \lambda \int_{\Omega} w_n h dx - \mu_n \int_{\Omega} (w_n^+)^{2\alpha-1} h dx \to 0. \]
Now, the assertion follows by Lemma \( \ref{2.1} \). \( \square \)

Let us define
\[ Q_{\lambda} = \inf_{w \in V} \varphi_{\lambda}(w) = \inf_{w \in V} \left\{ k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w|^2 dx - \lambda \int_{\Omega} |w(x,0)|^2 dx \right\}. \]
Denote by \( \eta_0(t) \in C^\infty(\mathbb{R}_+) \) a cut-off function, which is non-increasing and satisfies
\[ \eta_0(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq 1. \end{cases} \]
Assume \( 0 \in \Omega \), for fixed \( \rho > 0 \) small enough such that \( \overline{B}_\rho \subset \Omega \), we define the function \( \eta(x,y) = \eta(\frac{y}{\rho}) = \eta_0(\frac{|(x,y)|}{\rho}) \). Then \( \eta w_{\epsilon} \in H_{0,L}^1(\Omega) \). It is standard to establish the following estimates, see \( \cite{2} \) for details.

**Lemma 2.3.** The family \( \{ \eta w_{\epsilon} \} \subset H_{0,L}^1(\Omega) \) and its trace on \( y = 0 \) satisfy
\[ \| \eta w_{\epsilon} \|_s^2 = \| w_{\epsilon} \|_s^2 + O(\epsilon^{N-\alpha}), \quad (2.2) \]
If \( N > 2\alpha \),
\[ \| \eta w_{\epsilon} \|_{L_2(\Omega)}^2 = C \epsilon^\alpha + O(\epsilon^{N-\alpha}), \quad (2.3) \]
If \( N = 2\alpha \),
\[ \| \eta w_{\epsilon} \|_{L_2(\Omega)}^2 = C \epsilon^\alpha \log^\frac{1}{\epsilon} + O(\epsilon^\alpha) \quad (2.4) \]
for \( \epsilon > 0 \) small enough and some \( C > 0 \).
Lemma 2.4. Assume $N \geq 2\alpha$, $0 < \lambda < \lambda_1$, then
\[ Q_\lambda = \inf_{w \in V} \varphi_\lambda(w) < S_{\alpha,N}. \quad (2.5) \]
Moreover, there exists $u \in V$ such that $\varphi_\lambda(u) = Q_\lambda$.

Proof. We first show that (2.5) holds if $N \geq 2\alpha$ and $0 < \lambda < \lambda_1$. Since
\[ \int_{|x| > \frac{\epsilon}{2}} |u_\epsilon|^{2^*_\alpha} dx = \int_{|x| > \frac{\epsilon}{2}} \frac{\epsilon^N}{(|x|^2 + \epsilon^2)^N} dx \leq \frac{N2^*_\alpha}{\rho^N-\epsilon^N}, \]
we have
\[ \int_\Omega |\eta u_\epsilon|^{2^*_\alpha} dx \geq \int_{\{|x| \leq \frac{\epsilon}{2}\}} |u_\epsilon|^{2^*_\alpha} dx = \|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} - \int_{\{|x| \geq \frac{\epsilon}{2}\}} |u_\epsilon|^{2^*_\alpha} dx \geq \|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + O(\epsilon^N). \]
By Lemma 2.3 for $N > 2\alpha$, we have
\[ k_\alpha \int_{C_\Omega} y^{1-\alpha} |\nabla(\eta u_\epsilon)|^2 dx dy - \lambda \int_\Omega |\eta u_\epsilon|^2 dx \leq \frac{(\int_\Omega |\eta u_\epsilon|^{2^*_\alpha} dx)^{\frac{2}{2^*_\alpha}}}{\|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + O(\epsilon^N)} \leq S_{\alpha,N} - \frac{\lambda C\epsilon^\alpha}{\|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + O(\epsilon^N)} < S_{\alpha,N}. \]
Similarly, for $N = 2\alpha$, we find for $\epsilon$ small enough such that
\[ Q_\lambda \leq S_{\alpha,N} - \frac{\lambda C\epsilon^\alpha \log(\frac{1}{\epsilon})}{\|u_\epsilon\|^{2^*_\alpha}_{L^{2^*_\alpha}(\Omega)} + O(\epsilon^\alpha)} < S_{\alpha,N}. \]
Consequently, inequality (2.5) holds.

Next, we show that $Q_\lambda$ is achieved if $0 < \lambda < \lambda_1$. Obviously, $Q_\lambda > 0$. Now, let $\{w_n\} \subset H^1_{0,L}(C_\Omega)$ be a minimizing sequence of $Q_\lambda > 0$ such that $w_n \geq 0$ and $\|w_n(x,0)\|_{L^{2^*_\alpha}(\Omega)} = 1$. The boundedness of $\{w_n\}$ implies that
\[
\begin{align*}
  w_n(x,y) &\to w(x,y) \quad \text{in} \quad H^1_{0,L}(C_\Omega), \\
  w_n(x,0) &\to w(x,0) \quad \text{in} \quad L^q(\Omega), \\
  w_n(x,0) &\to w(x,0) \quad \text{a.e. in} \quad \Omega,
\end{align*}
\]
where $1 \leq q \leq 2^*_\alpha$. Since
\[ \|w_n\|^2 = \|w_n - w\|^2 + \|w\|^2 + o(1), \]
by the Brezis-Lieb Lemma,
\[
\begin{align*}
  \|w_n\|^2 - \lambda \|w_n(x,0)\|^2_{L^2(\Omega)} \\
  &\geq S_{\alpha,N} \|w_n(x,0) - w(x,0)\|^2_{L^{2^*_\alpha}(\Omega)} + Q_\lambda \|w(x,0)\|^2_{L^{2^*_\alpha}(\Omega)} + o(1) \\
  &\geq (S_{\alpha,N} - Q_\lambda) \|w_n(x,0) - w(x,0)\|^2_{L^{2^*_\alpha}(\Omega)} + Q_\lambda \|w_n(x,0)\|^2_{L^{2^*_\alpha}(\Omega)} + o(1) \\
  &= (S_{\alpha,N} - Q_\lambda) \|w_n(x,0) - w(x,0)\|^2_{L^{2^*_\alpha}(\Omega)} + Q_\lambda + o(1).
\end{align*}
\]
Hence, we obtain
\[ o(1) + Q_\lambda \geq (S_{\alpha,N} - Q_\lambda)\|w_n(x,0) - w(x,0)\|_{L^2(\Omega)}^2 + Q_\lambda + o(1). \]
The $S_{\alpha,N} > Q_\lambda$ implies $w_n(x,0) \to w(x,0)$ in $L^2(\Omega)$ and $\|w(x,0)\|_{L^2(\Omega)} = 1$. This yields
\[ Q_\lambda \leq \|w\|^2 - \lambda\|w(x,0)\|_{L^2(\Omega)}^2 \leq \lim_{n \to +\infty} (\|w_n\|^2 - \lambda\|w_n(x,0)\|_{L^2(\Omega)}^2) \leq Q_\lambda; \]
that is, $w$ is a minimizer for $Q_\lambda$. \hfill \Box

3. Proof of main theorem

Taking into account the concentration-compactness principle in [11], we may derive the following result, its proof can be found in [2].

Lemma 3.1. Suppose $w_n \to w$ in $H^1_{0,L}(\Omega)$, and the sequence $\{y^{1-\alpha}|\nabla w_n|^2\}$ is tight, i.e. for any $\eta > 0$ there exists $\rho_0 > 0$ such that for all $n$,
\[ \int_{\{y > \rho_0\}} \int_{\Omega} y^{1-\alpha}|\nabla w_n|^2 \, dx \, dy < \eta. \]
Let $u_n = t_r w_n$ and $u = t_r w$ and let $\mu$, $\nu$ be two non negative measures such that
\[ y^{1-\alpha}|\nabla w_n|^2 \to \mu \quad \text{and} \quad |u_n|^{2^*} \to \nu \] (3.1)
in the sense of measures as $n \to \infty$. Then, there exists an at most countable set $I$ and points $x_i \in \Omega$ with $i \in I$ such that
1. $\nu = |u|^2 + \sum_{k \in I} \nu_k \delta_{x_k}$, $\nu_k > 0$,
2. $\mu = y^{1-\alpha}|\nabla w|^2 + \sum_{k \in I} \mu_k \delta_{x_k}$, $\mu_k > 0$,
3. $\mu_k \geq S_{\alpha,N} \nu_k^{\frac{2}{2^*}}$.

On the manifold $V$, we define the mapping $\beta : V \to \mathbb{R}^N$ by
\[ \beta(w) := \int_{\Omega} x(w^+(x,0))^{2^*} \, dx, \]
which has the following properties.

Lemma 3.2. Let $\{w_n\} \subset V$ be a sequence such that
\[ \|w_n\|_{H^1_{0,L}(\Omega)}^2 = \int_{\Omega} k_\alpha y^{1-\alpha}|\nabla w_n|^2 \, dx \, dy \to S_{\alpha,N} \]
as $n \to \infty$, then $\operatorname{dist}(\beta(w_n), \Omega) \to 0$, as $n \to \infty$.

Proof. Suppose by contradiction that $\operatorname{dist}(\beta(w_n), \Omega) \not\to 0$ as $n \to \infty$. We may verify that $\{w_n\}$ is tight. By Lemma 3.1 there exist sequences $\{\mu_k\}$ and $\{\nu_k\}$ such that
\[ S_{\alpha,N} = \lim_{n \to \infty} \|w_n\|^2 = k_\alpha \int_{\Omega} y^{1-\alpha}|\nabla w|^2 \, dx \, dy + \sum_{k \in I} \mu_k, \] (3.2)
\[ 1 = \lim_{n \to \infty} \int_{\Omega} \|u_n\|^2 = \int_{\Omega} |u|^{2^*} \, dx + \sum_{k \in I} \nu_k. \] (3.3)
By the Sobolev inequality and Lemma 3.1 from (3.2) we deduce that
\[ S_{\alpha,N} = \|w\|_{H^1_{0,L}(\Omega)}^2 + \sum_{k \in I} \mu_k \geq S_{\alpha,N} \|u\|_{L^2(\Omega)}^2 + S_{\alpha,N} (\sum_{k \in I} \nu_k)^\frac{2}{2^*}. \]
Hence,
\[ \|u\|_{L^2_{\alpha_k}(\Omega)}^2 + (\Sigma_{k \in I} w_k)^{-2} \leq 1. \]  \hspace{1cm} (3.4)

Equations (3.3) and (3.4) imply either \( \Sigma_{k \in I} w_k = 0 \) or \( \|u\|_{L^2_{\alpha_k}(\Omega)}^2 = 0 \).

If \( \Sigma_{k \in I} w_k = 0 \), that is \( \|u\|_{L^2_{\alpha_k}(\Omega)} = 1 \), the lower semi-continuity of norms yields
\[ S_{\alpha,N} \geq \|w\|_{H^1_{\alpha}(\partial \Omega)}^2 \geq \frac{k_{\alpha} \int_{\Omega} y^{1-\alpha} |\nabla w|^2 \, dx \, dy}{(\int_{\Omega} |u|^{2_\alpha} \, dx)^{\frac{2}{2_\alpha}}}. \]

While by the Sobolev trace inequality,
\[ S_{\alpha,N} \leq \frac{k_{\alpha} \int_{\Omega} y^{1-\alpha} |\nabla w|^2 \, dx \, dy}{(\int_{\Omega} |u|^{2_\alpha} \, dx)^{\frac{2}{2_\alpha}}}, \]
it then implies that \( S_{\alpha,N} \) is achieved, which is a contradiction to the fact that \( S_{\alpha,N} \) is not achieved unless \( C_\Omega = \mathbb{R}^{N+1}_+ \). Thus, \( \|u\|_{L^2_{\alpha_k}(\Omega)}^2 \neq 0 \). Consequently, \( \Sigma_{k \in I} w_k = 1 \) and \( u = 0 \). Furthermore, by the uniqueness of the extension of \( u \), we have \( u = 0 \). Now, it is standard to show that \( \nu \) is concentrated at a single \( x_0 \) of \( \bar{\Omega} \).

So we have
\[ \beta(w_n) \to \int_{\Omega} x \, d\nu(x) = x_0 \in \bar{\Omega}, \]
this is a contradiction. \( \Box \)

Since \( \Omega \) is a smooth bounded domain of \( \mathbb{R}^N \), we choose \( r > 0 \) small enough so that
\( \Omega^+ = \{ x \in \mathbb{R}^N : \text{dist}(x, \Omega) < r \} \) and \( \Omega^- = \{ x \in \Omega : \text{dist}(x, \partial \Omega) > r \} \)
are homotopically equivalent to \( \Omega \). Moreover we assume that the ball \( B_r(0) \subset \Omega \), and then \( C_{B_r(0)} := B_r(0) \times (0, + \infty) \subset \bar{C}_{\Omega} \). We define
\[ V_0 := \{ w \in H^1_{0,L}(C_{B_r(0)}) : \int_{C_{B_r(0)}} w_{2_\alpha}^* (x, 0) \, dx = 1 \} \subset V \]
as well as
\[ Q_0 = \inf_{w \in V_0} \varphi_{\lambda}(w). \]
Denote by \( \varphi_{Q_0} := \{ w \in V : \varphi_{\lambda}(w) < Q_0 \} \) the level set below \( Q_0 \). We may verify as in Lemma 3.2 that \( Q_0 < S_{\alpha,N} \).

**Lemma 3.3.** There exists a \( \lambda^* \), \( 0 < \lambda^* < \lambda_1 \) such that for \( 0 \leq \lambda < \lambda^* \), if \( w \in \varphi_{Q_0} \), then \( \beta(w) \in \Omega^+_\lambda \).

**Proof.** By Hölder’s inequality, for every \( w \in V \),
\[ \int_{\Omega} |w(x,0)|^2 \, dx \leq \left( \int_{\Omega} |w(x,0)|^{2_\alpha} \, dx \right)^{\frac{2}{2_\alpha}} |\Omega|^{\alpha/N} = |\Omega|^{\alpha/N}. \]
Let \( \lambda^* = \frac{\epsilon}{|\Omega|^{\alpha/N}} \). If \( 0 < \lambda < \lambda^* \) and \( w \in \varphi_{Q_0} \), we have
\[ \|w\|^2 \leq \lambda \int_{\Omega} |w(x,0)|^2 \, dx + Q_0 \leq \lambda^* |\Omega|^{\alpha/N} + S_{\alpha,N} = S_{\alpha,N} + \epsilon. \]
Therefore, we conclude by Lemma 3.2 that \( \beta(w) \in \Omega^+_\lambda \). \( \Box \)
Now, we establish the relation of category between the domain $\Omega$ and the level set $\varphi_{\lambda}^{Q_0}$.

**Lemma 3.4.** If $N \geq 2\alpha$ and $0 < \lambda < \lambda^*$, then we have $\text{cat}_{\varphi_{\lambda}^{Q_0}} \varphi_{\lambda}^{Q_0} \geq \text{cat}_{\Omega}(\Omega)$.

**Proof.** Let $w_0 \in H^1_0([0, \infty))$ be a minimizer of $Q_0$. Hence, we may assume that $w_0 > 0$ is cylinder symmetric and $\|w_0\|_{H^2_{\text{loc}}(B_{1}((0,0)))} = 1$,

\[
Q_0 = \int_{C_{B_{r}(0)}} k_{\alpha} y^{1-\alpha} |\nabla w_0|^2 \, dy - \lambda \int_{B_{r}(0)} |w_0(x,0)|^2 \, dx.
\]

For $z \in \Omega^\cap$, we define $\gamma : \Omega^\cap \to \varphi_{\lambda}^{Q_0}$ by

\[
\gamma(z) = \begin{cases} w_0(x - z, y), & (x, y) \in B_r(z) \times (0, +\infty), \\ 0, & (x, y) \notin B_r(z) \times (0, +\infty). \end{cases}
\]

Since $w_0(x, 0)$ is a radial function,

\[
\beta \circ \gamma(z) = \int_{B_{r}(z)} x(w_0)^{2_\varphi_{\lambda}^{Q_0}}(x - z, 0) \, dx = \int_{B_{r}(0)} x(w_0)^{2_\varphi_{\lambda}^{Q_0}}(x, 0) \, dx + z = z.
\]

Hence, $\beta \circ \gamma = \text{id}$.

Assume that $\varphi_{\lambda}^{Q_0} = A_1 \cup A_2 \cup \cdots \cup A_n$, where $A_j, j = 1, 2, \ldots, n$, is closed and contractible in $\varphi_{\lambda}^{Q_0}$, i.e. there exists $h_j \in C([0, 1] \times A_j, \varphi_{\lambda}^{Q_0})$ such that, for every $u, v \in A_j$,

\[
h_j(0, u) = u, \quad h_j(1, u) = h_j(1, v).
\]

Let $B_j := \gamma^{-1}(A_j), 1 \leq j \leq n$. The sets $B_j$ are closed and $\Omega^\cap = B_1 \cup B_2 \cdots \cup B_n$. By Lemma 3.3, we know $\beta(h_j(t, \gamma(x))) \in \Omega^\cap$. Using the deformation $g_j(t, x) = \beta(h_j(t, \gamma(x)))$, we see that $B_j$ is contractible in $\Omega^\cap$. Indeed, for every $x, y \in B_j$, there exist $\gamma(x), \gamma(y) \in A_j$ such that

\[
g_j(0, x) = \beta(h_j(0, \gamma(x))) = \beta(\gamma(x)) = x,
\]

\[
g_j(1, x) = \beta(h_j(1, \gamma(x))) = \beta(h_j(1, \gamma(y))) = g_j(1, y).
\]

It follows that $\text{cat}_{\varphi_{\lambda}^{Q_0}} \varphi_{\lambda}^{Q_0} \geq \text{cat}_{\Omega^\cap}(\Omega^\cap) = \text{cat}_{\Omega}(\Omega)$. □

**Lemma 3.5.** If $\varphi_{\lambda}|_{V}$ is bounded from below and satisfies the (PS)$_c$ condition for any

\[
c \in \left[ \inf_{w \in V} \varphi_{\lambda}, Q_0 \right],
\]

then $\varphi_{\lambda}|_{V}$ has a minimum and level set $\varphi_{\lambda}^{Q_0}$ contains at least $\text{cat}_{\varphi_{\lambda}^{Q_0}} \varphi_{\lambda}^{Q_0}$ critical points of $\varphi_{\lambda}|_{V}$.

The proof of the above lemma can be found in [14].

**Proof of Theorem 1.1.** By Lemma 3.5, for $0 < \lambda < \lambda^*$, the level set $\varphi_{\lambda}^{Q_0}$ contains at least $m := \text{cat}_{\varphi_{\lambda}^{Q_0}} \varphi_{\lambda}^{Q_0}$ critical points $w_1, w_2, \ldots, w_m$ of $\varphi_{\lambda}|_{V}$.

For $j = 1, 2, \ldots, m$, there exist $\mu_j \in \mathbb{R}$ such that, for $h \in H^1_0([0, \infty))$,

\[
k_{\alpha} \int_{C_{\Omega}} y^{1-\alpha} \nabla w_j \nabla h \, dy - \lambda \int_{\Omega} w h \, dx - \mu_j \int_{\Omega} (w_j^+)^{2\varphi_{\lambda}^{Q_0}-1} h \, dx = 0.
\]
Choosing $h = w_j – j$, we have

$$0 = k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w_j|^2 dx dy - \lambda \int_{\Omega} |w_j|^2 dx.$$ 

Since $0 < \lambda < \lambda_1$, it implies $w_j = 0$ and

$$k_\alpha \int_{\Omega} y^{1-\alpha} |\nabla w_j|^2 dx dy - \lambda \int_{\Omega} |w_j|^2 dx - \mu_j \int_{\Omega} (w_j^+)^{2^*_s} dx = 0.$$ 

Therefore, $\mu_j = \varphi_\lambda(w_j)$ and $v_j := \mu_j^{\frac{N}{2^*_s} - \alpha} w_j$ is a positive solution of (1.4), $tr_\Omega(v_j)$ is a solution of (1.1). By Lemma 3.4, problems (1.4) and (1.1) have at least $\text{cat}_\Omega(\Omega)$ positive solutions.

References


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