GLOBAL REGULARITY CRITERIA FOR THE n-DIMENSIONAL BOUSSINESQ EQUATIONS WITH FRACTIONAL DISSIPATION

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ABSTRACT. We consider the n-dimensional Boussinesq equations with fractional dissipation, and establish a regularity criterion in terms of the velocity gradient in Besov spaces with negative order.

1. Introduction

In this article, we study the n-dimensional Boussinesq equations with fractional dissipation,

\[ \begin{aligned}
\partial_t u + (u \cdot \nabla) u + \Lambda^{2\alpha} u + \nabla \Pi &= \vartheta \mathbf{e}_n, \\
\partial_t \vartheta + (u \cdot \nabla) \vartheta &= 0, \\
\nabla \cdot u &= 0,
\end{aligned} \tag{1.1} \]

where \( u : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) is the velocity field; \( \vartheta : \mathbb{R}^+ \times \mathbb{R}^n \rightarrow \mathbb{R} \) is a scalar function representing the temperature in the context of thermal convection (see [8]) and the density in the modeling of geophysical fluids (see [9]); \( \Pi \) is the fluid pressure; \( \mathbf{e}_n \) is the unit vector in the \( x_n \) direction; and \( \Lambda := (-\Delta)^{1/2}, \alpha \geq 0 \) is a real number.

When \( \alpha = 1 \), Equation (1.1) reduces to the classical Boussinesq equations, which are frequently used in the atmospheric sciences and oceanographic turbulence where rotation and stratification are important (see [8, 9]). If \( \vartheta = 0 \), then (1.1) becomes the generalized Navier-Stokes equation, which was first considered by Lions [7], where he showed the global regularity once \( \alpha \geq \frac{1}{2} + \frac{n}{4} \). One may refer the reader to [5, 10] for recent advances. Xiang-Yan [12], Yamazaki [13] and Ye [14] were able to extend Lions’s result to system (1.1), where there is no diffusion in the \( \vartheta \) equation. And it remains an open problem for the global-in-time smooth for (1.1) with \( 0 < \alpha < \frac{1}{2} + \frac{n}{4} \). The purpose of the present paper is to establish a blow-up criterion as follows.

**Theorem 1.1.** Let \( 0 < \alpha < \frac{1}{2} + \frac{n}{4} \), \((u_0, \vartheta_0) \in H^s(\mathbb{R}^n) \) with \( s > 1 + \frac{n}{2} \) and \( \nabla \cdot u_0 = 0 \). Assume that \((u, \vartheta)\) be the smooth local unique solution pair to (1.1) with initial data
If additionally,\[
\nabla u \in L_{2\alpha-\gamma}^{2\alpha-\gamma}(0,T; \dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n))
\]
for some $0 < \gamma < 2\alpha$, then the solution $(u, \vartheta)$ can be extended smoothly beyond $T$.

Here, $\dot{B}^{-\gamma}_{\infty,\infty}(\mathbb{R}^n)$ is the homogeneous Besov space with negative order, which contains classical Lebesgue space $L^{2\alpha}_\gamma(\mathbb{R}^n)$, see [1, Chapter 2]. In the proof of Theorem 1.1 in Section 2, we shall frequently use the following refined Gagliardo-Nirenberg inequality.

**Lemma 1.2** ([1, Theorem 2.42]). Let $2 < q < \infty$ and $\gamma$ be a positive real number. Then a constant $C$ exists such that\[
\|f\|_{L^q} \leq C \|f\|_{1-2q}^{1/2} \|f\|_{\dot{H}^\gamma(\frac{2}{q}-1)}^{2/q}.
\]

**Remark 1.3.** Our result extends that of Kozono-Shimada [6]. Indeed, the Navier-Stokes equations corresponds to (1.1) with $\vartheta = 0$ and $\alpha = 1$.

**Remark 1.4.** In [3] (see also the end-point smallness condition in [2]), Geng-Fan proved a regularity criterion\[
u \in L^{2r}(-\infty,\infty; \dot{B}^{-r}_{\infty,\infty}(\mathbb{R}^3)) \quad (-1 < r < 1, \ r \neq 0)
\]
for system (1.1) with $\alpha = 1$ and $n = 3$. Thus our result generalizes (1.4) also, in view of the fact that\[
\|f\|_{\dot{B}^{-1-\gamma}_{\infty,\infty}} \leq \|f\|_{\dot{B}^{-1-\gamma}_{\infty,\infty}} \leq \|f\|_{\dot{B}^{-1-\gamma}_{\infty,\infty}}.
\]
Moreover, our result (1.2) is valid for (1.1) with arbitrarily large $n$ and arbitrarily small $\alpha$.

Interested readers are referred to [11] for blow-up criterion for (1.1) without diffusion in the $u$ equation.

## 2. Proof of Theorem 1.1

It is not difficult to prove that there exists a $T_0 > 0$ and a unique smooth solution $(u, \vartheta)$ to (1.1) on $[0, T_0]$. We only need to establish the a priori estimates. Therefore, in the following calculations, we assume that the solution $(u, \vartheta)$ is sufficiently smooth on $[0, T]$.

First, taking the inner product of (1.1)$_1$ and (1.1)$_2$ with $u, \vartheta$ in $L^2(\mathbb{R}^n)$ respectively, we obtain\[
\frac{1}{2} \frac{d}{dt} \|u, \vartheta\|^2_{L^2} + \|A^\alpha u\|^2_{L^2} = \int_{\mathbb{R}^n} \vartheta e_n \cdot u \ dx \leq \frac{1}{2} \|u, \vartheta\|^2_{L^2}.
\]
Applying Gronwall inequality, we deduce\[
\|u, \vartheta\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|A^\alpha u\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C.
\]
For $k > 0$, applying $A^k$ to \([1.1]_1\), and testing the resulting equations by $A^k \mathbf{u}$ respectively, we obtain
\[
\frac{1}{2} \frac{d}{dt} \|A^k \mathbf{u}\|^2_{L^2} + \|A^{k+\alpha} \mathbf{u}\|^2_{L^2} = - \int_{\mathbb{R}^n} A^k \left((\mathbf{u} \cdot \nabla) \mathbf{u}\right) \cdot A^k \mathbf{u} \, dx + \int_{\mathbb{R}^n} A^k (\partial \mathbf{e}_n) \cdot A^k \mathbf{u} \, dx
\]
\[= - \int_{\mathbb{R}^3} \{A^k((\mathbf{u} \cdot \nabla) \mathbf{u}) - (\mathbf{u} \cdot \nabla)(A^k \mathbf{u})\} \cdot A^k \mathbf{u} \, dx + \int_{\mathbb{R}^n} A^k (\partial \mathbf{e}_n) \cdot A^k \mathbf{u} \, dx
\]
\[\equiv I_1^k + I_2^k.\]

We may use the following commutator estimates of Kato-Ponce \([4]\):
\[
\|A^k(fg) - f A^k g\|_{L^p} \leq C \left(\|f\|_{L^{p_1}} \|A^{k-1} g\|_{L^{p_2}} + \|A^k f\|_{L^{p_3}} \|g\|_{L^{p_4}}\right)
\]
with
\[1 < p, p_2, p_3 < \infty, \quad 1 \leq p_1, p_4 \leq \infty, \quad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p_3} + \frac{1}{p_4}\]
to bound $I_1^k$ as
\[
I_1^k \leq C \|A^k((\mathbf{u} \cdot \nabla) \mathbf{u}) - (\mathbf{u} \cdot \nabla)(A^k \mathbf{u})\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^{k+\alpha} \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^{k+\alpha} \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^{k+\alpha} \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^{k+\alpha} \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \|\nabla \mathbf{u}\|_{L^{\frac{2(k+\gamma+\alpha-1)}{\gamma}}} \|A^k \mathbf{u}\|_{L^{\frac{4(k+\gamma+\alpha-1)}{2(k+\gamma+\alpha-1)}}} \\
\leq C \|\nabla \mathbf{u}\|_{B^{\infty}_{\infty}} \|A^{k+\alpha} \mathbf{u}\|_{L^{2}} \|\nabla \mathbf{u}\|_{B^{\infty}_{\infty}} \|A^k \mathbf{u}\|_{L^{2}} \\
\leq C \|\nabla \mathbf{u}\|_{B^{\infty}_{\infty}} \|A^{k+\alpha} \mathbf{u}\|_{L^{2}} + \|\nabla \mathbf{u}\|_{B^{\infty}_{\infty}} \|A^k \mathbf{u}\|_{L^{2}}^2.\]

Substituting (2.4) in (2.2), we find
\[
\frac{d}{dt} \|A^k \mathbf{u}\|^2_{L^2} + \|A^{k+\alpha} \mathbf{u}\|^2_{L^2} \leq C \|\nabla \mathbf{u}\|_{B^{\infty}_{\infty}} \|A^k \mathbf{u}\|_{L^2}^2 + 2I_2^k.\]

Now, we treat $2I_2^k$ step by step. If $0 < k \leq \alpha$, then
\[
2I_2^k = 2 \int_{\mathbb{R}^n} \partial \mathbf{e}_n \cdot A^{2k} \mathbf{u} \, dx
\]
\[\leq 2 \|\partial \mathbf{e}_n\|_{L^2} \|A^{2k} \mathbf{u}\|_{L^2} \leq C \|\partial \mathbf{e}_n\|_{L^2} \left(\|A^{k+\alpha} \mathbf{u}\|_{L^2} \right) \left(H^{k+\alpha}(\mathbb{R}^n) \subset \dot{H}^{2k}(\mathbb{R}^n)\right) \leq C + \frac{1}{2} \|A^{k+\alpha} \mathbf{u}\|_{L^2}^2 \] (by (2.1)).

Substituting (2.6) into (2.5), we apply Gronwall inequality to deduce
\[
\|A^k(\mathbf{u}, \partial)\|_{L^\infty(0,T;L^2(\mathbb{R}^n))} + \|A^{k+\alpha} \mathbf{u}\|_{L^2(0,T;L^2(\mathbb{R}^n))} \leq C \quad (0 < k \leq \alpha).\]
Suppose we have already the statement for some \( 0 \leq l \in \mathbb{N}, \)
\[ \|A^l(u, \vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} + \|A^{l+\alpha}u\|_{L^2(0,t;L^2(\mathbb{R}^n))} \leq C \quad (\forall \ l \alpha < k \leq (l+1)\alpha), \quad (2.8) \]
we wish to deduce higher-order estimate
\[ \|A^{k+\alpha}(u, \vartheta)\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} + \|A^{k+2\alpha}u\|_{L^2(0,t;L^2(\mathbb{R}^n))} \leq C. \quad (2.9) \]
Indeed, as long as (2.8) holds, we may dominate \( 2I_2^{k+\alpha} \) as
\[ 2I_2^{k+\alpha} = 2 \int_{\mathbb{R}^n} A^{k+\alpha}(\vartheta e_n) \cdot A^{k+\alpha}u \, dx \]
\[ = 2 \int_{\mathbb{R}^n} A^k(\vartheta e_n) \cdot A^{k+2\alpha}u \, dx \quad (2.10) \]
\[ \leq 2\|A^k\|_{L^2} \|A^{k+2\alpha}u\|_{L^2} \]
\[ \leq 2\|A^k\|_{L^2}^{2} + \frac{1}{2}\|A^{k+2\alpha}u\|_{L^2}^{2}. \]
Putting (2.10) into (2.5) with \( k \) replaced by \( k + \alpha \), and using (2.8), we deduce (2.9) as desired.

Now prove that (2.7) and (2.8) imply (2.9), we see readily that
\[ \|A^\alpha u\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} + \|A^{1+\alpha}u\|_{L^2(0,t;L^2(\mathbb{R}^n))} \leq C. \quad (2.11) \]
With this good estimate of the velocity field, we are now in a position to treat that of \( \vartheta \). Applying \( A^\alpha \) to \((1.1)_2\), and testing the resultant equation by \( A^\alpha \vartheta \), we obtain
\[ \frac{1}{2} \frac{d}{dt} \|A^\alpha \vartheta\|_{L^2}^{2} \]
\[ = - \int_{\mathbb{R}^n} A^\alpha[(u \cdot \nabla)\vartheta] \cdot A^\alpha \vartheta \, dx \]
\[ = - \int_{\mathbb{R}^n} \{A^\alpha[(u \cdot \nabla)\vartheta] - (u \cdot \nabla)A^\alpha \vartheta\} \cdot A^\alpha \vartheta \, dx \quad (2.12) \]
\[ \leq C \left( \|\nabla u\|_{L^\infty} + \|A^\alpha \vartheta\|_{L^2} + \|\nabla \vartheta\|_{L^\infty} + \|A^\alpha u\|_{L^2} \right) \|A^\alpha \vartheta\|_{L^2} \quad \text{(by (2.3))} \]
\[ \leq C \left( \|u\|_{L^2} + \|A^\alpha u\|_{L^2} \right) \|A^\alpha \vartheta\|_{L^2}^{2} + \left( \|\vartheta\|_{L^2} + \|A^\alpha \vartheta\|_{L^2} \right) \|A^\alpha u\|_{L^2} \|A^\alpha \vartheta\|_{L^2} \]
\[ \text{(by } H^\alpha(\mathbb{R}^n) \subset W^{1,\infty}(\mathbb{R}^n)\text{)} \]
\[ \leq C + C\|A^\alpha \vartheta\|_{L^2}^{2} \quad \text{(by (2.1) and (2.11)).} \]
Applying Gronwall inequality, we obtain
\[ \|A^\alpha \vartheta\|_{L^\infty(0,t;L^2(\mathbb{R}^n))} \leq C. \]
With this estimate and (2.11), we complete the proof.

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