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EXACT BOUNDARY CONTROLLABILITY FOR A SERIES OF MEMBRANES ELASTICALLY CONNECTED

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Abstract. In this article we study the exact controllability with Neumann boundary controls for a system of linear wave equations coupled in parallel by lower order terms on piecewise smooth domains of the plane. We obtain square integrable controls for initial state with finite energy and time of controllability near the optimal value.

1. Introduction

In this article we study a system of $m \geq 2$ coupled wave equations

$$\begin{gathered} U_{tt} - \Delta U + AU = 0 \quad \text{in } \Omega\times]0,T[, \\ U(\cdot,0) = U_0 \quad \text{in } \Omega, \\ U_t(\cdot,0) = U_1 \quad \text{in } \Omega, \\ \frac{\partial U}{\partial \eta} = f \quad \text{on } \partial\Omega\times]0,T[, \end{gathered} \tag{1.1}$$

where Ω is a bounded simply connected domain of the plane with piecewise smooth boundary $\partial\Omega$. The exterior unit normal vector, defined almost everywhere in $\partial\Omega$ is denoted by η and by $\frac{\partial U}{\partial \eta}$ we denote the normal derivative of $U = (u^1, \dots, u^m)^T$ where ^{T} stands for transpose. As usual we write $U_{tt} = (u^1_{tt}, \dots, u^m_{tt})^T$, $\Delta U = (\Delta u^1, \dots, \Delta u^m)^T$ where the subscript $_{tt}$ denotes the second derivative with respect to t and Δ the Laplacian with respect to space variables. $A = [a_{ij}]_{m\times m}$ is assumed to be a real diagonalizable matrix with nonnegative eigenvalues.

The system (1.1) includes those used to model physical phenomena in which several elastic bodies are attached together by elastic layers. For instance, following [9, 17] we see that a system composed by m identical membranes connected in parallel by elastic layers having the transverse displacement given by $U = (u^1, \dots, u^m)^T$ must satisfy the relations

$$\begin{gathered} u^i_{tt} - \Delta u^i + k_{i-1}(u^i - u^{i-1}) + k_i(u^i - u^{i+1}) = 0 \quad i = 1, \dots, m, \\ u^0 \equiv u^{m+1} \equiv 0, \quad k_i \text{ real constants } i = 0, 1, \dots, m. \end{gathered} \tag{1.2}$$

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In this case the coupling matrix is the tridiagonal symmetric matrix

$$A = \begin{bmatrix} k_0 + k_1 & -k_1 & & & & \\ -k_1 & k_1 + k_2 & -k_2 & & & \\ & -k_2 & k_2 + k_3 & -k_3 & & \\ & & -k_3 & \ddots & & \\ & & & & k_{m-2} + k_{m-1} & -k_{m-1} \\ & & & & -k_{m-1} & k_{m-1} + k_m \end{bmatrix}.$$

In this work we are interested in studying exact boundary controllability for system (1.1). Before announcing the results of this work we establish some notation. For a general bounded domain $\Xi \subset \mathbb{R}^N$ where N is a positive integer, we denote $(L^2(\Xi), \|\cdot\|_{L^2(\Xi)})$ and $(H^1(\Xi), \|\cdot\|_{H^1(\Xi)})$ the usual Lebesgue space of square integrable functions on Ξ and the Sobolev space of functions of $L^2(\Xi)$ whose first order distributional derivatives are also in $L^2(\Xi)$. These spaces are endowed with their usual norms [7]. We also introduce the product space $\mathcal{H}^1(\Xi) = H^1(\Xi) \times \cdots \times H^1(\Xi) = [H^1(\Xi)]^m$ with norm given by

$$\|U\|_{\mathcal{H}^1(\Xi)} = \big[\|u^1\|^2_{H^1(\Xi)} + \cdots + \|u^m\|^2_{H^1(\Xi)}\big]^{1/2}, \quad U = (u^1, \dots, u^m)^T \in \mathcal{H}^1(\Xi)$$

and the space $\mathcal{L}^2(\Xi) = L^2(\Xi) \times \cdots \times L^2(\Xi) = [L^2(\Xi)]^m$ with norm

$$\|U\|_{\mathcal{L}^2(\Xi)} = \big[\|u^1\|^2_{L^2(\Xi)} + \cdots + \|u^m\|^2_{L^2(\Xi)}\big]^{1/2}, \quad U = (u^1, \dots, u^m)^T \in \mathcal{L}^2(\Xi).$$

The product space $\mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi)$ is endowed with the natural norm

$$\|(U, V)\|^2_{\mathcal{H}^1(\Xi)\times\mathcal{L}^2(\Xi)} = \|U\|^2_{\mathcal{H}^1(\Xi)} + \|V\|^2_{\mathcal{L}^2(\Xi)}.$$

The spaces $\mathcal{H}^1_{\rm loc}(\Xi)$, $\mathcal{L}^2_{\rm loc}(\Xi)$ and $\mathcal{H}^1_0(\Xi)$ have their usual meaning. The theorem bellow is the main result of this article.

Theorem 1.1. *Let $A = [a_{ij}]_{m\times m}$ be a real diagonalizable matrix with nonnegative eigenvalues and Ω be a bounded simply connected domain of the plane with piecewise smooth boundary $\partial\Omega$. It is assumed that $\partial\Omega$ has no cuspid point and that Ω lays in one side of $\partial\Omega$. Then, for any $T_* > \operatorname{diam}(\Omega)$ and any initial state $U_0 \in \mathcal{H}^1(\Omega)$, $U_1 \in \mathcal{L}^2(\Omega)$ there exist a control $f \in \mathcal{L}^2(\partial\Omega\times]0, T_*[)$ so that the solution $U \in \mathcal{H}^1(\Omega\times]0, T_*[)$ of (1.1) satisfies the final condition $U(\cdot, T_*) = U_t(\cdot, T_*) = 0$ on Ω.*

This theorem extends to systems of $m > 2$ equations and time of control near optimal value the results presented in [6] for a system with two equations. We employ the extension method introduced by Russell [18] as improved by Lagnese [12] for the wave equation with smooth initial state. We explore properties of the solutions of the Cauchy problem for $U_{tt} - \Delta U + AU = 0$ such as local energy decay, regularity on noncharacteristic surfaces and time analyticity. Following the evolution of the method to be employed and also the didactic aspect we firstly prove controllability for a large enough time interval following Russell's original procedure [18]. After, following the lines of [16] we obtain time analyticity for the solution operator associated to the Cauchy problem for $U_{tt} - \Delta U + AU = 0$ and proceed as in [12] to prove Theorem 1.1.

The literature on control and stabilization for coupled systems has increased enormously in the last two decades. Just to mention, energy decay and boundary stabilization for the system (1.2) with $m = 2$, $k_0 = k_2 = 0$ has been treated by several authors. Indeed, in [1, 2, 11] the authors estimate the energy decay rate inside a bounded smooth domain $\Omega \subset \mathbb{R}^N$ under nonlinear boundary feedbacks. Similar questions for two waves with different speed has been studied in [14] for smooth domains of $\mathbb{R}^N$, $(N \leq 3)$. Controllability for a system like (1.1) with $m = 2$ and Dirichlet type control has been studied in [3] by employing the Two Level Energy Method. Concerning controllability results for coupled parabolic problems we mention the survey [10]. A variety of problems for systems highly coupled of partial differential equations, possibly of different type, has been considered in the treatise [13].

This article is organized as follows. In Section 2 we obtain a local energy decay for solutions to the Cauchy problem for the system $U_{tt} - \Delta U + AU = 0$ and obtain controllability in large enough time interval. Section 3 is dedicated to prove time analyticity of solutions to the system $U_{tt} - \Delta U + AU = 0$. Finally, in Section 4 we prove Theorem 1.1 and illustrate how the method works for the case in which the control acts only in a part of the membrane's boundary.

2. Energy decay and controllability in large time

In this section we prove the local energy decay of solutions to the Cauchy problem

$$\begin{gathered} U_{tt} - \Delta U + AU = 0 \quad \text{in } \mathbb{R}^{2+1}, \\ U(\cdot, 0) = U_0, \quad U_t(\cdot, 0) = U_1 \quad \text{in } \mathbb{R}^2, \end{gathered} \tag{2.1}$$

and use it to obtain controllability for the system (1.1) in an interval of time sufficiently large. Let $0 \leq \lambda_1 \leq \cdots \leq \lambda_m$ be non negative eigenvalues of the real diagonalizable matrix $A = [a_{ij}]_{m\times m}$.

Theorem 2.1. *Let $\Xi \subset \mathbb{R}^2$, be a bounded domain. There exists constants $T_0 >$ $\operatorname{diam}(\Xi)$ and $K = K(\Xi, A, T_0) > 0$ such that the solution $U \in \mathcal{H}^1_{\rm loc}(\mathbb{R}^{2+1})$ of the Cauchy problem* (2.1) *satisfies*

$$\|U(\cdot,t)\|^2_{\mathcal{H}^1(\Xi)} + \|U_t(\cdot,t)\|^2_{\mathcal{L}^2(\Xi)} \leq \frac{K}{t^2}\big\{\|U_0\|^2_{\mathcal{H}^1(\Xi)} + \|U_1\|^2_{\mathcal{L}^2(\Xi)}\big\} \tag{2.2}$$

for every $t > T_0$ whenever the initial data $(U_0, U_1) \in \mathcal{H}^1(\mathbb{R}^2) \times \mathcal{L}^2(\mathbb{R}^2)$ has compact support in Ξ.

Proof. First observe that the transposed matrix A^T is also diagonalizable and has the same eigenvalues of A. Let $\{v^i = (\alpha^i_1, \dots, \alpha^i_m) : i = 1, \dots, m\}$ be a basis for $\mathbb{R}^m$ whose elements are eigenvectors of A^T with v^i associated to λ_i. We have for each $i = 1, \dots, m$ the set of relations

$$a_{1j}\alpha^i_1 + a_{2j}\alpha^i_2 + \cdots + a_{mj}\alpha^i_m = \lambda_i\alpha^i_j, \quad j = 1, \dots, m. \tag{2.3}$$

Let B be the real matrix $[\alpha^i_j]_{m\times m}$ and let $W = (w^1, \dots, w^m)^T$ be given by $W = BU$ where $U = (u^1, \dots, u^m)^T$ is the solution of the Cauchy problem (2.1). Hence

$$w^i = \alpha^i_1 u^1 + \alpha^i_2 u^2 + \cdots + \alpha^i_m u^m, \quad i = 1, \dots, m. \tag{2.4}$$

For each $i = 1, \dots, m$, an easy computation leads us to

$$w^i_{tt} = \Delta(\alpha^i_1 u^1 + \alpha^i_2 u^2 + \cdots + \alpha^i_m u^m) - \sum_{j=1}^m (a_{1j}\alpha^i_1 + a_{2j}\alpha^i_2 + \cdots + a_{mj}\alpha^i_m)u^j.$$

Now, by using the relations (2.3) and (2.4) we obtain

$$w^i_{tt} = \Delta w^i - \lambda_i(\alpha^i_1 u^1 + \alpha^i_2 u^2 + \cdots + \alpha^i_m u^m) = \Delta w^i - \lambda_i w^i, \quad i = 1, \dots, m.$$

Putting $U_0 = (u_0^1, u_0^2, \dots, u_0^m)^T$, $U_1 = (u_1^1, u_1^2, \dots, u_1^m)^T$ and noting that W must satisfy the initial conditions $W(\cdot, 0) = BU_0$ and $W_t(\cdot, 0) = BU_1$ we see that each entry w^i of the new independent variable W satisfies the Cauchy problem

$$\begin{aligned} w_{tt}^i - \Delta w^i + \mu_i^2 w^i = 0 \quad &\text{in } \mathbb{R}^{2+1}, \\ w^i(\cdot, 0) = \alpha_1^i u_0^1 + \alpha_2^i u_0^2 + \dots + \alpha_m^i u_0^m \quad &\text{in } \mathbb{R}^2, \\ w_t^i(\cdot, 0) = \alpha_1^i u_1^1 + \alpha_2^i u_1^2 + \dots + \alpha_m^i u_1^m \quad &\text{in } \mathbb{R}^2, \end{aligned} \tag{2.5}$$

where $\mu_i^2 = \lambda_i$ for $i = 1, \dots, m$.

Since, $w^i(\cdot, 0)$ and $w_t^i(\cdot, 0)$ have compact support in Ξ, using [6, Lemma 2.1] we obtain the estimate

$$\|w^i(\cdot, t)\|_{H^1(\Xi)}^2 + \|w_t^i(\cdot, t)\|_{L^2(\Xi)}^2 \le \frac{K_i}{t^2} \big\{ \|w^i(\cdot, 0)\|_{H^1(\Xi)}^2 + \|w_t^i(\cdot, 0)\|_{L^2(\Xi)}^2 \big\}$$

for every $t > T_0^i$, for some $T_0^i > \operatorname{diam}(\Xi)$ and some constant $K_i = K_i(\Xi, T_0^i, \lambda_i) > 0$. The constant T_0^i depends on λ_i and $\operatorname{diam}(\Xi)$. From the above inequality it follows

$$\begin{aligned} &\|w^i(\cdot, t)\|_{H^1(\Xi)}^2 + \|w_t^i(\cdot, t)\|_{L^2(\Xi)}^2 \\ &\le \frac{K_i}{t^2} (|\alpha_1^i|^2 + |\alpha_2^i|^2 + \dots + |\alpha_m^i|^2) \big\{ \|U_0\|_{\mathcal{H}^1(\Xi)}^2 + \|U_1\|_{\mathcal{L}^2(\Xi)}^2 \big\} \end{aligned} \tag{2.6}$$

for every $t > T_0^i$, $i = 1, \dots, m$.

Now we set $\|B\|^2 = \sum |\alpha_j^i|^2$, $M = \max\{K_i; i = 1, \dots, m\}$ and $T_0 = \max\{T_0^i; i = 1, \dots, m\}$, and by summing up inequalities (2.6) we obtain

$$\|W(\cdot, t)\|_{\mathcal{H}^1(\Xi)}^2 + \|W_t(\cdot, t)\|_{\mathcal{L}^2(\Xi)}^2 \le \frac{M}{t^2} \|B\|^2 \big\{ \|U_0\|_{\mathcal{H}^1(\Xi)}^2 + \|U_1\|_{\mathcal{L}^2(\Xi)}^2 \big\} \tag{2.7}$$

for every $t > T_0$.

By setting $B^{-1} = [\beta_j^i]_{m \times m}$ and observing that $u^i(\cdot, t) = \beta_1^i w^1(\cdot, t) + \dots + \beta_m^i w^m(\cdot, t)$, $i = 1, \dots, m$, we obtain

$$\begin{aligned} &\|U(\cdot, t)\|_{\mathcal{H}^1(\Xi)}^2 + \|U_t(\cdot, t)\|_{\mathcal{L}^2(\Xi)}^2 \\ &\le \|B^{-1}\|^2 \big\{ \|W(\cdot, t)\|_{\mathcal{H}^1(\Xi)}^2 + \|W_t(\cdot, t)\|_{\mathcal{L}^2(\Xi)}^2 \big\}. \end{aligned} \tag{2.8}$$

Now we put $K = M\|B\|^2 \|B^{-1}\|^2$ and combine (2.7) and (2.8) to obtain (2.2). $\square$

We can now prove the first result on exact controllability for the system (1.1).

Theorem 2.2. *Let $A = [a_{ij}]_{m \times m}$ be a real diagonalizable matrix with nonnegative eigenvalues and $\Omega \subset \mathbb{R}^2$, be a bounded simply connected domain with piecewise smooth boundary $\partial\Omega$. It is assumed that $\partial\Omega$ has no cuspid point and that Ω lays in one side of $\partial\Omega$. Then, there exists a large enough $T > 0$ such that for each initial state $U_0 \in \mathcal{H}^1(\Omega)$, $U_1 \in \mathcal{L}^2(\Omega)$ there exist a control $f \in \mathcal{L}^2(\partial\Omega \times]0, T[)$ so that the solution $U \in \mathcal{H}^1(\Omega \times]0, T[)$ of the system (1.1) satisfies the final condition $U(\cdot, T) = U_t(\cdot, T) = 0$ on Ω.*

Proof. For a fixed $\delta > 0$ we set $\Omega_\delta = \{x + y : x \in \Omega, |y| < \delta\}$. Associated with the Cauchy problem (2.1) we define, for $t > 0$, the bounded linear operator $\mathbf{S}_t : \mathcal{H}_0^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta) \to \mathcal{H}^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$ by setting

$$\mathbf{S}_t(V_0, V_1)(x) = (V(x, t), V_t(x, t)), \quad x \in \Omega_\delta \tag{2.9}$$

where V is the solution to the Cauchy problem (2.1) with initial data $(V_0, V_1) \in \mathcal{H}_0^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$ extended by zero outside Ω_δ.

Now, let T_0 be given by Theorem 2.1 with $\Xi = \Omega_\delta$. It follows from the Theorem 2.1 the estimate

$$\|\mathbf{S}_T(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)} \le \frac{K}{T^2}\big\{\|V_0\|^2_{\mathcal{H}^1(\Omega_\delta)} + \|V_1\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}, \tag{2.10}$$

for every $T > T_0$, $(V_0,V_1) \in \mathcal{H}^1_0(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)$ and a constant K independent of (V_0,V_1).

For arbitrary $T > 0$ consider the Cauchy problem

$$\begin{gathered} Z_{tt} - \Delta Z + AZ = 0 \quad \text{in } \mathbb{R}^{2+1}, \\ Z(\cdot,T) = Z_0, \quad Z_t(\cdot,T) = Z_1 \quad \text{in } \mathbb{R}^2, \end{gathered} \tag{2.11}$$

with initial state $(Z_0,Z_1) \in \mathcal{H}^1_0(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)$ extended by zero outside Ω_δ, at initial time T. Associated to the Cauchy problem (2.11) we define the bounded linear operator $\widehat{\mathbf{S}}_T : \mathcal{H}^1_0(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta) \to \mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)$ by setting

$$\widehat{\mathbf{S}}_T(Z_0,Z_1)(x) = (Z(x,0), Z_t(x,0)), \quad x \in \Omega_\delta. \tag{2.12}$$

Note that the function $Z(\cdot, T-\tau)$ satisfies $Z_{\tau\tau} - \Delta Z + AZ = 0$ with data (Z_0,Z_1) at time $\tau = 0$. Now, assuming that $T > T_0$ and applying the Theorem 2.1 to the solution of (2.11) we obtain

$$\|Z(\cdot,0)\|^2_{\mathcal{H}^1(\Omega_\delta)} + \|Z_\tau(\cdot,0)\|^2_{\mathcal{L}^2(\Omega_\delta)} \le \frac{K}{T^2}\big\{\|Z_0\|^2_{\mathcal{H}^1(\Omega_\delta)} + \|Z_1\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}.$$

Hence,

$$\|\widehat{\mathbf{S}}_T(Z_0,Z_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)} \le \frac{K}{T^2}\big\{\|Z_0\|^2_{\mathcal{H}^1(\Omega_\delta)} + \|Z_1\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}, \tag{2.13}$$

for every $T > T_0$ and $(Z_0,Z_1) \in \mathcal{H}^1_0(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)$.

Let E be an extension operator taking an arbitrary initial state $(V_0,V_1) \in \mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$ into its extension $(\widetilde{V}_0,\widetilde{V}_1) \in \mathcal{H}^1(\mathbb{R}^2)\times\mathcal{L}^2(\mathbb{R}^2)$ with compact support in Ω_δ. Let $\widetilde{V} \in \mathcal{H}^1_{\rm loc}(\mathbb{R}^{2+1})$ be the solution of (2.1) with initial state $(\widetilde{V}_0,\widetilde{V}_1)$.

Let $\varphi \in C_0^\infty(\mathbb{R}^2)$ be a cut off function such that $\varphi = 1$ on $\Omega_{\delta/2}$ and $\varphi = 0$ on $\mathbb{R}^2\backslash\Omega_\delta$. Let $\widetilde{Z} \in \mathcal{H}^1_{\rm loc}(\mathbb{R}^{2+1})$ be the solution of the problem (2.11) with $(Z_0,Z_1) = \varphi(\widetilde{V}(\cdot,T), \widetilde{V}_t(\cdot,T))$. Notice that the state $(\widetilde{Z}(\cdot,0), \widetilde{Z}_t(\cdot,0))$ is expressed in terms of the operators $\mathbf{S}_T$, $\widehat{\mathbf{S}}_T$ and the extension E as

$$(\widetilde{Z}(\cdot,0), \widetilde{Z}_t(\cdot,0)) = (\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E)(V_0,V_1)$$

for every $T > T_0$ and arbitrary initial data $(V_0,V_1) \in \mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$. Here φ stands for the operator multiplication by φ.

Now let us define the function $U^\circ = \widetilde{V} - \widetilde{Z}$. Observe that U° satisfies

$$U^\circ_{tt} - \Delta U^\circ + AU^\circ = 0 \quad \text{in } \mathbb{R}^{2+1}. \tag{2.14}$$

Since $\widetilde{Z}(\cdot,T) = \varphi\widetilde{V}(\cdot,T)$ and $\widetilde{Z}_t(\cdot,T) = \varphi\widetilde{V}_t(\cdot,T)$ and $\varphi = 1$ on $\Omega_{\delta/2}$ it follows that $U^\circ(\cdot,T) = U^\circ_t(\cdot,T) = 0$ on Ω. Consider $U_0 \in \mathcal{H}^1(\Omega)$, $U_1 \in \mathcal{L}^2(\Omega)$ given in the statement of the Theorem 2.2. To have $U^\circ(\cdot,0) = U_0$ and $U^\circ_t(\cdot,0) = U_1$ in Ω we might solve for the variable $(V_0,V_1) \in \mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$ the equation $(V_0,V_1) - (\widetilde{Z}(\cdot,0), \widetilde{Z}_t(\cdot,0)) = (U_0,U_1)$ in Ω. More precisely we must solve the equation

$$(V_0,V_1) - (R\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E)(V_0,V_1) = (U_0,U_1) \tag{2.15}$$

where R denotes the operator restriction to Ω. Denoting $\mathbf{K}_T = R\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E$, we prove that $Id - \mathbf{K}_T$ is invertible. We use the energy decay to prove that $\mathbf{K}_T$ is a contraction in $\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$ if the parameter T is taken sufficiently large. Indeed, by using (2.13) we have

$$\begin{aligned}
\|R\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)} &= \|(\widetilde{Z}(\cdot,0),\widetilde{Z}_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)}\\
&\le \|(\widetilde{Z}(\cdot,0),\widetilde{Z}_t(\cdot,0))\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)}\\
&= \|\widehat{\mathbf{S}}_T\varphi(\widetilde{V}(\cdot,T),\widetilde{V}_t(\cdot,T))\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)}\\
&\le \frac{K}{T^2}\big\{\|\varphi\widetilde{V}(\cdot,T)\|^2_{\mathcal{H}^1(\Omega_\delta)}+\|\varphi\widetilde{V}_t(\cdot,T)\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}\\
&\le \frac{\widetilde{K}}{T^2}\big\{\|\widetilde{V}(\cdot,T)\|^2_{\mathcal{H}^1(\Omega_\delta)}+\|\widetilde{V}_t(\cdot,T)\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}\\
&= \frac{\widetilde{K}}{T^2}\|\mathbf{S}_T E(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)}
\end{aligned}$$

where $\widetilde{K}$ depends on K and φ. Now, using (2.10) we obtain

$$\begin{aligned}
\frac{\widetilde{K}}{T^2}\big\{\|\widetilde{V}(\cdot,T)\|^2_{\mathcal{H}^1(\Omega_\delta)}+\|\widetilde{V}_t(\cdot,T)\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\} &\le \frac{\widetilde{K}}{T^2}\frac{K}{T^2}\big\{\|\widetilde{V}_0\|^2_{\mathcal{H}^1(\Omega_\delta)}+\|\widetilde{V}_1\|^2_{\mathcal{L}^2(\Omega_\delta)}\big\}\\
&= \frac{\widetilde{K}K}{T^4}\|E(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)}\\
&\le \frac{\text{Const.}}{T^4}\|(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)}
\end{aligned}$$

where Const. represents a convenient constant. Hence

$$\|\mathbf{K}_T(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)} \le \frac{\text{Const.}}{T^4}\|(V_0,V_1)\|^2_{\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)},$$

for all $(V_0,V_1)\in\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$ and $T>T_0$.

At this point we choose and fix a value for $T>T_0$ such that $\frac{\text{Const.}}{T^4}<1$. For such T the operator $\mathbf{K}_T$ is a contraction on $\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$. Let $(\mathcal{V}_0,\mathcal{V}_1)\in\mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega)$ be the unique solution of (2.15). From the construction we see that $E(\mathcal{V}_0,\mathcal{V}_1)-(\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E)(\mathcal{V}_0,\mathcal{V}_1)$ is an extension of (U_0,U_1) to the entire space $\mathbb{R}^2$. Let us define $(U_0^\circ,U_1^\circ)=: E(\mathcal{V}_0,\mathcal{V}_1)-(\widehat{\mathbf{S}}_T\varphi\mathbf{S}_T E)(\mathcal{V}_0,\mathcal{V}_1)$. Since all the Cauchy problems involved in the construction of (U_0°,U_1°) have the property of finite velocity of propagation and all the initial states considered have compact support we conclude that (U_0°,U_1°) also have compact support. Once we have the appropriate extension to the initial data we return to the beginning of the proof and start solving the problem (2.1) with initial data (U_0°,U_1°), then we localize its state at the time T to obtain the correct initial state to the backward problem (2.11). We then built the solution U° to the system (2.14) satisfying

$$U^\circ(\cdot,0)=U_0,\quad U_t^\circ(\cdot,0)=U_1,\quad U^\circ(\cdot,T)=0,\quad U_t^\circ(\cdot,T)=0\quad\text{on }\Omega. \tag{2.16}$$

Observe that $U_{tt}^\circ-\Delta U^\circ\in\mathcal{L}^2_{\rm loc}(\mathbb{R}^2\times\mathbb{R})$. From [19, Theorem 2] it follows that trace of the normal derivative of U° along the surface $\partial\Omega\times]0,T[$ is well defined and is locally square integrable. Hence $\frac{\partial U^\circ}{\partial\eta}\in\mathcal{L}^2(\partial\Omega\times]0,T[)$. To finish the proof we define $U=: U^\circ|_{\Omega\times]0,T[}$; the restriction of U to the domain $\Omega\times]0,T[$ and $f=:\frac{\partial U^\circ}{\partial\eta}$ and observe that they all meet the conditions of the theorem. $\square$

3. Time analyticity of solutions to the system $U_{tt} - \Delta U + AU = 0$

Let $\Xi \subset \mathbb{R}^2$ be a bounded domain, $\mu > 0$ and v_0, $v_1 \in C_0^\infty(\mathbb{R}^2)$ be functions such that $\operatorname{supp} v_0$, $\operatorname{supp} v_1 \subset \Xi$. If v is the solution of the Cauchy problem

$$\begin{gathered} v_{tt} - \Delta v + \mu^2 v = 0 \quad \text{in } \mathbb{R}^{2+1}, \\ v(\cdot,0) = v_0, \quad v_t(\cdot,0) = v_1 \quad \text{in } \mathbb{R}^{2+1}, \end{gathered} \tag{3.1}$$

then v is given, for every $t > 0$, by the formula

$$\begin{aligned} v(x,t) = {}& \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{|y-x|<t} \frac{\cos(\mu\sqrt{t^2-|y-x|^2})}{\sqrt{t^2-|y-x|^2}} v_0(y)dy \\ &+ \frac{1}{2\pi}\int_{|y-x|<t} \frac{\cos(\mu\sqrt{t^2-|y-x|^2})}{\sqrt{t^2-|y-x|^2}} v_1(y)dy. \end{aligned} \tag{3.2}$$

If the initial data are such that $v_0 \in H^1(\mathbb{R}^2)$, $v_1 \in L^2(\mathbb{R}^2)$ and $\operatorname{supp} v_0, \operatorname{supp} v_1 \subset \Xi$, the solution $v \in H^1_{\rm loc}(\mathbb{R}^{2+1})$ to the Cauchy problem (3.1) is obtained as the limit of a sequence of smooth solutions and hence it is also represented (almost everywhere) by (3.2).

Now we fix $T_0 > \operatorname{diam}(\Xi)$ and observe that for $t \geq T_0$ we have $\Xi \subset \{y : |y-x| < t\}$ for every $x \in \overline{\Xi}$. Since the initial data in (3.1) is assumed to have compact support in Ξ it follows that, for $t \geq T_0$, the domain of integration in (3.2) can be changed by Ξ. Or else

$$\begin{aligned} v(x,t) = {}& \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{\Xi} \frac{\cos(\mu\sqrt{t^2-|y-x|^2})}{\sqrt{t^2-|y-x|^2}} v_0(y)dy \\ &+ \frac{1}{2\pi}\int_{\Xi} \frac{\cos(\mu\sqrt{t^2-|y-x|^2})}{\sqrt{t^2-|y-x|^2}} v_1(y)dy, \end{aligned} \tag{3.3}$$

for $x \in \overline{\Xi}$ and $t \geq T_0$. In [6], by using (3.3) it was proved the energy decay estimate (2.2), (for space dimension > 2, see [15]).

For each $t > 0$ we define the bounded linear operator $S_t : H_0^1(\Xi) \times L^2(\Xi) \to H^1(\Xi) \times L^2(\Xi)$ by setting

$$S_t(v_0,v_1)(x) = (v(x,t), v_t(x,t)), \quad x \in \Xi \tag{3.4}$$

where v is the solution to the Cauchy problem (3.1) with initial data $(v_0,v_1) \in H_0^1(\Xi) \times L^2(\Xi)$ extended by zero outside Ξ.

Our goal now is to extend the family $\{S_t : t > T_0\}$ to the complex parameter $t = \zeta \in \Sigma$ where Σ is an appropriate domain in the complex plane. To do so we start by exploiting the explicit formula for $v(x,t)$ and $v_t(x,t))$, $x \in \overline{\Xi}$ and $t \geq T_0 > \operatorname{diam}(\Xi)$. To handle properly the formula (3.3) we introduce the functions

$$\gamma_k(x,y,t) = \frac{\cos(\mu\sqrt{t^2-|y-x|^2})}{(t^2-|y-x|^2)^{k/2}}, \quad \theta_k(x,y,t) = \frac{\sin(\mu\sqrt{t^2-|y-x|^2})}{(t^2-|y-x|^2)^{k/2}} \tag{3.5}$$

$x,y \in \overline{\Xi}$, $t \geq T_0$ and $k = 1,2,\dots$. Observe that (3.3) reduces to

$$v(x,t) = \frac{1}{2\pi}\frac{\partial}{\partial t}\int_{\Xi}\gamma_1(x,y,t)v_0(y)dy + \frac{1}{2\pi}\int_{\Xi}\gamma_1(x,y,t)v_1(y)dy.$$

Firstly we claim that the functions (3.5) satisfy

$$|\gamma_k(x,y,t)|,\ |\theta_k(x,y,t)| \leq \frac{\varrho^k}{T_0^k}, \quad k = 1,2,\dots \tag{3.6}$$

for some $\varrho > 0$ and every $x, y \in \overline{\Xi}$ and $t \geq T_0$. Indeed, if we introduce the function $\chi(s) = \frac{1}{\sqrt{1-s^2}}$, $-1 < s < 1$ and observe that for $x, y \in \overline{\Xi}$ and $t > \operatorname{diam}(\overline{\Xi})$ we have $0 \leq |\frac{y-x}{t}| < 1$, we then rewrite formulas (3.5) as

$$\gamma_k(x,y,t) = \frac{1}{t^k}\cos(\mu\sqrt{t^2 - |y-x|^2})\chi(|\frac{y-x}{t}|)^k, \quad k = 1, 2, \dots, \tag{3.7}$$

$$\theta_k(x,y,t) = \frac{1}{t^k}\sin(\mu\sqrt{t^2 - |y-x|^2})\chi(|\frac{y-x}{t}|)^k, \quad k = 1, 2, \dots \tag{3.8}$$

Now we choose and fix κ such that $\frac{\operatorname{diam}(\Xi)}{T_0} < \kappa < 1$ and let ϱ be the maximum of the function χ in $[-\kappa, \kappa]$. For $x, y \in \overline{\Xi}$ and $t \geq T_0$ we have $|\frac{y-x}{t}| \leq \frac{\operatorname{diam}(\Xi)}{T_0} < \kappa$ which implies $\chi(|\frac{y-x}{t}|) \leq \varrho$. Hence $|\gamma_k(x,y,t)|$, $|\theta_k(x,y,t)| \leq \frac{\varrho^k}{t^k} \leq \frac{\varrho^k}{T_0^k}$ for every $x, y \in \overline{\Xi}$ and $t \geq T_0$. This shows our claim.

Now observe that the derivatives of the functions (3.7) and (3.8) are expressed in terms of themselves. Indeed, some computation led us

$$\frac{\partial}{\partial t}\gamma_k = -\mu t\theta_{k+1} - kt\gamma_{k+2}, \tag{3.9}$$

$$\frac{\partial}{\partial t}\theta_k = \mu t\gamma_{k+1} - kt\theta_{k+2}, \tag{3.10}$$

$$\frac{\partial}{\partial x_i}\gamma_k = -\mu(y_i - x_i)\theta_{k+1} - k(y_i - x_i)\gamma_{k+2}, \quad i = 1, 2, \tag{3.11}$$

$$\frac{\partial}{\partial x_i}\theta_k = \mu(y_i - x_i)\gamma_{k+1} - k(y_i - x_i)\theta_{k+2}, \quad i = 1, 2. \tag{3.12}$$

By using (3.9)-(3.12) to obtain higher order derivatives of γ_1 and using the estimates (3.6) we conclude that $v \in C^\infty(\overline{\Xi} \times [T_0, \infty))$ and differentiations in (3.3) may be carried out beneath the integral. For instance, observing that

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\gamma_k = &-\mu\theta_{k+1} - \mu^2t^2\gamma_{k+2} + \mu t^2(k+1)\theta_{k+3} \\ &- k\gamma_{k+2} + \mu kt^2\theta_{k+3} + k(k+2)t^2\gamma_{k+4}, \end{aligned} \tag{3.13}$$

$$\begin{aligned} \frac{\partial^2}{\partial t^2}\theta_k = &\mu\gamma_{k+1} - \mu^2t^2\theta_{k+2} - \mu t^2(k+1)\gamma_{k+3} \\ &- k\theta_{k+2} - \mu kt^2\gamma_{k+3} + k(k+2)t^2\theta_{k+4} \end{aligned} \tag{3.14}$$

we obtain

$$\begin{aligned} v(x,t) = \frac{1}{2\pi}\Big[&-\mu t\int_\Xi \theta_2(x,y,t)v_0(y)dy \\ &- t\int_\Xi \gamma_3(x,y,t)v_0(y)dy + \int_\Xi \gamma_1(x,y,t)v_1(y)dy\Big] \end{aligned} \tag{3.15}$$

and

$$
\begin{aligned}
v_t(x,t) = \frac{1}{2\pi}\Big[& -\mu \int_\Xi \theta_2(x,y,t)v_0(y)dy - \mu^2 t^2 \int_\Xi \gamma_3(x,y,t)v_0(y)dy \\
& + 3\mu t^2 \int_\Xi \theta_4(x,y,t)v_0(y)dy - \int_\Xi \gamma_3(x,y,t)v_0(y)dy \\
& + 3t^2 \int_\Xi \gamma_5(x,y,t)v_0(y)dy - \mu t \int_\Xi \theta_2(x,y,t)v_1(y)dy \\
& - t \int_\Xi \gamma_3(x,y,t)v_1(y)dy\Big]
\end{aligned} \tag{3.16}
$$

as long as $t \geq T_0$ and $x \in \overline{\Xi}$.

From the discussion above we see that the bounded linear operator S_t defined in (3.4) is compact, for each $t \geq T_0$.

Let Σ_0 be the region of the complex plane given by

$$
\Sigma_0 = \big\{\zeta : \zeta = T_0 + z, |\arg(z)| \leq \frac{\pi}{4}\big\}.
$$

Let $B(H_0^1(\Xi) \times L^2(\Xi), H^1(\Xi) \times L^2(\Xi))$ be the space of all bounded linear operators from $H_0^1(\Xi) \times L^2(\Xi)$ into $H^1(\Xi) \times L^2(\Xi)$. We claim that the map

$$
[T_0, \infty) \ni t \to S_t \in B(H_0^1(\Xi) \times L^2(\Xi), H^1(\Xi) \times L^2(\Xi))
$$

extends to Σ_0 by setting

$$
S_\zeta(v_0, v_1)(x) = (v(x,\zeta), v_t(x,\zeta)), \quad x \in \Xi \tag{3.17}
$$

where $v(x,\zeta)$ and $v_t(x,\zeta)$ are obtained from (3.15) and (3.16) by changing t by ζ. Moreover, S_ζ is compact, for every $\zeta \in \Sigma_0$.

Observe that formulas (3.15) and (3.16) show that $v(x,t)$ and $v_t(x,t)$ are expressed as linear combinations of terms of the form

$$
t^p \int_\Xi \gamma_\nu(x,y,t)f(y)dy \quad \text{and} \quad t^q \int_\Xi \theta_k(x,y,t)f(y)dy \tag{3.18}
$$

where $p, q \in \{0,1,2\}$, $\nu \in \{1,3,5\}$, $k \in \{2,4\}$, $f \in \{v_0, v_1\}$ and $x \in \overline{\Xi}$, $t \geq T_0$.To prove our claim it suffices to extend the integrals in (3.18) to the parameter $t = \zeta \in \Sigma_0$ and certify that the resulting integrals have the regularity to make sure that S_ζ take values in $H^1(\Xi) \times L^2(\Xi)$.

To begin, let $\kappa > 0$ be such that $\frac{\operatorname{diam}(\Xi)}{T_0} < \kappa < 1$. For each $\zeta = T_0 + z \in \Sigma_0$ and all $x, y \in \overline{\Xi}$ we have

$$
\operatorname{Re}(\zeta^2 - |x-y|^2) = T_0^2 + 2T_0 \operatorname{Re}(z) - \operatorname{Im}(z)^2 + \operatorname{Re}(z)^2 - |x-y|^2 \geq T_0^2 - |x-y|^2
$$

since $|\arg(z)| \leq \pi/4$. It follows

$$
\operatorname{Re}(\zeta^2 - |x-y|^2) \geq T_0^2\Big(1 - |\frac{x-y}{T_0}|^2\Big) > T_0^2(1-\kappa^2) > 0 \tag{3.19}
$$

and $|\arg(\zeta^2 - |x-y|^2)| \leq \pi/2$. By choosing that value of $(\zeta^2 - |x-y|^2)^{1/2}$ which have positive real part, one obtains, for fixed $x, y \in \overline{\Xi}$, an analytic function of ζ whose values lie in the sector

$$
\Sigma_\kappa = \big\{\zeta : \zeta = \sqrt{1-\kappa^2}T_0 + z, |\arg(z)| \leq \frac{\pi}{4}\big\}
$$

for all $x, y \in \overline{\Xi}$. Now, by using (3.19) we obtain

$$|\gamma_k(x,y,\zeta)| \leq \frac{|\cos(\mu\sqrt{\zeta^2-|y-x|^2})|}{T_0^k(1-\kappa^2)^{k/2}}, \quad |\theta_k(x,y,\zeta)| \leq \frac{|\sin(\mu\sqrt{\zeta^2-|y-x|^2})|}{T_0^k(1-\kappa^2)^{k/2}},$$

for every $x, y \in \overline{\Xi}$, $\zeta \in \Sigma_0$ and $k = 1, 2, \dots$. Since $(\zeta^2-|x-y|^2)^{1/2}$ takes values in Σ_κ for every $x, y \in \overline{\Xi}$, it follows that $\cos(\mu\sqrt{\zeta^2-|y-x|^2})$ and $\sin(\mu\sqrt{\zeta^2-|y-x|^2})$ are locally bounded as functions of ζ, for every $x, y \in \overline{\Xi}$, i.e., for each open ball $V_{\zeta_0} = \{\zeta \in \mathbb{C} : |\zeta - \zeta_0| < r\}$ contained in the interior of Σ_0 there exist a constant $C_{V_{\zeta_0}} > 0$ such that

$$|\cos(\mu\sqrt{\zeta^2-|y-x|^2})|, |\sin(\mu\sqrt{\zeta^2-|y-x|^2})| \leq C_{V_{\zeta_0}}$$

for every $\zeta \in V_{\zeta_0}$ and $x, y \in \overline{\Xi}$. Hence,

$$|\gamma_k(x,y,\zeta)|, |\theta_k(x,y,\zeta)| \leq \frac{C_{V_{\zeta_0}}}{T_0^k(1-\kappa^2)^{k/2}}, \quad k = 1, 2, \dots \tag{3.20}$$

for every $x, y \in \overline{\Xi}$, and $\zeta \in V_{\zeta_0}$.

Hence, formulas (3.9)-(3.12) together with (3.20) shows that integrals (3.18) are well defined for $t = \zeta \in \Sigma_0$ as well as all differentiations in (3.18) may be carried out beneath the integral. Hence, the map defined in (3.17) is well defined in Σ_0 and it takes values in $B(H_0^1(\Xi)\times L^2(\Xi), H^1(\Xi)\times L^2(\Xi))$. Moreover, the maps $x \to v(x,\zeta)$, $x \to v_t(x,\zeta)$ are in $C^\infty(\overline{\Xi})$ for every $\zeta \in \Sigma_0$. It follows from this discussion that S_ζ is compact, for every $\zeta \in \Sigma_0$.

Even though we obtain the analyticity of the maps $\zeta \to v(x,\zeta)$, $\zeta \to v_t(x,\zeta)$ for each $x \in \overline{\Xi}$ the regularity with respect to ζ is even better as the next theorem shows.

Theorem 3.1. *Let $\Xi \subset \mathbb{R}^2$ be a bounded domain, $T_0 > \operatorname{diam}(\Xi)$ a real number and $\Sigma_0 = \{\zeta : \zeta = T_0 + z, |\arg(z)| \leq \frac{\pi}{4}\}$. The map*

$$\Sigma_0 \ni \zeta \to S_\zeta \in B(H_0^1(\Xi) \times L^2(\Xi), H^1(\Xi) \times L^2(\Xi))$$

defined by

$$S_\zeta(v_0, v_1)(x) = (v(x,\zeta), v_t(x,\zeta)), \quad x \in \Xi$$

where $v(x,\zeta)$ and $v_t(x,\zeta)$ are given by (3.15) *and* (3.16) *respectively ($t = \zeta$) is analytic inside the sector Σ_0.*

Proof. Let $P_0 : H^1(\Xi) \times L^2(\Xi) \to H^1(\Xi)$, $P_1 : H^1(\Xi) \times L^2(\Xi) \to L^2(\Xi)$ be the projections of $H^1(\Xi) \times L^2(\Xi)$ onto $H^1(\Xi)$ and $L^2(\Xi)$ respectively. It suffices to prove analyticity of the maps

$$\Sigma_0 \ni \zeta \to P_0 S_\zeta \in B(H_0^1(\Xi) \times L^2(\Xi), H^1(\Xi)),$$
$$\Sigma_0 \ni \zeta \to P_1 S_\zeta \in B(H_0^1(\Xi) \times L^2(\Xi), L^2(\Xi)),$$

respectively. Let $\langle\cdot,\cdot\rangle_{H^1(\Xi)}$ and $\langle\cdot,\cdot\rangle_{L^2(\Xi)}$ denote the inner product in $H^1(\Xi)$ and $L^2(\Xi)$ respectively. As weak analyticity implies strong analyticity (see [8, theorem 3.12, p. 152]) it is enough to show that the complex functions

$$\Sigma_0 \ni \zeta \to F_0(\zeta) =: \langle P_0 S_\zeta(v_0, v_1), w\rangle_{H^1(\Xi)} \in \mathbb{C}, \quad w \in H^1(\Xi),$$
$$\Sigma_0 \ni \zeta \to F_1(\zeta) =: \langle P_1 S_\zeta(v_0, v_1), w\rangle_{L^2(\Xi)} \in \mathbb{C}, \quad w \in L^2(\Xi),$$

for $(v_0, v_1) \in H_0^1(\Xi)\times L^2(\Xi)$ are analytic in the interior of Σ_0. It suffices to prove it only for F_0, to F_1 the procedure is analogous. Remember that $P_0 S_\zeta(v_0, v_1) = v(\cdot,\zeta)$

where v is given in (3.15). Hence, for each $(v_0, v_1) \in H_0^1(\Xi) \times L^2(\Xi)$ and $w \in H^1(\Xi)$ we have

$$\langle P_0 S_\zeta(v_0, v_1), w\rangle_{H^1(\Xi)} = \int_\Xi v(x,\zeta)\overline{w(x)}dx + \sum_{i=1}^2 \int_\Xi \frac{\partial v}{\partial x_i}(x,\zeta)\overline{\frac{\partial w}{\partial x_i}(x)}dx.$$

Now, considering that $v(x,\zeta)$ is a linear combination of terms of the form (3.18) (with $t = \zeta$) and taking into account the formulas (3.11) and (3.12) for the derivatives $\frac{\partial}{\partial x_i}\gamma_k$ and $\frac{\partial}{\partial x_i}\theta_k$ respectively, we see that $v(x,\zeta)$ and $\frac{\partial v}{\partial x_i}(x,\zeta)$ are linear combinations of terms of the form

$$\zeta^p \int_\Xi \gamma_\nu(x,y,\zeta)(y_i - x_i)^s f(y)dy \text{ and } \zeta^q \int_\Xi \theta_k(x,y,\zeta)(y_i - x_i)^s f(y)dy$$

where $s \in \{0,1\}$, $f \in \{v_0, v_1\}$ and p, q, k and ν are nonnegative integers. It follows that $\langle P_0 S_\zeta(v_0, v_1), w\rangle$ is a linear combination of terms of the form

$$\int_\Xi \zeta^p \int_\Xi \gamma_\nu(x,y,\zeta)(y_i - x_i)^s f(y)\overline{\Big(\frac{\partial}{\partial x_i}\Big)^l w(x)}\, dy\, dx, \tag{3.21}$$

$$\int_\Xi \zeta^q \int_\Xi \theta_k(x,y,\zeta)(y_i - x_i)^s f(y)\overline{\Big(\frac{\partial}{\partial x_i}\Big)^l w(x)}\, dy\, dx, \tag{3.22}$$

where $s, l \in \{0,1\}$, $f \in \{v_0, v_1\}$ and p, q, k and ν are nonnegative integers.

Now, by setting $\psi(x,y) = (y_i - x_i)^s f(y)\overline{(\frac{\partial}{\partial x_i})^l w(x)}$ in (3.21) and (3.22) we conclude that to prove that F_0 is analytic inside Σ_0 it suffices to prove analyticity of

$$\Sigma_0 \ni \zeta \to \zeta^p \int_{\Xi\times\Xi} \psi(x,y)\gamma_\nu(x,y,\zeta)\, dx\, dy \tag{3.23}$$

$$\Sigma_0 \ni \zeta \to \zeta^p \int_{\Xi\times\Xi} \psi(x,y)\theta_k(x,y,\zeta)\, dx\, dy \tag{3.24}$$

inside the sector Σ_0.

Let ζ_0 be an arbitrary point in the interior of Σ_0 and V_{ζ_0} some open ball centered in ζ_0 and contained in the interior of Σ_0. Now, observing that

$$\frac{\partial\gamma_\nu}{\partial\zeta}(x,y,\zeta) = -\mu\zeta\theta_{\nu+1}(x,y,\zeta) - \nu\zeta\gamma_{\nu+2}(x,y,\zeta), \quad \nu = 1,2,\dots$$

$$\frac{\partial\theta_k}{\partial\zeta}(x,y,\zeta) = \mu\zeta\gamma_{k+1}(x,y,\zeta) - k\zeta\theta_{k+2}(x,y,\zeta), \quad k = 1,2,\dots,$$

for every $x, y \in \Xi$ and $\zeta \in \Sigma_0$, and using the estimate (3.20) we conclude that $|\frac{\partial\gamma_\nu}{\partial\zeta}(x,y,\zeta)|$ and $|\frac{\partial\theta_k}{\partial\zeta}(x,y,\zeta)|$ are bounded in $\overline{\Xi} \times \overline{\Xi} \times V_{\zeta_0}$. Since ψ is integrable in the bounded set $\Xi \times \Xi$ we have

$$\int_{\Xi\times\Xi} |\psi(x,y)\gamma_\nu(x,y,\zeta)|\, dx\, dy \le \text{Const.} \int_{\Xi\times\Xi} |\psi(x,y)|\, dx\, dy,$$

$$\int_{\Xi\times\Xi} |\psi(x,y)\theta_k(x,y,\zeta)|\, dx\, dy \le \text{Const.} \int_{\Xi\times\Xi} |\psi(x,y)|\, dx\, dy,$$

for every $x, y \in \Xi$, $\zeta \in V_{\zeta_0}$. Hence, we may differentiate (3.23) and (3.24) with respect to $\zeta \in V_{\zeta_0}$ under the integrals. From this, it follows that F_0 is analytic inside Σ_0. $\square$

Now, for each $i = 1, \dots, m$ and $t > 0$ we define the bounded linear operator $S_t^i : H_0^1(\Xi) \times L^2(\Xi) \to H^1(\Xi) \times L^2(\Xi)$ by setting

$$S_t^i(w_0^i, w_1^i)(x) = (w^i(x,t), w_t^i(x,t)), \quad x \in \Xi \tag{3.25}$$

where w^i is the solution to the Cauchy problem

$$\begin{gathered} w_{tt}^i - \Delta w^i + \mu_i^2 w^i = 0 \quad \text{in } \mathbb{R}^{2+1}, \\ w^i(\cdot,0) = w_0^i \quad \text{in } \mathbb{R}^2, \\ w_t^i(\cdot,0) = w_1^i \quad \text{in } \mathbb{R}^2, \end{gathered} \tag{3.26}$$

where $\mu_i^2 = \lambda_i$ and initial data $(w_0^i, w_1^i) \in H_0^1(\Xi) \times L^2(\Xi)$ extended by zero outside Ξ.

Now fix $T_0 > \operatorname{diam}(\Xi)$. Then applying Theorem 3.1 to the family of operators $\{S_t^i : t > T_0\}$ shows that it admits an extension to the sector Σ_0, analytic in its interior. As in Theorem 3.1, for each $\zeta \in \Sigma_0$, S_ζ^i is given by

$$S_\zeta^i(w_0^i, w_1^i)(x) = (w^i(x,\zeta), w_t^i(x,\zeta)), \quad x \in \Xi \tag{3.27}$$

where $w^i(x,\zeta)$ and $w_t^i(x,\zeta)$ are given by (3.15) and (3.16) respectively after appropriate adjustments.

Now we define

$$\mathbf{S}_\zeta^* : \mathcal{H}_0^1(\Xi) \times \mathcal{L}^2(\Xi) \to \mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi) \tag{3.28}$$

by setting

$$\begin{aligned} &\mathbf{S}_\zeta^*((w_0^1, \dots, w_0^m), (w_1^1, \dots, w_1^m)) \\ &= ((P_0 S_\zeta^1(w_0^1, w_1^1), \dots, P_0 S_\zeta^m(w_0^m, w_1^m)), (P_1 S_\zeta^1(w_0^1, w_1^1), \dots, \\ &\quad P_1 S_\zeta^m(w_0^m, w_1^m))) \\ &= ((w^1(\cdot,\zeta), \dots, w^m(\cdot,\zeta)), (w_t^1(\cdot,\zeta), \dots, w_t^m(\cdot,\zeta))) \end{aligned} \tag{3.29}$$

where P_0 and P_1 are the projections introduced in the proof of Theorem 3.1. It follows from that theorem that the map

$$\Sigma_0 \ni \zeta \to \mathbf{S}_\zeta^* \in B(\mathcal{H}_0^1(\Xi) \times \mathcal{L}^2(\Xi), \mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi)) \tag{3.30}$$

is analytic inside Σ_0.

Now consider the Cauchy problem

$$\begin{gathered} U_{tt} - \Delta U + AU = 0 \quad \text{in } \mathbb{R}^{2+1}, \\ U(\cdot,0) = U_0, \quad U_t(\cdot,0) = U_1 \quad \text{in } \mathbb{R}^2, \end{gathered} \tag{3.31}$$

with initial state $(U_0, U_1) \in \mathcal{H}_0^1(\Xi) \times \mathcal{L}^2(\Xi)$. For $t > 0$ we define the bounded linear operator $\mathbf{S}_t : \mathcal{H}_0^1(\Xi) \times \mathcal{L}^2(\Xi) \to \mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi)$ by setting

$$\mathbf{S}_t(U_0, U_1)(x) = (U(x,t), U_t(x,t)), \quad x \in \Xi \tag{3.32}$$

where U is the solution to the Cauchy problem (3.31).

Theorem 3.2. *The family of operators $\{\boldsymbol{S}_t; t > T_0\}$ defined by* (3.32) *extends to the complex parameter $\xi \in \Sigma_0$ as a family of compact operators that is analytic in the interior of the sector Σ_0.*

Proof. As in section 2, we denote $U_0 = (u_0^1, u_0^2, \dots, u_0^m)^T$, $U_1 = (u_1^1, u_1^2, \dots, u_1^m)^T$, $W = (w^1, \dots, w^m)^T$ where $W = BU$. Note that w^i satisfy (3.26) with initial conditions $w_0^i = \alpha_1^i u_0^1 + \alpha_2^i u_0^2 + \dots + \alpha_m^i u_0^m$ and $w_1^i = \alpha_1^i u_1^1 + \alpha_2^i u_1^2 + \dots + \alpha_m^i u_1^m$, $i = 1, \dots, m$. From (3.29) we have

$$\begin{aligned} &\mathbf{S}_t^*((w_0^1, \dots, w_0^m), (w_1^1, \dots, w_1^m)) \\ &= ((w^1(\cdot,t), \dots, w^m(\cdot,t)), (w_t^1(\cdot,t), \dots, w_t^m(\cdot,t))) \\ &= (W(\cdot,t), W_t(\cdot,t)) = (BU(\cdot,t), BU_t(\cdot,t)). \end{aligned} \tag{3.33}$$

Let us introduce the projections $\mathbf{P}_0 : \mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi) \to \mathcal{H}^1(\Xi)$ and $\mathbf{P}_1 : \mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi) \to \mathcal{L}^2(\Xi)$ of $\mathcal{H}^1(\Xi) \times \mathcal{L}^2(\Xi)$ onto $\mathcal{H}^1(\Xi)$ and $\mathcal{L}^2(\Xi)$ respectively. From (3.33) it follows

$$\begin{aligned} BU(\cdot,t) &= \mathbf{P}_0 \mathbf{S}_t^*((w_0^1, \dots, w_0^m), (w_1^1, \dots, w_1^m)), \\ BU_t(\cdot,t) &= \mathbf{P}_1 \mathbf{S}_t^*((w_0^1, \dots, w_0^m), (w_1^1, \dots, w_1^m)). \end{aligned}$$

Now, observing that $(w_0^1, \dots, w_0^m)^T = BU_0$ and $(w_1^1, \dots, w_1^m)^T = BU_1$ we obtain

$$U(\cdot,t) = [B^{-1}\mathbf{P}_0 \mathbf{S}_t^* B] U_0 \quad \text{and} \quad U_t(\cdot,t) = [B^{-1}\mathbf{P}_1 \mathbf{S}_t^* B] U_1.$$

Therefore

$$\mathbf{S}_t(U_0, U_1) = ([B^{-1}\mathbf{P}_0 \mathbf{S}_t^* B] U_0, [B^{-1}\mathbf{P}_1 \mathbf{S}_t^* B] U_1) \tag{3.34}$$

for all $(U_0, U_1) \in \mathcal{H}_0^1(\Xi) \times \mathcal{L}^2(\Xi)$ and $t \geq T_0$. From the relationship (3.34) among the operators $\mathbf{S}_t$ and $\mathbf{S}_t^*$ and the analyticity of the map (3.30) the result follows. $\square$

4. Control near optimal time

Now we go over the proof of Theorem 1.1. Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with piecewise smooth boundary $\partial\Omega$. Given any $T_* > \operatorname{diam}(\Omega)$ we choose $\delta > 0$ and $T_0 > 0$ such that

$$\operatorname{diam}(\Omega) < \operatorname{diam}(\Omega_\delta) < T_0 < T_*.$$

To take advantage of all the discussion up to this point, we put $\Xi = \Omega_\delta$.

From now on we proceed as in the proof of the Theorem 1.1 by considering the compact operators $\mathbf{S}_T$ and $\widehat{\mathbf{S}}_T$ from $\mathcal{H}_0^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$ into $\mathcal{H}^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$ given by (2.9) and (2.12) respectively and $\mathbf{K}_T = R\widehat{\mathbf{S}}_T \varphi \mathbf{S}_T E$ on $\mathcal{H}^1(\Omega) \times \mathcal{L}^2(\Omega)$.

The crux of the proof of the Theorem 2.2 was to prove the invertibility of $Id - \mathbf{K}_T$ for some T. Our goal is to prove that $Id - \mathbf{K}_T$ is invertible for a value of T such that $T_0 < T < T_*$. Once this is done, by proceeding exactly like in the proof of Theorem 2.2 we obtain the exact controllability for the system (1.1) at the time T less than T_*. Hence, by using a standard procedure (linearity and uniqueness in particular) we obtain the controllability at the desired moment T_*.

To achieve our goal we use a theorem of alternatives due to Atikinson [4]. However, before we have to extend the family $\{\mathbf{K}_T : T > T_0\}$ to the sector Σ_0, analytically in its interior. Note that operators $\mathbf{S}_T$ and $\widehat{\mathbf{S}}_T$ obey the following relations

$$\mathbf{P}_0 \widehat{\mathbf{S}}_T(V_0, V_1) = \mathbf{P}_0 \mathbf{S}_T(V_0, -V_1), \quad \mathbf{P}_1 \widehat{\mathbf{S}}_T(V_0, V_1) = \mathbf{P}_1 \mathbf{S}_T(-V_0, V_1)$$

for $(V_0, V_1) \in \mathcal{H}_0^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$.

Since, by Theorem 3.2 $\{\mathbf{S}_T : T > T_0\}$ admits such an extension, the same applies to $\{\widehat{\mathbf{S}}_T : T > T_0\}$ and hence to $\{\mathbf{K}_T : T > T_0\}$. Now we apply to the family of compact operators $\{\mathbf{K}_\zeta : \zeta \in \Sigma_0\}$ Atkinson's result as in [8, Theorem 1.9, page 370]. This theorem states that either 1 is eigenvalue of $\mathbf{K}_\zeta$ for every $\zeta \in \Sigma_0$ or $Id - \mathbf{K}_\zeta$

is invertible for all but at most a finite number of values of ζ in each compact subset of Σ_0. From the proof of Theorem 2.2 we know that $\mathbf{K}_\zeta$ is a contraction if $\zeta \in \mathbb{R} \cap \Sigma_0$ is sufficiently large. This excludes the first alternative. It follows that for a convenient $\varepsilon > 0$, the compact set $[T_0 + \varepsilon, T_* - \varepsilon] \subset \Sigma_0$ includes some T for which $Id - \mathbf{K}_T$ is invertible. As observed before, this suffices to conclude the proof of Theorem 1.1. □

We end up this section showing how to apply the procedure above to treat the case in which part of membranes boundary is hold fixed. Let

$$\Omega = \{(r\cos\theta, r\sin\theta); 0 < r_1 < r < r_2,\ 0 < \theta < \pi/n\}, \tag{4.1}$$

$$\Gamma_0 = \{(r\cos\theta, r\sin\theta); r_1 < r < r_2, \theta = 0 \text{ or } \theta = \pi/n\}, \tag{4.2}$$

where n is a positive integer.

Given any $T_* > 2r_2$ and initial state $U_0 \in \mathcal{H}^1(\Omega)$, $U_1 \in \mathcal{L}^2(\Omega)$, $U_0 = 0$ on Γ_0, there exist a control $f \in \mathcal{L}^2(\partial\Omega/\Gamma_0 \times]0, T_*[)$ so that the solution $U \in \mathcal{H}^1(\Omega \times]0, T_*[)$ of the system

$$\begin{gathered} U_{tt} - \Delta U + AU = 0 \quad \text{in } \Omega \times]0, T[, \\ U(\cdot, 0) = U_0 \quad \text{in } \Omega, \\ U_t(\cdot, 0) = U_1 \quad \text{in } \Omega, \\ U = 0 \quad \text{on } \Gamma_0 \times]0, T[, \\ \frac{\partial U}{\partial \eta} = f \quad \text{on } \partial\Omega/\Gamma_0 \times]0, T[, \end{gathered} \tag{4.3}$$

satisfies the final condition $U(\cdot, T_*) = U_t(\cdot, T_*) = 0$ on Ω.

The proof of this result follows the same lines of the proof of Theorem 1.1 with additional care on the extension operator E used in the beginning.

Consider the angular sector $\Omega_\infty = \{(r\cos\theta, r\sin\theta); r > 0,\ 0 < \theta < \pi/n\}$. Fix T_0 and $\delta > 0$ such that $\delta < r_1$ and $r_2 + \delta/2 < T_0 < T_*$. By using standard techniques of the theory of Sobolev spaces we can extend an arbitrary initial data $(V_0, V_1) \in \mathcal{H}^1(\Omega) \times \mathcal{L}^2(\Omega)$ with $V_0 = 0$ on Γ_0 to the angular sector Ω_∞ in such a way that the extension (V_0, V_1) results in $\mathcal{H}^1(\Omega_\infty) \times \mathcal{L}^2(\Omega_\infty)$, vanish for $|x| < r_1 - \delta/2$ and for $|x| > r_2 + \delta/2$, and more: $V_0 = 0$ on the edges $\theta = 0$ and $\theta = \pi/n$ of Ω_∞.

Next we extend each entry of (V_0, V_1) to the plane as odd functions with respect to each entire line determined by the angles $\theta = i(\pi/n)$, $i = 1, \dots, n$. We denote $(\widetilde{V}_0, \widetilde{V}_1)$ the resulting extensions and define $E(V_0, V_1) := (\widetilde{V}_0, \widetilde{V}_1)$. Observe that $E(V_0, V_1)$ has compact support in the set defined by $\Omega_\delta =: \{x \in \mathbb{R}^2 : r_1 - \delta < |x| < r_2 + \delta\}$. Clearly $E : \mathcal{H}^1(\Omega) \times \mathcal{L}^2(\Omega) \to \mathcal{H}^1(\Omega_\delta) \times \mathcal{L}^2(\Omega_\delta)$ is a bounded linear operator with the properties we need to proceed the proof. Indeed, solving a Cauchy problem (2.1) with initial state $(\widetilde{V}_0, \widetilde{V}_1)$ is equivalent to solve a Cauchy problem for a system of decoupled Klein-Gordon equations with initial state of same regularity and odd with respect to the lines $\theta = i(\pi/n)$, $i = 1, \dots, n$. From the explicit formula for the solution to the Klein-Gordon equation we see that the solution of 2.1 vanishes on the lines $\theta = i(\pi/n)$, $i = 1, \dots, n$ and has trace (together with its time derivative) as odd function with respect to those lines on each plane $t = T > 0$. By using a localizing function $\varphi \in C_0^\infty(\mathbb{R}^2)$ of radial type satisfying $\varphi = 1$ on $\Omega_{\delta/2}$ and vanishing outside Ω_δ we see that the solution to the backward problem (2.11) also vanish on the lines $\theta = i(\pi/n)$, $i = 1, \dots, n$. From this we conclude that if we use $\Omega_\delta =: \{x \in \mathbb{R}^2 : r_1 - \delta < |x| < r_2 + \delta\}$ and the extension operator

$E : \mathcal{H}^1(\Omega)\times\mathcal{L}^2(\Omega) \to \mathcal{H}^1(\Omega_\delta)\times\mathcal{L}^2(\Omega_\delta)$ constructed above, and proceed as in the proof of the Theorem 1.1 we obtain the desired result.

Similar result holds if Ω is a rectangle with sides parallel to coordinate axis and Γ_0 is one its sides or even two consecutive sides.

Controllability for a single wave equation in the domain Ω given in (4.1) was considered in [5].

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