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ANALYTIC SOLUTIONS FOR HAMILTON-JACOBI-BELLMAN EQUATIONS

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ABSTRACT. Closed form solutions are found for a particular class of Hamilton-Jacobi-Bellman equations emerging from a differential game among firms competing over quantities in a simultaneous oligopoly framework. After the derivation of the solutions, a microeconomic example in a non-standard market is presented where feedback equilibrium is calculated with the help of one of the previous formulas.

1. INTRODUCTION

In differential games and optimal control theory, Hamilton-Jacobi-Bellman (HJB) equations have played a major role in the previous decades. A wide range of models have been introduced and analyzed whose solutions have been either determined or approximated with the help of Dynamic Programming techniques, and especially feedback strategies are considered a key solution concept.

Just to cite a few fundamental recent textbooks on the different aspects of this issue: the main theoretical contribution is probably the volume by Seierstad and Sydsaeter ([21], 1986), whereas the textbook by Dockner et al. [9] is a major contribution including a wide range of applications to a lot of economic models. Jørgensen and Zaccour [13] focus on marketing models especially, furthermore a very rich treatment on HJB equations is provided in Bardi and Capuzzo Dolcetta [1].

In recent decades, a variegated stream of literature has found a relevant development on several aspects of HJB equations: a theoretical investigation involving results on viscosity solutions can be found in Lions and Souganidis [18], relevant properties of the value function in infinite horizon optimal control problems are found out in Baumeister et al. [2], an iterative dynamic programming method for 2-agent games is introduced by Zhang et al. [23], a study on differential games with non-constant discounting is proposed by Marin-Solano and Shevkoplyas [19], necessary and sufficient conditions for feedback equilibria in linear-quadratic games are established in [10].

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As far as economic models are concerned, in the latest years the attention of economic literature has been directed towards oligopoly models with several different market effects: the survey carried out by Jørgensen and Zaccour [12] provides an extensive outline of applications until 2007. Other relevant contributions are due to Wirl [22], who discusses the properties of the optimal value function in a scenario with polluting emissions, whereas Erickson [11] studies a dynamic advertising framework. Prasad et al. [20] derive feedback equilibrium in an advertising model where customers switch to competing brands, and HJB equations are solved also by Colombo and Dawid [8] in a scenario where technological spillovers appear. On the other hand, Lambertini and Mantovani [17] derive feedback strategies in a dynamic renewable resource oligopoly under pre-emption and subject to voracity effects. All these papers, and many others, feature solutions to the related HJB equations in linear-quadratic structures and standard demand on the markets.

However, relatively little attention has been paid to industrial organization models where the inverse demand function of the market has a non-standard form such as the hyperbolic one. A preliminary microfoundation and some results in such a setting can be found in Lambertini [15] (dynamic framework) and Colombo [5] (static framework), whereas in Lambertini and Palestini [16], the derivation and solution of the HJB equations originating from this framework are presented and discussed.

In this article, I would like to extend such treatment by focussing on a class of HJB equations which are strictly connected to oligopoly differential games with hyperbolic inverse demand functions. Differently from recent papers which analyze several formulations of linear-quadratic differential games (i.e. [5, 6]), I take into account games whose structure is based on polynomials having degree higher than 2.

To solve the HJB equation, the approach I adopt is the same as in most literature on differential games and optimal control applied to economics (see, for example [9]): a guess for $V^*(\cdot)$ is chosen and then the explicit formulation of its coefficients is established. Here is a brief outline of the main results:

- A class of HJB equations having a polynomial term in one of the two arguments of the unknown function is taken into account and solved in closed form.
- An oligopoly differential game is introduced with a hyperbolic inverse demand function. By deriving the HJB equation for this model, the structure we obtain is the one that can be solved.
- The application of the formula is exhibited and the Nash feedback strategy is determined.

The remainder of this article is as follows. Section 2 features the main findings on the solution of a class of HJB equations with two different choices of parameters. Section 3 introduces an application to a 3-firm differential game where the inverse demand function of the market is hyperbolic. The value function and the Nash feedback strategy are explicitly calculated. Section 4 concludes and outlines some possible future improvements.

2. Analytic solutions

Consider $m \in \mathbb{Z}_+$ such that $m \geq 1$. Let us introduce a family of HJB equations, having $V(x,t) \in C^1((0,+\infty)\times[0,T])$ as their unknowns:

$$\frac{\partial V(x,t)}{\partial t} - \rho V(x,t) = \sum_{l=0}^{m} \beta_l x^l + \alpha x^{\gamma} \frac{\partial V(x,t)}{\partial x}, \tag{2.1}$$

where $\gamma \in \{0,1\}$. Functions V are defined on $(x,t) \in (0,+\infty)\times[0,T]$, such that the boundary condition $V(x,T)=0$ holds for all $x>0$.

To solve (2.1), we choose a suitable guess, turning out to be a polynomial in x having the same degree of the polynomial which appears at the right-hand side of (2.1). The two cases will be separated based on the value of γ. The arguments of V will often be omitted for simplicity during the derivation of the solutions to (2.1).

2.1. **Solution for $\gamma = 0$.** When $\gamma = 0$, α is the only coefficient of the first-order partial derivative of $V(\cdot)$ with respect to x, i.e.

$$\frac{\partial V}{\partial t} - \rho V = \sum_{l=0}^{m} \beta_l x^l + \alpha \frac{\partial V}{\partial x}. \tag{2.2}$$

To establish the solution to (2.2), a preliminary Lemma is helpful, to provide the solution formula for the linear dynamic system whose unknowns are going to be the coefficients of the value function.

Lemma 2.1. *The solution to the dynamic system*

$$\begin{pmatrix} \dot y_0(t) \\ \dot y_1(t) \\ \dots \\ \dot y_{m-1}(t) \\ \dot y_m(t) \end{pmatrix} = \begin{pmatrix} \rho & \alpha & 0 & \cdots & \cdots & 0 \\ 0 & \rho & 2\alpha & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & \rho & m\alpha \\ 0 & 0 & \cdots & 0 & 0 & \rho \end{pmatrix} \begin{pmatrix} y_0(t) \\ y_1(t) \\ \dots \\ y_{m-1}(t) \\ y_m(t) \end{pmatrix} + \begin{pmatrix} \beta_0 \\ \beta_1 \\ \dots \\ \beta_{m-1} \\ \beta_m \end{pmatrix}, \tag{2.3}$$

endowed with final conditions $y_j(T) = 0$, for all $j = 0,1,\dots,m$, and for all $\alpha,\beta_0,\beta_1,\dots,\beta_m \in \mathbb{R}$, is given by

$$y_j^*(t) = \sum_{k=0}^{m} \gamma_{j,k}(t-T)^k e^{\rho(t-T)} + C_j, \tag{2.4}$$

where:

- *if $j \in \{0,1,\dots,m-2\}$, then*
$$C_j = -\gamma_{j0} = -\frac{1}{\rho}\Big(\beta_j + \sum_{k=1}^{m-j}(-1)^k\big(\frac{\alpha}{\rho}\big)^k\big(\Pi_{l=j+1}^{j+k} l\big)\beta_{j-k}\Big),$$
and the following recurrence relation among coefficients holds:
$$\gamma_{j,k} = \frac{(j+1)\alpha\gamma_{j+1,k+1}}{k} \quad \text{for } k = 1,\dots,m-j,$$
$$\gamma_{j,k} = 0 \quad \text{for } k = m-j,\dots,m;$$
- *if $j = m-1$, then*
$$C_{m-1} = -\gamma_{m-1,0} = -\frac{1}{\rho}\Big(\beta_{m-1} - \frac{m\alpha}{\rho}\beta_m\Big),$$

$$\gamma_{m-1,1} = \frac{m\alpha\beta_m}{\rho}, \quad \gamma_{m-1,k} = 0 \text{for } k = 2, \dots, m;$$

- *if $j = m$, then $C_m = -\beta_m/\rho$, $\gamma_{m,0} = \beta_m/\rho$, $\gamma_{m,k} = 0$ for $k = 1, \dots, m$.*

Proof. Since the matrix in (2.3) has $\lambda = \rho$ as its unique eigenvalue, having algebraic multiplicity $m+1$ and geometric multiplicity 1, all the solutions to the dynamic system can be expressed in the form

$$y_j^*(t) = \sum_{k=0}^{m} \gamma_{j,k}(t-T)^k e^{\rho(t-T)} + C_j,$$

where, by imposing the final condition $y_j^*(T) = 0$, we obtain that $C_j = -\gamma_{j0}$ for all $j = 0, 1, \dots, m$. The system can be solved by starting from the last equation, which is solvable by separation of variables, and then proceeding upwards by successive substitutions. It is easy to work out the initial two solutions:

$$y_m^*(t) = \frac{\beta_m}{\rho}\left(e^{\rho(t-T)} - 1\right),$$
$$y_{m-1}^*(t) = \frac{1}{\rho}\Big(\beta_{m-1} - \frac{m\alpha\beta_m}{\rho}\Big)\left(e^{\rho(t-T)} - 1\right) + \frac{m\alpha\beta_m}{\rho}(t-T)e^{\rho(t-T)}.$$

Subsequently, solving upwards and employing substitutions, we can determine the iteration for all the solutions. The general Cauchy problem

$$\dot{y}_j(t) = \rho y_j(t) + (j+1)y_{j+1}^*(t) + \beta_j$$
$$y_j(T) = 0$$

has the unique solution

$$y_j^*(t) = \frac{\beta_j}{\rho}\big(e^{\rho(t-T)} - 1\big) + (j+1)\alpha\Big(\int_T^t y_{j+1}^*(s)e^{-\rho s}ds\Big)e^{\rho t} \tag{2.5}$$

for $j = 0, \dots, m-1$. To determine the recurrence relations between coefficients, consider the general formulation (2.4) of the j-th solution. By integrating, we have

$$\begin{aligned}
&y_j^*(t)\\
&= \frac{\beta_j}{\rho}\big(e^{\rho(t-T)} - 1\big) + (j+1)\alpha e^{\rho t}\int_T^t \Big(\sum_{k=0}^{m}\gamma_{j+1,k}(s-T)^k e^{-\rho T} + C_{j+1}e^{-\rho s}\Big)ds\\
&= \frac{\beta_j}{\rho}\big(e^{\rho(t-T)} - 1\big) + (j+1)\alpha\Big[\sum_{k=0}^{m}\gamma_{j+1,k}\frac{(t-T)^{k+1}}{k+1}e^{\rho(t-T)} + \frac{C_{j+1}}{\rho}\big(1 - e^{\rho(t-T)}\big)\Big]\\
&= -\frac{1}{\rho}(\beta_j - (j+1)\alpha C_{j+1})(1 - e^{\rho(t-T)}) + (j+1)\alpha\sum_{k=0}^{m}\gamma_{j+1,k}\frac{(t-T)^{k+1}}{k+1}e^{\rho(t-T)},
\end{aligned}$$

which gives the recurrence relations among coefficients:

$$C_j = -\frac{1}{\rho}(\beta_j - (j+1)\alpha C_{j+1}), \quad \gamma_{j,0} = -C_j, \quad \gamma_{j,k} = \frac{(j+1)\alpha\gamma_{j+1,k+1}}{k}.$$

□

Theorem 2.2. *The function $V^*(x,t) = \sum_{l=0}^{m} A_l^*(t)x^l$ with*

$$A_0^*(t) = \sum_{k=0}^{m}\gamma_{0,k}(t-T)^k e^{\rho(t-T)} + C_0$$

$$
\begin{gathered}
A_1^*(t)=\sum_{k=0}^{m}\gamma_{1,k}(t-T)^k e^{\rho(t-T)}+C_1\\
\dots\\
A_{m-1}^*(t)=\frac{m\alpha\beta_m}{\rho}(t-T)e^{\rho(t-T)}+\frac{1}{\rho}\big[\frac{m\alpha\beta_m}{\rho}-\beta_{m-1}\big]\Big(1-e^{\rho(t-T)}\Big)\\
A_m^*(t)=\frac{\beta_m(e^{\rho(t-T)}-1)}{\rho}
\end{gathered}
$$

is a solution of (2.2).

Proof. Consider the guess $V(x,t)=\sum_{l=0}^{m}A_i(t)x^l$ and substitute it into (2.2) to obtain

$$
\sum_{l=0}^{m}\dot{A}_l(t)x^l-\rho\sum_{l=0}^{m}A_l(t)x^l=\sum_{l=0}^{m}\beta_l x^l+\alpha\Big(\sum_{l=1}^{m}lx^{l-1}A_l(t)\Big),
$$

after eliminating the first term of the sum on the right-hand side. Collecting terms with all the powers of x in both sides leads to the dynamic system

$$
\begin{gathered}
\dot{A}_0(t)=\rho A_0(t)+\alpha A_1(t)+\beta_0\\
\dot{A}_1(t)=\rho A_1(t)+2\alpha A_2(t)+\beta_1\\
\dot{A}_2(t)=\rho A_2(t)+3\alpha A_3(t)+\beta_2\\
\dots\\
\dot{A}_{m-1}(t)=\rho A_{m-1}(t)+m\alpha A_m(t)+\beta_{m-1}\\
\dot{A}_m(t)=\rho A_m(t)+\beta_m\,,
\end{gathered}
\tag{2.6}
$$

which should be endowed with the following set of final conditions satisfying the boundary condition: $A_j(T)=0$, for $j=0,1,\dots,m$. By Lemma 2.1, the solution of (2.6) amounts to

$$
\begin{gathered}
A_0^*(t)=\sum_{k=0}^{m}\gamma_{0,k}(t-T)^k e^{\rho(t-T)}+C_0\\
\dots\\
A_{m-1}^*(t)=\frac{m\alpha\beta_m}{\rho}(t-T)e^{\rho(t-T)}+\frac{1}{\rho}\big[\frac{m\alpha\beta_m}{\rho}-\beta_{m-1}\big]\Big(1-e^{\rho(t-T)}\Big)\\
A_m^*(t)=\frac{\beta_m(e^{\rho(t-T)}-1)}{\rho},
\end{gathered}
\tag{2.7}
$$

where coefficients C_j and $\gamma_{j,k}$ are defined as in Lemma 2.1. □

2.2. **Solution for $\gamma=1$.** In this case, (2.1) takes the form

$$
\frac{\partial V}{\partial t}-\rho V=\sum_{l=0}^{m}\beta_l x^l+\alpha x\frac{\partial V}{\partial x},
\tag{2.8}
$$

The next theorem intends to exhibit the solution strategy.

Theorem 2.3. *The function $V^*(x,t)=\sum_{l=0}^{m}A_l^*(t)x^l$ with*

$$
A_0^*(t)=\frac{\beta_0\left(e^{\rho(t-T)}-1\right)}{\rho}
$$

$$
\begin{gathered}
A_1^*(t) = \frac{\beta_1\left(e^{(\rho+\alpha)(t-T)}-1\right)}{\rho+\alpha} \\
A_2^*(t) = \frac{\beta_2\left(e^{(\rho+2\alpha)(t-T)}-1\right)}{\rho+2\alpha} \\
\dots \\
A_m^*(t) = \frac{\beta_m\left(e^{(\rho+m\alpha)(t-T)}-1\right)}{\rho+m\alpha},
\end{gathered}
$$

is a solution of (2.8).

Proof. Call $V(x,t) = A_0(t) + A_1(t)x + \dots + A_m(t)x^m$, where $A_j(t) \in C^1([0,T])$, for all $j = 0, \dots, m$. By replacing it in (2.1), we have

$$
\sum_{l=0}^{m} \dot{A}_l(t)x^l - \rho \sum_{l=0}^{m} A_l(t)x^l = \sum_{l=0}^{m} \beta_l x^l + \alpha\Big(\sum_{l=0}^{m} l x^l A_l(t)\Big).
$$

Collecting terms with powers of x yields an $m+1$-equations dynamic system

$$
\begin{gathered}
\dot{A}_0(t) = \rho A_0(t) + \beta_0 \\
\dot{A}_1(t) = (\rho+\alpha)A_1(t) + \beta_1 \\
\dot{A}_2(t) = (\rho+2\alpha)A_2(t) + \beta_2 \quad , \\
\dots \\
\dot{A}_m(t) = (\rho+m\alpha)A_m(t) + \beta_m
\end{gathered}
\tag{2.9}
$$

subject to the set of final conditions satisfying the boundary data: $A_j(T) = 0$, for $j = 0, 1, \dots, m$.

System (2.9) can be easily solved by separation of variables in each ODE. Plugging the solutions into the expression of $V(x,t)$, we achieve the solution to (2.8):

$$
V^*(x,t) = \frac{\beta_0\left(e^{\rho(t-T)}-1\right)}{\rho} + \sum_{l=1}^{m} \frac{\beta_l\left(e^{(\rho+l\alpha)(t-T)}-1\right)x^l}{\rho+l\alpha}. \tag{2.10}
$$

□

3. A microeconomic application

Consider N firms engaging in a Cournot competition, producing homogeneous goods and bearing a cost for developing R&D in their own sectors. This typical setup describes an oligopolistic game evolving over time, where players aim to maximize their own payoff (For a rich overview of such models, see [9].). In a simplified version of this scenario, each player chooses a strategy, denoted by a control variable, to maximize an objective function which is the integral of the discounted flows of her profits. The notation to be employed is standard for industrial organization models. I am going to borrow it mainly from [15]. It is exposed in the following list together with some hypotheses

- $u_i(t) \in U_i \subseteq \mathbb{R}_+$ is the strategic variable for the i-th player, representing output level, and $u = (u_1, \dots, u_N)$ is a vector of strategies. Each control set U_i may be either bounded, such as $[0, \overline{u}]$, or unbounded, such as $[0, +\infty)$, in compliance with the inverse demand structure of the market;
- $p(u(t))$ is the inverse demand function of the market, decreasing in the sum of all outputs;

- $cu_i(t)$ is the linear production cost borne by the i-th player, where $c > 0$ is the marginal cost parameter;
- $\pi_i(u(t), t) = (p(u(t)) - c)u_i(t)$ is the profit gained by the i-th firm at time t;
- the horizon of the competition is finite, i.e. the game evolves over a compact time interval $[0, T]$;
- the discount factor of profits is the same for all players: $e^{-\rho t}$, where $\rho > 0$ is the force of interest on the market, also considered as a measure of how much players discount their future profits;
- $k_i(t)$ is player i's state variable, describing physical capital or capacity, which accumulates over time in compliance with a given dynamics $G(k_i(t))$ to allow continuous production. Consider the most general case for the state set, i.e. $k_i(t) \in K_i \subseteq \mathbb{R}_+$, meaning that, depending on the cost structure, the state set may be either bounded or unbounded. The initial conditions of such accumulation process are k_i, for $i = 1, \dots, N$, i.e. k_i are the capacity levels at instant t, where $t \in [0, T)$ is the initial instant of the game;
- the i-th firm bears a further cost induced by accumulation of its own physical capital. The cost function $C_i(k_i)$ is a non-negative function of the i-th physical capital. Generally, $C_i(k_i)$ is either a linear or a convex function;
- there is no scrap value or salvage value at time T (This requirement is equivalent to considering a no prize game.);
- the game is played simultaneously;
- players are symmetric, meaning that their productive characteristics make the oligopoly symmetric. In an oligopolistic competition, symmetry among firms can be described in several ways: same initial capital endowment, same number of workers earning the same wages, same output having the same production costs, and so on. Consequently, they cannot be distinguished and for this reason we can search for a symmetric solution of the game. Asymmetric scenarios are more complex, and such frameworks may be investigated in future research.

Firm i solves the following optimization program:

$$\max_{u_i \geq 0} \int_0^T e^{-\delta t}[\pi_i(u_1(t), \dots, u_N(t)) - C_i(k_i(t))]dt, \tag{3.1}$$

subject to

$$\begin{gathered} \dot{k}_i(t) = G(k_i(t)) - u_i(t), \\ k_i(0) = k_{i0}. \end{gathered} \tag{3.2}$$

Usually, the search for the feedback (or Markov-perfect) equilibrium is pursued by solving the related system of HJB equations, having the optimal value functions $V_i(k_i, t)$ as its unknowns.

I am going to confine my attention to the explicit solution to the related HJB equations, without taking into account the issue of sufficient conditions. Such a topic is widely discussed in many important contributions such as the textbooks by Bertsekas [3, 4], and the volumes treating dynamic programming with applications to economics and management science, i.e. Kamien and Schwartz ([14] is the most recent edition) and Dockner et al. [9]. Basically, under simple regularity assumptions which are verified in most solvable models, the existence of a solution to the HJB equations corresponds to the existence of a feedback solution to the differential game.

Call k_i the i-th level of capacity at time t, where $t \in [0, T)$ is the initial time instant of the game. Hence, the 2 arguments of V_i are k_i, i.e. the initial level of capital, and t, i.e. the initial time. Hence, $k = (k_1, \dots, k_N)$ is the vector of initial data, and the i-th player's optimal value function is

$$V_i(k_i, t) = \int_t^T e^{-\delta s} \left[\pi_i(u(s), s) - C_i(k_i)\right] ds. \tag{3.3}$$

The i-th HJB equation reads

$$\begin{aligned} \frac{\partial V_i(k,t)}{\partial t} - \rho V_i(k,t) = \max_{u_i \geq 0} \Big\{ &\pi_i(u(t), t) - C_i(k_i) + \frac{\partial V_i(k,t)}{\partial k_i}(G(k_i) - u_i) \\ &+ \sum_{j \neq i} \frac{\partial V(k_i, t)}{\partial k_j}(G(k_j) - u_j) \Big\}, \end{aligned} \tag{3.4}$$

endowed with the transversality condition $V_i(k_i, T) = 0$, representing the vanishing of (3.3) at the final instant of the game.

The concept of Nash feedback equilibrium of a game deserves to be briefly recalled: an N-tuple $(u_1^*(k,t), \dots, u_N^*(k,t))$ is a feedback Nash equilibrium if for all $j = 1, \dots, N$, $u_j^*(k,t)$ is a maximizer of V_j when all the remaining players play strategy $u_l^*(k,t)$, for all $l \neq j$.

An approach which is commonly adopted in such problems involves the determination of symmetric solutions, i.e. such as $u_1^* = u_2^* = \dots = u_N^*$, which requires suitable symmetry assumptions and basically transforms a differential game into an optimal control problem with a single agent. What follows is an Example showing the derivation of a feedback equilibrium in a problem where the inverse demand function of the market is hyperbolic.

Example 3.1. Sticking to the above notation, consider a market in which 3 firms compete over quantity and where the inverse demand function is hyperbolic, i.e.

$$p(u_1, u_2, u_3) = \frac{A}{\sum_{j=1}^3 u_j}, \tag{3.5}$$

where $A > 0$ is the market reservation price. See [15] and [16] for a theoretical explanation. The dynamic constraints are the kinematic equations:

$$\dot{k}_i(t) = \alpha k_i(t) - u_i(t),$$

where $\alpha > 0$ indicates the growth rate of the physical capital. The cost induced by the development of k_i for the i-th firm is $C_i(k_i) = \frac{k_i^2}{5} + \frac{k_i^4}{10}$, which is convex for $k_i \geq 0$. Given such data, the PDE (3.4) becomes

$$\begin{aligned} &\frac{\partial V_i(k,t)}{\partial t} - \rho V_i(k,t) \\ &= \max_{u_i \geq 0} \Big\{ \Big(\frac{A}{\sum_{j=1}^3 u_j} - c \Big) u_i - \frac{k_i^2}{5} - \frac{k_i^4}{10} + \frac{\partial V_i(k_i,t)}{\partial k_i}(\alpha k_i - u_i) \\ &\quad + \sum_{j \neq i} \frac{\partial V_i(k_i,t)}{\partial k_j}(\alpha k_j - u_j) \Big\}. \end{aligned}$$

As can be simply verified, the expression to be maximized on the right-hand side is concave in variables u_i, meaning that the existence of maximizers is ensured. Maximizing the expression on the right-hand side yields (whenever possible, arguments are omitted to lighten the notation):

$$-c + A\frac{\sum_{j\neq i} u_j}{(u_1+u_2+u_3)^2} - \frac{\partial V_i}{\partial k_i} = 0,$$

leading to the 3-equation system

$$\begin{aligned}
-c + A\frac{u_2+u_3}{(u_1+u_2+u_3)^2} - \frac{\partial V_1}{\partial k_1} &= 0,\\
-c + A\frac{u_1+u_3}{(u_1+u_2+u_3)^2} - \frac{\partial V_2}{\partial k_2} &= 0,\\
-c + A\frac{u_1+u_2}{(u_1+u_2+u_3)^2} - \frac{\partial V_3}{\partial k_3} &= 0.
\end{aligned}$$

Summing the equations yields

$$u_1+u_2+u_3 = \frac{2A}{3c + \frac{\partial V_1}{\partial k_1} + \frac{\partial V_2}{\partial k_2} + \frac{\partial V_3}{\partial k_3}}.$$

We impose symmetry in order to find out the symmetric solution, so we call $u := u_1 = u_2 = u_3$, $k := k_1 = k_2 = k_3$, and $V := V_1 = V_2 = V_3$. Consequently, we achieve a unique relation leading to the optimal output u^*:

$$u^* = \frac{2A}{9(c+\frac{\partial V}{\partial k})}.$$

Before proceeding, we should note that the search for a symmetric solution transforms the differential game into an optimal control problem, having only $V(k,t)$ as its unknown. However, if we assumed symmetry a priori, the quantity to be maximized would have become linear in the unique strategic variable u, and consequently would have admitted no maximum points. When such a maximization problem does not admit stationary points, we have to discuss the behaviour at the boundary of the strategy space.

On the other hand, the cross derivatives $\frac{\partial V_i}{\partial k_j}$ are not meaningful any longer, hence they should be removed from the unique HJB equation. By replacing u^* into the unique HJB we achieve the following:

$$\begin{aligned}
&\frac{\partial V}{\partial t} - \rho V = \frac{A}{3} - \frac{2Ac}{9\big(c+\frac{\partial V}{\partial k}\big)} - \frac{k_i^2}{5} - \frac{k_i^4}{10} + \alpha k\frac{\partial V}{\partial k} - \frac{2A}{9\big(c+\frac{\partial V}{\partial k}\big)}\frac{\partial V}{\partial k}\\
&\iff \frac{\partial V}{\partial t} - \rho V = \frac{A}{3} - \frac{2A}{9\big(c+\frac{\partial V}{\partial k}\big)}\big(c+\frac{\partial V}{\partial k}\big) - \frac{k_i^2}{5} - \frac{k_i^4}{10} + \alpha k\frac{\partial V}{\partial k}\\
&\iff \frac{\partial V(k,t)}{\partial t} - \rho V(k,t) = \frac{A}{9} - \frac{k_i^2}{5} - \frac{k_i^4}{10} + \alpha k\frac{\partial V(k,t)}{\partial k}.
\end{aligned}$$

which is a PDE belonging to the class of (2.8), with parameters $\beta_0 = \frac{A}{9}$, $\beta_2 = -\frac{1}{5}$, $\beta_4 = -\frac{1}{10}$, $\beta_j = 0$ for all $j \in \mathbb{Z}_+ \setminus \{0,2,4\}$. We can directly apply formula (2.10) to achieve the optimal value function

$$V^*(k,t) = \frac{A\left(e^{\rho(t-T)}-1\right)}{9\rho} - \frac{\left(e^{(\rho+2\alpha)(t-T)}-1\right)k^2}{5(\rho+2\alpha)} - \frac{\left(e^{(\rho+4\alpha)(t-T)}-1\right)k^4}{10(\rho+4\alpha)},$$

and finally, by substitution, the optimal feedback strategy:

$$u^*(k,t) = \frac{2A}{9\Big(c - \frac{2\left(e^{(\rho+2\alpha)(t-T)}-1\right)k}{5(\rho+2\alpha)} - \frac{2\left(e^{(\rho+4\alpha)(t-T)}-1\right)k^3}{5(\rho+4\alpha)}\Big)}.$$

Concluding Remarks. Analytic solutions have been worked out for a class of HJB equations arising from a differential game of oligopoly among firms engaging in competition over outputs in a market subject to a hyperbolic inverse demand function.

Possible future developments of the present work are either the derivation of the solutions to (2.1) if parameter γ is different from 0 and 1 or a discussion on the same problem played over an infinite time horizon. Furthermore, hyperbolic demand structures are an issue which deserves to be further developed in general, also in settings which are far from differential games scenarios.

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