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MULTIPLE POSITIVE SOLUTIONS FOR NONLOCAL PROBLEMS INVOLVING A SIGN-CHANGING POTENTIAL

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Abstract. In this article we show the existence and multiplicity of positive solutions for the nonlocal problem with a sign-changing weight function,

$$-(a-b\int_\Omega |\nabla u|^2dx)\Delta u = f_\lambda(x)|u|^{q-2}u, \quad \text{in } \Omega,$$
$$u=0, \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a,b>0$, $1<q<2$. Our technical approach is based on the variational method.

1. Introduction and statement of main results

In this article, we are interested in finding the existence of positive solutions to the nonlocal problem

$$\begin{gathered} -M(\int_\Omega |\nabla u|^2dx)\Delta u = \lambda f(x,u) \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N\ge 3$. When the continuous function $M:\mathbb{R}^+\to\mathbb{R}^+$ satisfies certain conditions, (1.1) has been investigated by many researchers by imposing different types of hypotheses on $f(x,u)$; see for example [3, 7, 8, 6, 5, 9, 10, 12, 15, 11, 14]. However, observing the all above studies, we see that the function M is assumed to be bounded from below. Recently, Yin and Liu [13] investigated the existence and multiplicity of nontrivial solutions for the a nonlocal problem

$$\begin{gathered} -(a-b\int_\Omega |\nabla u|^2dx)\Delta u = |u|^{p-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $N\ge 1, a,b>0$ are constants and $2<p<p^*$. They obtained a nontrivial non-negative solution and a nontrivial

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non-positive solution by using the mountain-pass lemma. Motivated by their work, we consider the equation

$$\begin{gathered}-(a-b\int_\Omega |\nabla u|^2dx)\Delta u = f_\lambda(x)|u|^{q-2}u, \quad \text{in } \Omega,\\ u=0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $a,b>0, 1<q<2$, the weight function $f_\lambda \in L^\infty(\Omega)$, defined by $f_\lambda = \lambda f_+ + f_-$, $\lambda>0$, with $f_\pm = \pm\max\{\pm f, 0\} \not\equiv 0$. An interesting question is whether multiplicity of positive solutions can be established for (1.3). We shall give a positive answer to this question.

Our main existence and multiplicity results for (1.3) can be stated as follows.

Theorem 1.1. *Assume that $a,b>0$, $1<q<2$ and $f\in L^\infty(\Omega)$. Then there exists $\lambda_*>0$, such that for any $\lambda\in(0,\lambda_*)$, problem (1.3) has at least two positive solutions.*

In this article, we use the following notation: The space $H_0^1(\Omega)$ is equipped with the norm $\|u\|^2 = \int_\Omega |\nabla u|^2dx$, the norm in $L^p(\Omega)$ is denoted by $\|\cdot\|_p$. $C, C_1, C_2, \dots$ denote various positive constants, which may vary from line to line. We denote by B_r (respectively, ∂B_r) the closed ball (respectively, the sphere) of center zero and radius r, i.e. $B_r = \{u\in H_0^1(\Omega) : \|u\|\le r\}$, $\partial B_r = \{u\in H_0^1(\Omega) : \|u\| = r\}$. Let S be the best Sobolev embedding constant for the embedding $H_0^1(\Omega) \hookrightarrow L^6(\Omega)$, namely

$$S = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2dx}{(\int_\Omega |u|^6dx)^{\frac{2}{3}}}.$$

2. Proof of main theorem

2.1. **Existence of a first positive solution of** (1.3)**.** We define the functional

$$I_\lambda(u) = \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx.$$

A function u is called a solution of (1.3) if $u\in H_0^1(\Omega)$ and for all $v\in H_0^1(\Omega)$ it holds

$$(a-b\|u\|^2)\int_\Omega (\nabla u, \nabla v)dx - \int_\Omega f_\lambda(x)|u|^{q-2}uv\,dx = 0.$$

To prove our main theorem, some preliminary results are needed. We first recall the following lemma from [4].

Lemma 2.1. *Let $r,s>1$, $\psi\in L^s(\Omega)$ and $\psi^+ = \max\{\psi, 0\}\neq 0$. Then there exists $\varphi_0 \in C_0^\infty(\Omega)$ such that $\int_\Omega \psi(x)|\varphi_0|^r dx > 0$.*

Lemma 2.2. *Assume $a,b>0, 1<q<2$ and $f\in L^\infty(\Omega)$, then I_λ satisfies the $(PS)_c$ condition with $c<\frac{a^2}{4b} - D\lambda$, where*

$$D = (\frac{1}{q}-\frac{1}{4})|f_+|S^{-q/2}|\Omega|^{(6-q)/6}m^q.$$

Proof. Let $\{u_n\}\subset H_0^1(\Omega)$ be a $(PS)_c$ sequence for I_λ, i. e.,

$$I_\lambda(u_n)\to c, \quad I'_\lambda(u_n)\to 0, \quad \text{as } n\to\infty. \tag{2.1}$$

From (2.1) it follows that

$$\begin{aligned} b\|u_n\|^4 &= a\|u_n\|^2 - \int_\Omega f_\lambda(x)|u_n|^q dx \\ &\le a\|u_n\|^2 - \int_\Omega f_-(x)|u_n|^q dx \\ &\le a\|u_n\|^2 + |f_-|S^{-q/2}|\Omega|^{(6-q)/6}\|u_n\|^q, \end{aligned}$$

so that

$$b\|u_n\|^{4-q} \le a\|u_n\|^2 + |f_-|S^{-q/2}|\Omega|^{(6-q)/6},$$

which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$; that is, there is $m > 0$ (independent of λ) such that $\|u_n\| \le m$ for every n. Moreover, there exist a subsequence (still denoted by $\{u_n\}$) and $u_* \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_* \quad \text{weakly in } H_0^1(\Omega), \\ u_n \to u_* \quad \text{strongly in } L^p(\Omega)\ (1 \le p < 6), \\ u_n(x) \to u_*(x) \quad \text{a.e. in } \Omega \end{gathered}$$

as $n \to \infty$. It follows easily from Vitali Convergence Theorem that

$$\lim_{n\to\infty} \int_\Omega f_\lambda(x)|u_n|^q dx = \int_\Omega f_\lambda(x)|u_*|^q dx.$$

Set $w_n = u_n - u_*$, then $\|w_n\| \to 0$. Otherwise, there exists a subsequence (still denoted by w_n) such that

$$\lim_{n\to\infty} \|w_n\| = l > 0.$$

From (2.1), for every $\phi \in H_0^1(\Omega)$, it holds

$$(a - b\|u_n\|^2) \int_\Omega (\nabla u_n, \nabla\phi) dx - \int_\Omega f_\lambda(x)|u_n|^{q-2}u_n\phi dx = o(1).$$

Letting $n \to \infty$, by using the Brézis-Lieb's lemma (see [2]), it holds

$$(a - bl^2 - b\|u_*\|^2) \int_\Omega (\nabla u_*, \nabla\phi) dx - \int_\Omega f_\lambda(x)|u_*|^{q-2}u_*\phi dx = 0. \tag{2.2}$$

Taking the test function $\phi = u_*$ in (2.2), it holds

$$(a - bl^2 - b\|u_*\|^2)\|u_*\|^2 - \int_\Omega f_\lambda(x)|u_*|^q dx = 0. \tag{2.3}$$

Note that $\langle I_\lambda'(u_n), u_n\rangle \to 0$ as $n \to \infty$, it holds

$$a\|w_n\|^2 + a\|u_*\|^2 - b\|w_n\|^4 - 2b\|w_n\|^2\|u_*\|^2 - b\|u_*\|^4 - \int_\Omega f_\lambda|u_*|^q dx = o(1). \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$a\|w_n\|^2 - b\|w_n\|^4 - b\|w_n\|^2\|u_*\|^2 = o(1). \tag{2.5}$$

Consequently $l^2(a - b\|u_*\|^2 - bl^2) = 0$, $l > 0$, so that

$$l^2 = \frac{a}{b} - \|u_*\|^2.$$

On the one hand, recalling that $\|u_n\| \le m$ and using (2.3), it follows

$$I_\lambda(u_*) = \frac{a}{2}\|u_*\|^2 - \frac{b}{4}\|u_*\|^4 - \frac{1}{q}\int_\Omega f_\lambda(x)|u_*|^q dx$$

$$
\begin{aligned}
&= \frac{a}{4}\|u_*\|^2 + \frac{b}{4}l^2\|u_*\|^2 - (\frac{1}{q} - \frac{1}{4})\int_\Omega f_\lambda(x)|u_*|^q dx \\
&\geq \frac{a}{4}\|u_*\|^2 + \frac{b}{4}l^2\|u_*\|^2 - \lambda(\frac{1}{q} - \frac{1}{4})|f_+|S^{-q/2}|\Omega|^{(6-q)/6}\|u_*\|^q \\
&\geq \frac{a}{4}\|u_*\|^2 + \frac{b}{4}l^2\|u_*\|^2 - \lambda(\frac{1}{q} - \frac{1}{4})|f_+|S^{-q/2}|\Omega|^{(6-q)/6}m^q \\
&= \frac{a}{4}\|u_*\|^2 + \frac{b}{4}l^2\|u_*\|^2 - D\lambda,
\end{aligned}
$$

where $D = \big(\frac{1}{q} - \frac{1}{4}\big)|f_+|S^{-q/2}|\Omega|^{(6-q)/6}m^q$.

On the other hand, by (2.1) and (2.5), it holds

$$
\begin{aligned}
&I_\lambda(u_*) \\
&= I_\lambda(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_*\|^2 + o(1) \\
&\leq I_\lambda(u_n) - \frac{a}{2}\|w_n\|^2 + \frac{1}{4}\left(a\|w_n\|^2 - b\|w_n\|^2\|u_*\|^2\right) + \frac{b}{2}\|w_n\|^2\|u_*\|^2 + o(1) \\
&= c - \frac{a}{4}\|w_n\|^2 + \frac{b}{4}\|w_n\|^2\|u_*\|^2 + o(1) \\
&< \frac{a^2}{4b} - D\lambda - \frac{a}{4}\left(\frac{a}{b} - \|u_*\|^2\right) + \frac{b}{4}l^2\|u_*\|^2 \\
&= \frac{a}{4}\|u_*\|^2 + \frac{b}{4}l^2\|u_*\|^2 - D\lambda.
\end{aligned}
$$

This is a contradiction. Therefore, $l = 0$, it implies that $u_n \to u_*$ in $H_0^1(\Omega)$. The proof is complete. □

Lemma 2.3. *There exist $R, \rho, \Lambda_0 > 0$, such that for each $\lambda \in (0, \Lambda_0)$, we have*

$$
\inf_{u\in\overline{B_R(0)}} I_\lambda(u) < 0 \quad \text{and} \quad I_\lambda|_{u\in\overline{\partial B_R(0)}} > \rho.
$$

Proof. For $u \in H_0^1(\Omega)$, it holds

$$
\begin{aligned}
I_\lambda(u) &= \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{1}{q}\int_\Omega f_\lambda(x)|u|^q dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}\int_\Omega f_+|u|^q dx \\
&\geq \frac{a}{2}\|u\|^2 - \frac{b}{4}\|u\|^4 - \frac{\lambda}{q}|f_+|S^{-q/2}|\Omega|^{(6-q)/6}\|u\|^q \\
&= \|u\|^q\big\{\frac{a}{2}\|u\|^{2-q} - \frac{b}{4}\|u\|^{4-q} - \frac{\lambda}{q}|f_+|S^{-q/2}|\Omega|^{(6-q)/6}\big\}.
\end{aligned}
$$

Set $h(t) = \frac{a}{2}t^{2-q} - \frac{b}{4}t^{4-q}$, we see that there exists a constant $R = \left(\frac{2a(2-q)}{b(4-q)}\right)^{1/2} > 0$ such that $\max_{t>0} h(t) = h(R) > 0$. Letting $\Lambda_0 = \frac{qS^{q/2}h(R)}{2|f_+||\Omega|^{(6-q)/6}}$, it follows that $I_\lambda|_{\|u\|=R} > 0$ for each $\lambda \in (0, \Lambda_0)$.

By Lemma 2.1, there exists $\varphi_0 \in C_0^\infty(\Omega) \subset H_0^1(\Omega)$ such that

$$
\int_\Omega f_\lambda(x)|\varphi_0|^q dx > 0.
$$

Applying the result, it holds

$$\lim_{t\to 0^+}\frac{I_\lambda(t\varphi_0)}{t^q}=-\frac{1}{q}\int_\Omega f_\lambda(x)|\varphi_0|^q dx<0.$$

therefore, when t is enough small, we have $I_\lambda(t\varphi_0)<0$. Thus there exists u small enough such that $I_\lambda(u)<0$. Then we deduce that

$$d=\inf_{u\in\overline{B_R(0)}} I_\lambda(u)<0<\inf_{u\in\overline{\partial B_R(0)}} I_\lambda(u). \tag{2.6}$$

□

Theorem 2.4. *Assume $a,b>0$, $1<q<2$ and $f\in L^\infty(\Omega)$, problem* (1.3) *has a positive solution u_λ with $I_\lambda(u_\lambda)<0$.*

Proof. From (2.6), by applying Ekeland's variational principle in $\overline{B_R(0)}$, there exists a minimizing sequence $\{u_n\}\subset\overline{B_R(0)}$ such that

$$I_\lambda(u_n)\le\inf_{u\in\overline{B_R(0)}} I_\lambda(u)+\frac{1}{n},\quad I_\lambda(v)\ge I_\lambda(u_n)-\frac{1}{n}\|v-u_n\|,\quad v\in\overline{B_R(0)}.$$

Therefore,

$$I'_\lambda(u_n)\to 0\quad\text{and}\quad I_\lambda(u_n)\to d.$$

Since $\{u_n\}$ is bounded and $\overline{B_R(0)}$ is a closed convex set, there exist $u_\lambda\in\overline{B_R(0)}\subset H_0^1(\Omega)$ and a subsequence still denoted by $\{u_n\}$, such that $u_n\rightharpoonup u_\lambda$ in $H_0^1(\Omega)$ as $n\to\infty$.

Note that $I_\lambda(|u_n|)=I_\lambda(u_n)$, by Lemma 2.2, we can obtain $u_n\to u_\lambda$ in $H_0^1(\Omega)$ and $d=\lim_{n\to\infty}I_\lambda(u_n)=I_\lambda(u_\lambda)<0$, which suggests that $u_\lambda\ge 0$ and $u_\lambda\not\equiv 0$. Since $u_\lambda\in H_0^1(\Omega)$, by the embedding theorem we get $u_\lambda\in L^6(\Omega)$. Besides, as $f_\lambda\in L^\infty(\infty)$, by the regularity of weak solutions, it holds $u_\lambda\in W^{2,\frac{6}{q}}(\Omega)$. By the embedding theorem again, it holds that $u_\lambda\in C^{1,\alpha}(\Omega)$. Therefore, by the Harnack inequality, we obtain $u_\lambda>0$ a.e. in Ω. The proof is complete. □

2.2. **Existence of a second positive solution of** (1.3)**.**

Lemma 2.5. *Assume that $\lambda\in(0,\Lambda_0)$, for given R, the functional I_λ satisfies the following conditions:*

(i) *$I_\lambda(u)>0$ if $u\in S_R$,*
(ii) *there exists $e\in H_0^1(\Omega)$ such that $I_\lambda(e)<0$ when $\|e\|>R$.*

Proof. (i) The conclusion follows from Lemma 2.3 when $\lambda<\Lambda_0$.

(ii) For $u\in H_0^1(\Omega)\backslash\{0\}$, it holds

$$\begin{aligned} I_\lambda(tu)&=\frac{at^2}{2}\|u\|^2-\frac{bt^4}{4}\|u\|^4-\frac{t^q}{q}\int_\Omega f_\lambda(x)|u|^q dx\\ &\le\frac{at^2}{2}\|u\|^2-\frac{bt^4}{4}\|u\|^4+\frac{t^q}{q}\int_\Omega |f_\lambda(x)||u|^q dx\to-\infty \end{aligned}$$

as $t\to+\infty$. Therefore we can find $e\in H_0^1(\Omega)$ such that $I_\lambda(e)<0$ when $\|e\|>R$. The proof is complete. □

It is well known that the function

$$U_\varepsilon(x)=\frac{(3\varepsilon^2)^{1/4}}{(\varepsilon^2+|x|^2)^{1/2}},\quad x\in\mathbb{R}^3,\ \varepsilon>0$$

satisfies

$$-\Delta U_\varepsilon = U_\varepsilon^5 \quad \text{in } \mathbb{R}^3,$$
$$\int_{\mathbb{R}^3} |U_\varepsilon|^6 = \int_{\mathbb{R}^3} |\nabla U_\varepsilon|^2 = S^{3/2}.$$

Let $\eta \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \eta \le 1$, $|\nabla\eta| \le C$ and $\eta(x) = 1$ for $|x| < R_0$ and $\eta(x) = 0$ for $|x| > 2R_0$, we set $u_\varepsilon(x) = \eta(x)U_\varepsilon(x)$. Then it holds

$$\|u_\varepsilon\|^2 = S^{3/2} + O(\varepsilon),$$
$$|u_\varepsilon|_6^6 = S^{3/2} + O(\varepsilon^3).$$

Lemma 2.6. *Assume $a, b > 0$, $1 < q < 2$ and $f \in L^\infty(\Omega)$, it holds*

$$\sup_{t\ge 0} I_\lambda(u_\lambda + tu_\varepsilon) < \frac{a^2}{4b} - D\lambda.$$

Proof. Since u_λ is a positive solution of (1.3) and $I_\lambda(u_\lambda) < 0$, it holds

$$\begin{aligned}
&I_\lambda(u_\lambda + tu_\varepsilon)\\
&= \frac{a}{2}\|u_\lambda + tu_\varepsilon\|^2 - \frac{b}{4}\|u_\lambda + tu_\varepsilon\|^4 - \frac{1}{q}\int_\Omega f_\lambda(x)|u_\lambda + tu_\varepsilon|^q dx\\
&= \frac{a}{2}\|u_\lambda\|^2 + at\int_\Omega (\nabla u_\lambda, \nabla u_\varepsilon)dx + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{b}{4}\|u_\lambda\|^4 - \frac{bt^4}{4}\|u_\varepsilon\|^4\\
&\quad - bt\|u_\lambda\|^2\int_\Omega (\nabla u_\lambda, \nabla u_\varepsilon)dx - bt^2\Big(\int_\Omega (\nabla u_\lambda, \nabla u_\varepsilon)dx\Big)^2\\
&\quad - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 - bt^3\|u_\varepsilon\|^2\int_\Omega (\nabla u_\lambda, \nabla u_\varepsilon)dx - \frac{1}{q}\int_\Omega f_\lambda|u_\lambda + tu_\varepsilon|^q dx\\
&\le I_\lambda(u_\lambda) + \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2\\
&\quad + \int_\Omega |f_-|\Big\{\int_0^{tu_\varepsilon} [(u_\lambda + \eta)^{q-1} - u_\lambda^{q-1}]d\eta\Big\}dx\\
&\le \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 + \int_\Omega |f_-|\Big\{\int_0^{tu_\varepsilon} \eta^{q-1}d\eta\Big\}dx\\
&\le \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 + Ct^q\int_\Omega u_\varepsilon^q dx.
\end{aligned}$$

Set

$$g(t) = \frac{at^2}{2}\|u_\varepsilon\|^2 - \frac{bt^4}{4}\|u_\varepsilon\|^4 - \frac{bt^2}{2}\|u_\lambda\|^2\|u_\varepsilon\|^2 + Ct^q\int_\Omega u_\varepsilon^q dx.$$

We prove that there exist $t_\varepsilon > 0$ and positive constants t_1, t_2 independent of ε, λ, such that $\sup_{t\ge 0} g(t) = g(t_\varepsilon)$ and

$$0 < t_1 \le t_\varepsilon \le t_2 < \infty. \tag{2.7}$$

In deed, since $\lim_{t\to 0^+} g(t) > 0$, $\lim_{t\to +\infty} g(t) = -\infty$, there exists $t_\varepsilon > 0$ such that

$$g(t_\varepsilon) = \sup_{t\ge 0} g(t) \quad \text{and} \quad \frac{dg}{dt}|_{t=t_\varepsilon} = 0.$$

As in [9] it follows that (2.7) holds. Note that $\int_\Omega u_\varepsilon^q dx \le c\varepsilon^{q/2}$, then it holds that

$$\sup_{t\ge 0} I_\lambda(u_\lambda + tu_\varepsilon) \le \sup_{t\ge 0} g(t)$$

$$
\begin{aligned}
&\le \sup_{t\ge 0}\Big\{\frac{at^2}{2}\|u_\varepsilon\|^2-\frac{bt^4}{4}\|u_\varepsilon\|^4\Big\}-C_1\|u_\varepsilon\|^2+C_2\varepsilon^{q/2}\\
&\le \frac{a^2}{4b}+C_3\varepsilon-C_1S^{3/2}+C_2\varepsilon^{q/2}\\
&\le \frac{a^2}{4b}+(C_2+C_3)\varepsilon^{q/2}-C_1S^{3/2},
\end{aligned}
$$

where $C_i>0, i=1,2,3$. Let $\varepsilon=\lambda^{\frac{2}{q}}$, when $0<\lambda<\Lambda_1:=\frac{C_1S^{3/2}}{C_2+C_3+D}$, it holds

$$
(C_2+C_3)\lambda-C_1S^{3/2}<(C_2+C_3)\lambda-(C_2+C_3+D)\lambda=-D\lambda.
$$

Consequently, $\sup_{t\ge 0}I_\lambda(u_\lambda+tu_\varepsilon)<\frac{a^2}{4b}-D\lambda$. The proof is complete. □

Theorem 2.7. *Assume that $b>0, 1<q<2$ and $f\in L^\infty(\Omega)$, there exists $\lambda_*>0$ such that for each $\lambda\in(0,\lambda_*)$, problem (1.3) admits a positive solution v_λ with $I_\lambda(v_\lambda)>0$.*

Proof. Let $\lambda_*=\min\{\Lambda_0,\Lambda_1,\frac{a^2}{4bD}\}$, then Lemmas 2.1, 2.2, 2.3 2.5, and 2.6 hold for $\lambda<\lambda_*$. Applying the mountain-pass lemma [1], there is a sequence $\{u_n\}\subset H_0^1(\Omega)$ such that

$$
I_\lambda(u_n)\to c>0 \quad\text{and}\quad I'_\lambda(u_n)\to 0,
$$

where

$$
c=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}I_\lambda(\gamma(t)),
$$
$$
\Gamma=\big\{\gamma\in C([0,1],H_0^1(\Omega)):\gamma(0)=u_\lambda,\gamma(1)=e\big\}.
$$

From Lemma 2.2, $\{u_n\}$ has a convergent subsequence (still denoted by $\{u_n\}$) and there exists $v_\lambda\in H_0^1(\Omega)$ such that $u_n\to v_\lambda$ in $H_0^1(\Omega)$. Moreover, we can obtain v_λ is a non-negative weak solution of (1.3) and

$$
I_\lambda(v_\lambda)=\lim_{n\to\infty}I_\lambda(u_n)=c>0.
$$

Therefore, we infer that $v_\lambda\not\equiv 0$. It is similar to Theorem 2.4 that $v_\lambda>0$ a.e. in Ω. The proof is complete. □

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