UNIQUENESS OF TRAVELING WAVE SOLUTIONS FOR NON-MONOTONE CELLULAR NEURAL NETWORKS WITH DISTRIBUTED DELAYS

HUI-LING ZHOU, ZHIXIAN YU

Abstract. In this article, we study the uniqueness of traveling wave solutions for non-monotone cellular neural networks with distributed delays. First we establish a priori asymptotic behavior of the traveling wave solutions at infinity. Then, based on Ikehara’s theorem, we prove the uniqueness of the solution \( \psi(n-ct) \) with \( c \leq c^* \), where \( c^* < 0 \) is the critical wave speed.

1. Introduction

In this article, we study the uniqueness of traveling wave solution for the non-monotone cellular neural networks with distributed delays

\[
x^\prime_n(t) = -x_n(t) + \sum_{i=1}^{m} a_i J_i(y) f(x_{n-i}(t-y))dy + \alpha \int_{0}^{\tau} J_{m+1}(y)f(x_n(t-y))dy + \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)f(x_{n+j}(t-y))dy,
\]

where the constants \( n \in \mathbb{Z}, m, l \in \mathbb{N}, \tau \geq 0 \), and the variable \( t \in \mathbb{R} \). We use the following assumptions:

(H0) (i) \( \alpha > 0, a_1 > 0, a_i \geq 0 \) \( (i = 2, \ldots, m), \beta_1 > 0 \) and \( \beta_j \geq 0 \) \( (j = 2, \ldots, l) \).

(ii) \( a = \sum_{i=1}^{m} a_i \) and \( \beta = \sum_{j=1}^{l} \beta_j \).

(iii) \( J_i : [0, \tau] \rightarrow (0, +\infty) \) is the piecewise continuous function satisfying \( \int_{0}^{\tau} J_i(y)dy = 1 \), where \( 0 < \tau < \infty \).

(H1) \( f \in C([0,b], [0, \frac{b}{y+\alpha+\beta}])), f(0) = 0, \alpha f'(0) \geq 1 \) and there exists \( K > 0 \) with \( K \leq b \) such that

\( (a + \alpha + \beta)f(K) = K, \quad |f(u) - f(v)| \leq f'(0)|u - v| \) for \( u, v \in [0, b] \).

(H2) \( (a + \alpha + \beta)f(u) > u \) for \( u \in (0, K) \) and \( (a + \alpha + \beta)f(u) < u \) for \( u \in (K, b] \).

(H3) There exist \( \sigma > 0, \delta > 0 \) and \( M > 0 \) such that

\( f(u) \geq f'(0)u - Mu^{1+\sigma} \) for \( u \in [0, \delta] \).

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A traveling wave solution \([1.1]\) with speed \(c\) is a nonnegative bounded solution of the form \(u_n(t) = \psi(n - ct)\) satisfying \(\psi(-\infty) = 0\) and \(\liminf_{\xi \to +\infty} \psi(\xi) > 0\). Substituting \(u_n(t) = \psi(n - ct)\) in \([1.1]\), we have the wave profile equation

\[
-c\psi'(\xi) = -\psi(\xi) + \sum_{i=1}^{m} a_i J_i(y) f(\psi(\xi - i + cy)) dy + \alpha \int_{0}^{\tau} J_{m+1}(y) f(\psi(\xi + cy)) dy + \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) f(\psi(\xi + j + cy)) dy.
\]  

(1.2)

When the output function \(f\) is monotone, the existence of traveling wave solutions for many versions of CNNs \([1.1]\) with delays or without delays has been widely investigated, see for example \([10, 11, 13, 14, 15, 16, 17, 18, 21, 23, 24, 28, 30, 26]\). The existence of entire solutions for \([1.1]\) has been investigated by Wu and Hsu \([23, 24]\). Letting \(J_i = \delta(y - \tau_i), i = 1, \ldots, m + l + 1\), \([1.1]\) reduces to the multiple discrete delays equation

\[
w_n'(t) = -w_n(t) + \sum_{i=1}^{m} a_i f(w_{n-i}(t - \tau_i)) + \alpha f(w_{n}(t - \tau_{m+1})) + \sum_{j=1}^{l} \beta_j f(w_{n+j}(t - \tau_{m+1+j})).
\]  

(1.3)

Yu and Mei \([28]\) investigated uniqueness and stability of traveling wave solutions for \([1.3]\) with the monotone output function. In \([28]\) the authors used the technique in \([3]\) to study uniqueness of travelling wave solutions for \([1.3]\) with discrete delays. We will extend this method to \([1.1]\) with distributed delays.

For the non-monotone output function \(f\), Yu et al. \([27]\) only established the existence of non-critical traveling wave solutions. Yu and Zhao \([31]\) further established the existence of the spreading speed, its coincidence with the minimal wave speed and the existence of critical waves for the non-monotone DCNNs \([1.1]\). We summarize the existence of traveling wave solutions of \([1.1]\) with the non-monotone output function in \([27, 31]\) as follows.

**Proposition 1.1.** Assume that \((H0)-(H3)\) hold. Then there exists \(c^* < 0\) (which is given in Lemma 2.1) such that for any \(c \leq c^*\), \([1.1]\) admits a non-negative traveling wave solution \(\psi(n - ct)\) with the wave speed \(c^* < 0\) and satisfying

\[
\psi(-\infty) = 0 \quad \text{and} \quad 0 < \liminf_{\xi \to +\infty} \psi(\xi) \leq \limsup_{\xi \to +\infty} \psi(\xi) \leq b.
\]  

(1.4)

The uniqueness of monotone travelling wave solutions for various evolution systems has been established; see for example \([11, 24, 25, 26, 29, 20]\) and the references therein. The proof of uniqueness strongly relies on the monotonicity of travelling waves. It seems very difficult to extend the techniques in those literatures to the non-monotone evolution systems because the wave profile may lose the monotonicity and the study of the corresponding uniqueness is very limited, see, e.g., \([6, 8, 9]\). Recently, the authors in \([25, 29]\) extend the technique in \([3]\) to non-monotone lattice equations with discrete delays. In this article, we extend the technique in \([3]\) to non-monotone CNNs with distributed delays.
The rest of this article is organized as follows. Section 2 is devoted to studying the asymptotic behavior of the traveling wave solutions. In Section 3, we prove the uniqueness of the solution.

2. Asymptotic behavior of traveling wave solutions

In this section, we consider the asymptotic behavior at negative infinity of any traveling wave solutions of (1.1). The characteristic equation of (1.2) at 0 is

$$\Delta(c, \lambda) = -c \lambda + 1 - f'(0) \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y) e^{\lambda(-i+c)y} dy \right. + \alpha \int_{0}^{\tau} J_{m+1}(y) e^{\lambda cy} dy + \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y) e^{\lambda(j+c)y} dy \left. \right].$$

(2.1)

Lemma 2.1 ([26, Lemma 2.1]). Assume that (H0) and $\alpha f'(0) \geq 1$ hold. Then there exist a unique pair of $c^* < 0$ and $\lambda^* > 0$ such that

(i) $\Delta(c^*, \lambda^*) = 0$,

(ii) for any $c > c^*$ and $\lambda \in [0, +\infty)$, $\Delta(c, \lambda) < 0$;

(iii) for any $c < c^*$, $\Delta(c, \lambda) = 0$ has two positive roots $\lambda_2 \geq \lambda_1 > 0$. Moreover, if $c < c^*$, then $\lambda_1 = \lambda_2 = \lambda^*$.

Now we give a different version of Ikehara’s Theorem, which can be found in [3].

Proposition 2.2. Let $F(\lambda) := \int_{0}^{+\infty} u(x) e^{-\lambda x} dx$, where $u(x)$ is a positive decreasing function. Assume $F(\lambda)$ can be written as

$$F(\lambda) = \frac{h(\lambda)}{(\lambda + \mu)^{k+1}},$$

where $k > -1$ and $h(\lambda)$ is analytic in the strip $-\mu \leq \Re \lambda < 0$. Then

$$\lim_{x \to +\infty} \frac{u(x)}{x^k e^{-\mu x}} = \frac{h(-\mu)}{\Gamma(\mu + 1)}.$$

Remark 2.3. Changing the variable $t = -x$, and modifying the proof for Ikehara’s Theorem given in [4], we can show the following version of Proposition 2.2. Let $F(\lambda) := \int_{-\infty}^{0} u(t) e^{-\lambda t} dt$, where $u(t)$ is a positive increasing function. Assume $F(\lambda)$ can be written as

$$F(\lambda) = \frac{h(\lambda)}{(\mu - \lambda)^{k+1}},$$

where $k > -1$ and $h(\lambda)$ is analytic in the strip $\mu - \epsilon < \Re \lambda \leq \mu$ for some $0 < \epsilon < \mu$. Then

$$\lim_{x \to -\infty} \frac{u(x)}{|x|^k e^{\mu x}} = \frac{h(\mu)}{\Gamma(\mu + 1)}.$$

To apply Ikehara’s Theorem, we need to assure that traveling wave solutions are positive.

Lemma 2.4. Assume that (H0)–(H3) hold and let $\psi(n - ct)$ be a non-negative traveling wave of (1.1) with $c \leq c^*$ satisfying (1.4). Then $\psi(\xi) > 0$ for $\xi \in \mathbb{R}$.
Proof. Assume that there exists $\xi_0$ such that $\psi(\xi_0) = 0$. Without loss of generality, we may assume $\xi_0$ is the left-most point. According to $\psi(\xi) \geq 0$ for $\xi \in \mathbb{R}$, we can easily see that $\psi(\xi)$ attains the minimum at $\xi_0$ and $\psi'(\xi_0) = 0$. According to (H0) and (H1), it follows from (1.2) that
\[
\int_0^\tau J_{m+1}(y) f(\psi(\xi_0 + cy)) dy = 0,
\]
which implies that $f(\psi(\xi_0 + cy)) = 0$ for any $y \in [0, \tau]$. Thus, choosing some sufficiently small number $y_0 > 0$, we can obtain $\psi(\xi_0 + cy_0) = 0$ according to the continuity of $\psi(\xi)$ and $c < 0$. This contradicts to the choice of $\xi_0$, and completes the proof. □

Lemma 2.5. Assume that (H1)--(H3) hold and let $\psi(n - ct)$ be any non-negative traveling wave of (1.1) with $c \leq c_*$ and satisfy (1.4). Then there exists a positive number $\rho > 0$ such that $\psi(\xi) = O(e^{\rho \xi})$ as $\xi \to -\infty$.

Proof. Since $f'(0)(a + \alpha + \beta) > 1$, there exists $\epsilon_0 > 0$ such that
\[
A := (1 - \epsilon_0)f'(0)(a + \alpha + \beta) - 1 > 0.
\]
For such $\epsilon_0 > 0$, there exist $\delta_1 > 0$ such that $f(u) \geq (1 - \epsilon_0)f'(0)u$ for any $u \in [0, \delta_1]$. Since $\psi(-\infty) = 0$, there exists $M > 0$ and $\forall \xi \leq -M$ such that $\psi(\xi) < \delta_1$. Integrating (1.2) from $\eta$ to $\xi$ with $\xi \leq -l - M$, it follows that
\[
- c[\psi(\xi) - \psi(\eta)] = - \int_\eta^\xi \psi(x) dx + \sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y) f(\psi(x - i + cy)) dy dx
\]
\[+ \alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y) f(\psi(x + cy)) dy dx
\]
\[+ \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y) f(\psi(x + j + cy)) dy dx
\]
\[\geq - \int_\eta^\xi \psi(x) dx + f'(0)(1 - \epsilon_0) \left[ \sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y) \psi(x - i + cy) dy dx
\]
\[+ \alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y) \psi(x + cy) dy dx
\]
\[+ \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y) \psi(x + j + cy) dy dx \right] (2.2)
\]
\[= A \int_\eta^\xi \psi(x) dx + f'(0)(1 - \epsilon_0) \left[ \alpha \int_\eta^\xi \int_0^\tau J_{m+1}(y) \psi(x + cy) - \psi(x) dy dx
\]
\[+ \sum_{i=1}^m a_i \int_\eta^\xi \int_0^\tau J_i(y) (\psi(x - i + cy) - \psi(x)) dy dx
\]
\[+ \sum_{j=1}^l \beta_j \int_\eta^\xi \int_0^\tau J_{m+1+j}(y) (\psi(x + j + cy) - \psi(x)) dy dx \right].
Since $\psi(x)$ is differentiable, we have
\[
\int_{\eta}^{\xi} (\psi(x - i + cy) - \psi(x))dx = \int_{\eta}^{\xi} \int_{0}^{i+cy} \psi'(x + s)dsdx
\]
\[
= \int_{0}^{i+cy} (\psi(\xi + s) - \psi(\eta + s))ds.
\]
Similarly,
\[
\int_{\eta}^{\xi} (\psi(x + cy) - \psi(x))dx = \int_{0}^{cy} (\psi(\xi + s) - \psi(\eta + s))ds,
\]
\[
\int_{\eta}^{\xi} (\psi(x + j + cy) - \psi(x))dx = \int_{0}^{j+cy} (\psi(\xi + s) - \psi(\eta + s))ds.
\]
Letting $\eta \to -\infty$ in (2.2), we obtain
\[
A \int_{-\infty}^{\xi} \psi(x)dx
\]
\[
\leq -c\psi(\xi) - f'(0)(1 - \epsilon_0) \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} \int_{0}^{i+cy} J_i(y)\psi(\xi + s)dy ds \right.
\]
\[
+ \alpha \int_{0}^{\tau} \int_{0}^{cy} J_{m+1}(y)\psi(\xi + s)dy ds
\]
\[
+ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} \int_{0}^{j+cy} J_{m+1+j}(y)\psi(\xi + s)dy ds \bigg] \tag{2.3}
\]
From (2.3), we know that $\int_{-\infty}^{\xi} \psi(x)dx < +\infty$. Letting $\Phi(\xi) = \int_{-\infty}^{\xi} \psi(x)dx$ and integrating (2.3) from $-\infty$ to $\xi$, we have
\[
A \int_{-\infty}^{\xi} \Phi(x)dx
\]
\[
\leq -c\Phi(\xi) - f'(0)(1 - \epsilon_0) \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} \int_{0}^{i+cy} J_i(y)\Phi(\xi + s)dy ds \right.
\]
\[
+ \alpha \int_{0}^{\tau} \int_{0}^{cy} J_{m+1}(y)\Phi(\xi + s)dy ds
\]
\[
+ \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} \int_{0}^{j+cy} J_{m+1+j}(y)\Phi(\xi + s)dy ds \bigg] \tag{2.4}
\]
for some $\kappa > 0$ and $\varrho > 0$ according to the monotonicity of $\Phi(\xi)$. Letting $\varpi > 0$ such that $\varrho < A\varpi$, and for $\xi \leq -l - M$, it follows that
\[
\Phi(\xi - \varpi) \leq \frac{1}{\varpi} \int_{\xi - \varpi}^{\xi} \Phi(x)dx \leq \frac{1}{\varpi} \int_{-\infty}^{\xi} \Phi(x)dx \leq \frac{\varrho}{A\varpi} \Phi(\xi + \kappa). \tag{2.5}
\]
Define $h(\xi) = \Phi(\xi)e^{-\rho\kappa}$, where $\rho = \frac{1}{\rho + \varpi} \ln \frac{4\varpi}{\varrho} > 0$. Hence,
\[
h(\xi - \varpi) = \Phi(\xi - \varpi)e^{-\rho(\xi - \varpi)} \leq \frac{\varrho}{A\varpi} e^{\rho(\kappa + \varpi)} h(\xi + \kappa) = h(\xi + \kappa),
\]
which implies \( h \) is bounded. Therefore, \( \Phi(\xi) = O(e^{\rho \xi}) \) when \( \xi \to -\infty \). Integrating (1.2) from \(-\infty\) to \( \xi \), it follows from (H2) that

\[
-c\psi(\xi) = \sum_{i=1}^{m} a_i \int_{0}^{T} \int_{-\infty}^{\xi} J_i(y) f(\psi(x - i + cy)) \, dx \, dy \\
+ \alpha \int_{0}^{T} \int_{-\infty}^{\xi} J_{m+1}(y) f(\psi(x + cy)) \, dx \, dy \\
+ \sum_{j=1}^{l} \beta_j \int_{0}^{T} \int_{-\infty}^{\xi} J_{m+1+j}(y) f(\psi(x + j + cy)) \, dx \, dy - \Phi(\xi)
\]

\[
\leq f'(0) \sum_{i=1}^{m} a_i \int_{0}^{T} \int_{-\infty}^{\xi} J_i(y) \psi(x - i + cy) \, dx \, dy \\
+ \alpha f'(0) \int_{0}^{T} \int_{-\infty}^{\xi} J_{m+1}(y) \psi(x + cy) \, dx \, dy \\
+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{T} \int_{-\infty}^{\xi} J_{m+1+j}(y) \psi(x + j + cy) \, dx \, dy - \Phi(\xi)
\]

\[
(2.6)
\]

Thus, we have \( \psi(\xi) = O(e^{\rho \xi}) \) when \( \xi \to -\infty \). With the help of Ikehara’s theorem, we obtain the asymptotic behavior of traveling wave solutions at \(-\infty\). □

**Proposition 2.6.** Assume that (H1)-(H3) hold and let \( \psi(n - ct) \) be any non-negative traveling wave of (1.1) with the wave speed \( c \leq c_* \) and satisfy (1.4). Then

\[
\lim_{\xi \to -\infty} \frac{\psi(\xi)}{e^{\lambda_1 \xi}} \text{ exists for } c < c_*, \quad \lim_{\xi \to -\infty} \frac{\psi(\xi)}{\xi \psi(\xi)} \text{ exists for } c = c_*. \quad (2.7)
\]

**Proof.** According to Lemma 2.5, we define the two-sided Laplace transform of \( \psi \) for \( 0 < \Re \lambda < \rho \),

\[
L(\lambda) = \int_{-\infty}^{+\infty} \psi(x) e^{-\lambda x} \, dx.
\]

We claim that \( L(\lambda) \) is analytic for \( 0 < \Re \lambda < \lambda_1 \) and has a singularity at \( \lambda = \lambda_1 \). Note that

\[
-c\psi'(\xi) + \psi(\xi) - f'(0) \sum_{i=1}^{m} a_i \int_{0}^{T} J_i(y) \psi(\xi - i + cy) \, dy \\
- \alpha f'(0) \int_{0}^{T} J_{m+1}(y) \psi(\xi + cy) \, dy
\]
\[-f'(0) \sum_{j=1}^{l} \beta_j \int_0^\tau J_i m + 1 + j\psi(\xi + j + cy)dy\]
\[= \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)[f(\psi(\xi - i + cy)) - f'(0)\psi(\xi - i + cy)]dy\]
\[+ \alpha \int_0^\tau J_{m+1}(y)[f(\psi(\xi + cy)) - f'(0)\psi(\xi - i + cy)]dy\]
\[+ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}[f(\psi(\xi + j + cy)) - f'(0)\psi(\xi - i + cy)]dy\]
\[=: Q(\psi)(\xi).\]

Multiplying the two sides of the above equality by $e^{-\lambda \xi}$ and integrating $\xi$ on $\mathbb{R}$, we obtain
\[
\Delta(c, \lambda)L(\lambda) = \int_{-\infty}^{+\infty} e^{-\lambda x} Q(\psi)(x)dx. \tag{2.8}
\]

We know that the left-hand side of (2.8) is analytic for $0 < \Re \lambda < \rho$. According to (H3), for any $\pi > 0$, there exists $\bar{d} > 0$ such that $f(u) \geq f'(0)u - \bar{d}u^{\sigma+1}$, for all $u \in [0, \pi]$, where $\bar{d} := \max\{d, \gamma^{-\sigma+1}\} \max_{u \in [\gamma, \pi]} \{f'(0)u - f(u)\}$. Thus,
\[
-\bar{d} \left[ \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)\psi^{\sigma+1}(\xi - i + cy)dy + \alpha \int_0^\tau J_{m+1}(y)\psi^{\sigma+1}(\xi + cy)dy \right. \\
+ \left. \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}\psi^{\sigma+1}(\xi + j + cy)dy \right]
\leq Q(\psi)(\xi) \leq 0. \tag{2.9}
\]

Choose $\nu > 0$ such that $\frac{\nu}{\sigma} < \rho$. Then for any $\Re \lambda \in (0, \rho + \nu)$, we have
\[
\left| \int_{-\infty}^{+\infty} e^{-\lambda x} Q(\psi)(x)dx \right|
\leq \bar{d} \int_{-\infty}^{+\infty} e^{-\lambda \xi} \left[ \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)\psi^{\sigma+1}(\xi - i + cy)dy \right. \\
+ \left. \alpha \int_0^\tau J_{m+1}(y)\psi^{\sigma+1}(\xi + cy)dy \right. \\
+ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}\psi^{\sigma+1}(\xi + j + cy)dy \left. \right] d\xi
\leq \bar{d} \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)e^{\lambda(-i+cy)}dy + \alpha \int_0^\tau J_{m+1}(y)e^{\lambda cy}dy \\
+ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}e^{\lambda(j+cy)}dy \int_{-\infty}^{+\infty} e^{-\lambda x} \psi^{\sigma+1}(x)dx
\leq \bar{d} \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)e^{\lambda(-i+cy)}dy + \alpha \int_0^\tau J_{m+1}(y)e^{\lambda cy}dy \\
+ \sum_{j=1}^{l} \beta_j \int_0^\tau J_{m+1+j}e^{\lambda(j+cy)}dy \int_{-\infty}^{+\infty} e^{-\lambda x} \psi^{\sigma+1}(x)dx
\leq \bar{d} \sum_{i=1}^{m} a_i \int_0^\tau J_i(y)e^{\lambda(-i+cy)}dy + \alpha \int_0^\tau J_{m+1}(y)e^{\lambda cy}dy.
has a singularity at $\lambda$ and $\psi > 0$ according to Lemma 2.4. Since $\psi > 0$ according to Lemma 2.4, there exists a real number $D$ such that $L(\lambda)$ is analytic for $0 < \Re \lambda < D$ and $L(\lambda)$ has a singularity at $\lambda = D$. Thus, when $c \leq c_*$, $L(\lambda)$ is analytic for $\Re \lambda \in (0, \lambda_1)$ and $L(\lambda)$ has a singularity at $\lambda = \lambda_1$.

According to (2.8), we have

$$ F(\lambda) := \int_0^\infty \psi(x)e^{-\lambda x}dx = \frac{\int_{-\infty}^{+\infty} e^{-\lambda x}Q(\psi)(x)dx}{\Delta(c, \lambda)} - \int_0^{+\infty} \psi(x)e^{-\lambda x}dx. $$

Define $H(\lambda) = F(\lambda)(\lambda_1 - \lambda)^{k+1}$, where $k = 0$ if $c < c_*$ and $k = 1$ if $c = c_*$. We claim that $H(\lambda)$ is analytic in the strip $S := \{ \lambda \in C| 0 < \Re \lambda \leq \lambda_1 \}$. Indeed, define

$$ G(\lambda) = \frac{\int_{-\infty}^{+\infty} e^{-\lambda x}Q(\psi)(x)dx}{\Delta(c, \lambda)/(\lambda_1 - \lambda)^{k+1}} = L(\lambda)(\lambda_1 - \lambda)^{k+1}. $$

It is easily seen that $G(\lambda)$ is analytic in the strip $\{ \lambda \in C| 0 < \Re \lambda < \lambda_1 \}$.

To prove that $G(\lambda)$ is analytic for $\Re \lambda = \lambda_1$, we only need to prove that $\Delta(c, \lambda) = 0$ does not have any zero with $\Re \lambda = \lambda_1$ other than $\lambda = \lambda_1$. Indeed, letting $\lambda = \lambda_1 + i\tilde{\lambda}$, we have

$$ 0 = -c\tilde{\lambda} + 1 - f'(0) \left[ \sum_{k=1}^m a_k \int_0^\tau J_k(y)e^{\lambda_1(-k+c) \tilde{\lambda}dy} + \alpha \int_0^\tau J_{m+1}(y)e^{\lambda_1c \tilde{\lambda}} \cos(cy)\tilde{\lambda}dy \right] $$

$$ + \sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y)e^{\lambda_1(j+c) \tilde{\lambda}} \cos(j + cy)\tilde{\lambda}dy \right] $$

and

$$ 0 = -c\tilde{\lambda} - f'(0) \left[ \sum_{k=1}^m a_k \int_0^\tau J_k(y)e^{\lambda_1(-k+c) \tilde{\lambda}dy} + \alpha \int_0^\tau J_{m+1}(y)e^{\lambda_1c \tilde{\lambda}} \sin(cy)\tilde{\lambda}dy \right] $$

$$ + \sum_{j=1}^l \beta_j \int_0^\tau J_{m+1+j}(y)e^{\lambda_1(j+c) \tilde{\lambda}} \sin(j + cy)\tilde{\lambda}dy \right]. $$

(2.10)

(2.11)

It follows from (2.10) and (2.11) that $\tilde{\lambda} = 0$.

According to the above argument, $G(\lambda)$ is analytic in $S$, and $H(\lambda)$ is also analytic in $S$. Moreover, we claim that $H(\lambda_1) > 0$. Indeed, notice that $H(\lambda_1) = G(\lambda_1)$. On the other hand, $\int_{-\infty}^{+\infty} e^{-\lambda x}Q(\psi)(x)dx < 0$ according to (2.9) and $\lim_{\lambda \to \lambda_1^-} \Delta(c, \lambda)/(\lambda_1 - \lambda)^{k+1} < 0$ according to Lemma 2.4.

Since $\psi(\xi)$ may be non-monotone, Ikehara’s Theorem could be directly used. Thus, we need to make a function transformation, i.e., $\hat{\psi}(\xi) = \psi(\xi)e^{p\xi}$, where
Proof. From Proposition 2.6, there exist two positive numbers \( \varphi \) is any non-negative traveling wave of (1.1) with the wave speed \( k \), where \( k < c < c^* \) are finite. Without loss of generality, we assume \( \sup_{\xi} \omega = 0 \), there exists \( \xi \in \mathbb{R} \) such that \( \varphi(n - ct) = \psi(n - ct + \xi) \).

First, we consider \( c < c^* \). Since \( \omega(\pm \infty) = 0 \), \( \sup_{\xi \in \mathbb{R}} \{\omega(\xi)\} \) and \( \inf_{\xi \in \mathbb{R}} \{\omega(\xi)\} \) are finite. Without loss of generality, we assume \( \sup_{\xi \in \mathbb{R}} \{\omega(\xi)\} \geq \inf_{\xi \in \mathbb{R}} \{\omega(\xi)\} \).

If \( \omega(\xi) \neq 0 \), there exists \( \xi_0 \) such that
\[
\omega(\xi_0) = \max_{\xi \in \mathbb{R}} \omega(\xi) = \sup_{\xi \in \mathbb{R}} \omega(\xi) > 0, \quad \omega'(\xi_0) = 0.
\]
We claim that for all \( i, j \in \mathbb{Z} \), we have
\[
\omega(\xi_0 - i + cy) = \omega(\xi_0 + cy) = \omega(\xi_0 + j + cy) = \omega(\xi_0)
\]
for \( y \in [0, \tau] \). Suppose on the contrary that one of three inequalities \( \omega(\xi_0 - i + cy) < \omega(\xi_0) \), \( \omega(\xi_0 + cy) < \omega(\xi_0) \) and \( \omega(\xi_0 + j + cy) < \omega(\xi_0) \) for some \( i_0, j_0 \) must hold. According to (1.2), (3.1) and (H2), we obtain
\[
0 = c\omega'(\xi_0)
\]
\[
= -c\lambda_1\omega(\xi_0) + \omega(\xi_0)
\]
\[
- e^{-\lambda_1\omega} \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y)[f(\phi(\xi_0 - i + cy)) - f(\psi(\xi_0 + \xi - i + cy))]dy
\]
\[
+ e^{-\lambda_1\omega} \alpha \int_{0}^{\tau} J_{m+1}(y)[f(\phi(\xi_0 + cy)) - f(\psi(\xi_0 + \xi + j + cy))]dy
\]
\[
+ e^{-\lambda_1\omega} \sum_{j=1}^{I} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)[f(\phi(\xi_0 + cy + j)) - f(\psi(\xi_0 + \xi + j + cy))]dy
\]
\[
\geq c\lambda_1\omega(\xi_0) + \omega(\xi_0) - f'(0)\omega(\xi_0) \left[ \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y)e^{\lambda_1(-i+cy)}dy \right]
\]
\[
+ \alpha \int_{0}^{\tau} J_{m+1}(y)e^{\lambda_1 cy}dy + \sum_{j=1}^{I} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)e^{\lambda_1 (cy+j)}dy
\]
\[
\geq -\omega(\xi_0) \Delta (c, \lambda_1) = 0,
\]
which is a contradiction. Thus, \( \omega(\xi_0 + cy) = \omega(\xi_0) \) also holds for \( y_0 \in (0, \tau) \). Again by bootstrapping, \( \omega(\xi_0 + kcy_0) = \omega(\xi_0) \) for all \( k \in \mathbb{Z} \) and \( \omega(+\infty) = 0 \). Therefore, we have \( \phi(\xi) \equiv \psi(\xi + \xi) \) for \( \xi \in \mathbb{R} \), which contradicts to \( \omega(\xi) \neq 0 \).

Next, we consider \( c = c_* \). Assume \( \sup_{\xi \in \mathbb{R}} \{ \omega(\xi) \} \geq \inf_{\xi \in \mathbb{R}} \{ \omega(\xi) \} \). If \( \omega_*(\xi) \neq 0 \), there exists \( \xi^*_0 \) such that
\[
\omega_*(\xi^*_0) = \max_{\xi \in \mathbb{R}} \{ \omega_*(\xi) \} = \sup_{\xi \in \mathbb{R}} \{ \omega_*(\xi) \} > 0, \quad \omega'_*(\xi^*_0) = 0.
\]

Now we divide this part into three cases:

**Case 1:** Suppose that \( \xi^*_0 \to +\infty \) as \( \epsilon \to 0 \). It follows from (3.1) and (1.2) that
\[
c_*[\phi'(\xi^*_0) - \psi'(\xi^*_0 + \xi)]
\]
\[
= \phi(\xi^*_0) - \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y)[f(\phi(\xi^*_0 - i + c_*y)) - f(\psi(\xi^*_0 + \xi - i + c_*y))]dy
\]
\[
- \alpha \int_{0}^{\tau} J_{m+1}(y)[f(\phi(\xi^*_0 + c_*y)) - f(\psi(\xi^*_0 + \xi + c_*y))]dy - \psi(\xi^*_0 + \xi)
\]
\[
- \sum_{j=1}^{I} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)[f(\phi(\xi^*_0 + cy + j)) - f(\psi(\xi^*_0 + \xi + j + c_*y))]dy.
\]

We claim that for all \( i, j \in \mathbb{Z} \),
\[
\omega(\xi^*_0 - i + c_*y) = \omega(\xi^*_0 + c_*y) = \omega(\xi^*_0 + j + c_*y) = \omega(\xi^*_0)
\]
for \( y \in [0, \tau] \). Suppose for the contrary that one of three inequalities \( \omega(\xi^*_0 - i + c_*y) < \omega(\xi^*_0) \), \( \omega(\xi^*_0 + c_*y) < \omega(\xi^*_0) \) and \( \omega(\xi^*_0 + j + c_*y) < \omega(\xi^*_0) \) for some \( i_0, j_0 \) and
\( j_0 \) must hold. Choose \( \epsilon > 0 \) sufficiently small such that \( \xi_0^\epsilon - m + c_\ast \tau > 0 \). Thus,

\[-c_\ast \omega_\epsilon(\xi_0^\epsilon) \epsilon - c_\ast \lambda_\epsilon(\xi_0^\epsilon)(\epsilon \xi_0^\epsilon + 1) \]

\[\leq -\omega_\epsilon(\xi_0^\epsilon)(\epsilon |\xi_0^\epsilon| + 1)\]

\[+ f'(0) \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y)e^{\lambda_\epsilon(-i+c_\ast y)[\epsilon |\xi_0^\epsilon| - i + c_\ast y] + 1}|\omega_\epsilon(\xi_0^\epsilon - i + c_\ast y)dy\]

\[+ \alpha f'(0) \int_{0}^{\tau} J_{m+1}(y)e^{\lambda_\epsilon(c_\ast y)[\epsilon |\xi_0^\epsilon + c_\ast y] + 1}|\omega_\epsilon(\xi_0^\epsilon + c_\ast y)dy\]

\[+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)e^{\lambda_\epsilon(j+c_\ast y)[\epsilon |\xi_0^\epsilon + j + c_\ast y] + 1}|\omega_\epsilon(\xi_0^\epsilon + j + c_\ast y)dy,\]

it follows that

\[-c_\ast \omega_\epsilon(\xi_0^\epsilon) \epsilon + \omega_\epsilon(\xi_0^\epsilon)(\epsilon \xi_0^\epsilon + 1)\Delta(c_\ast ; \lambda_\ast) \]

\[< f'(0) \sum_{i=1}^{m} a_i \int_{0}^{\tau} J_i(y)e^{\lambda_\epsilon(-i+c_\ast y)\epsilon(-i + c_\ast y)\omega_\epsilon(\xi_0^\epsilon)dy} \]

\[+ \alpha f'(0) \int_{0}^{\tau} J_{m+1}(y)e^{\lambda_\epsilon(c_\ast y)\epsilon c_\ast y\omega_\epsilon(\xi_0^\epsilon)dy}\]

\[+ f'(0) \sum_{j=1}^{l} \beta_j \int_{0}^{\tau} J_{m+1+j}(y)e^{\lambda_\epsilon(j+c_\ast y)\epsilon(j + c_\ast y)\omega_\epsilon(\xi_0^\epsilon)dy} \]

This contradicts \( \frac{\partial \Delta(c_\ast; \lambda_\ast)}{\partial \epsilon} |_{c_\ast; \lambda_\ast = \lambda_\ast} = 0 \). Repeating the arguments, we have \( \omega_\epsilon(\xi_0^\epsilon) = \omega_\epsilon(\xi_0^\epsilon + k_\ast y_0) \) for all \( k \in \mathbb{Z} \) and some \( y_0 \in (0, \tau) \). It follows from that \( \omega_\epsilon(+\infty) = 0 \), we can obtain \( \phi(\xi) \equiv \psi(\xi + \tilde{\xi}) \) for \( \xi \in \mathbb{R} \), which contradicts \( \omega_\epsilon(\xi) \neq 0 \).

Similar to the process in [3], \( \phi(\xi) \equiv \psi(\xi + \tilde{\xi}) \) for \( \xi \in \mathbb{R} \) still holds for

**Case 2:** Suppose \( \xi_0^\epsilon \rightarrow -\infty \) as \( \epsilon \rightarrow 0 \) and **Case 3:** Suppose \( \xi_0^\epsilon \) is bounded. This completes the proof. \( \square \)

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