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LIOUVILLE TYPE THEOREMS FOR ELLIPTIC EQUATIONS INVOLVING GRUSHIN OPERATOR AND ADVECTION

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Abstract. In this article, we study the equation

$$-G_\alpha u+\nabla_G w\cdot\nabla_G u=\|\mathbf{x}\|^s|u|^{p-1}u,\quad \mathbf{x}=(x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2},$$

where G_α (resp., ∇_G) is Grushin operator (resp. Grushin gradient), $p>1$ and $s\ge 0$. The scalar function w satisfies a decay condition, and $\|\mathbf{x}\|$ is the norm corresponding to the Grushin distance. Based on the approach by Farina [8], we establish a Liouville type theorem for the class of stable sign-changing weak solutions. In particular, we show that the nonexistence result for stable positive classical solutions in [4] is still valid for the above equation.

1. Introduction

In this article, we examine the nonexistence of stable sign-changing weak solutions of

$$-G_\alpha u+\nabla_G w\cdot\nabla_G u=\|\mathbf{x}\|^s|u|^{p-1}u,\quad \mathbf{x}=(x,y)\in\mathbb{R}^N=\mathbb{R}^{N_1}\times\mathbb{R}^{N_2},\tag{1.1}$$

where the constants p,α,s satisfy $p>1,\alpha\ge 0$ and $s\ge 0$. The Grushin operator G_α (resp. the Grushin gradient ∇_G) is defined by

$$G_\alpha u=\Delta_x u+|x|^{2\alpha}\Delta_y u\quad(\text{resp. }\nabla_G u=(\nabla_x u,|x|^\alpha\nabla_y u)).$$

The advection term w is smooth and satisfies

$$|\nabla_G w(\mathbf{x})|\le\frac{C}{\|\mathbf{x}\|^\theta+1}\quad\text{for some }\theta\ge 0.$$

Here

$$\|\mathbf{x}\|=\Big(|x|^{2(\alpha+1)}+|y|^2\Big)^{\frac{1}{2(\alpha+1)}}$$

corresponds to the Grushin distance.

Let us begin by noting that G_0 is just the Laplace operator. So far, there have been many works dealing with the stable solutions of (1.1) with $\alpha=0$ and $w=0$ (see [8, 5, 14] and the references therein). The pioneering work in this direction is due to Farina [8] where the classification of stable classical solutions was completely established in nonweighted case, i.e., $s=0$. One of the main results in [8] is the following.

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Theorem 1.1 ([8]). *Let $\alpha = s = 0$ and $w \equiv 0$. Let $u \in C^2(\mathbb{R}^N)$ be a stable classical solution of* (1.1) *with*

$$1 < p < +\infty \quad \text{if } N \le 10$$
$$1 < p < p_c(N) = \frac{(N-2)^2 - 4N + 8\sqrt{N-1}}{(N-2)(N-10)} \quad \text{if } N \ge 11.$$

Then $u \equiv 0$.

After that, Theorem 1.1 was generalized to the weighted case in [5, 14]. In [5], the authors proved the nonexistence of nontrivial stable weak solutions under the restriction that the solutions are locally bounded. This restriction was removed in [14].

Theorem 1.2 ([14]). *Let $\alpha = 0$ and $w \equiv 0$. Suppose that $s > -2$. Let u be a stable weak solution of* (1.1) *with $1 < p < p(N, s)$, where*

$$p(N,s) = \begin{cases} +\infty & \text{if } N \le 10 + 4s \\ \frac{(N-2)^2 - 2(s+2)(s+N) + 2\sqrt{(s+2)^3(s+2N-2)}}{(N-2)(N-10-4s)} & \text{if } N > 10 + 4s. \end{cases}$$

Then $u \equiv 0$.

It was also shown in [14] that there exists a family of stable solutions of (1.1) with $\alpha = 0$ and $w = 0$ if $p \ge p(N, s)$. From Theorem 1.2, one can see the explicit effect of the weight on the critical exponent.

We now turn to the case where $\alpha > 0$, $s = 0$ and $w \equiv 0$. Let us first recall some facts about the problem involving the Grushin operator. It is well-known that the operator G_α belongs to the wide class of subelliptic operators studied by Franchi et al. in [10] (see also [1, 2]). The Liouville type theorem has been recently proved by Monticelli [12] for nonnegative classical solutions, and by Yu [15] for nonnegative weak solutions of the problem

$$-G_\alpha u = u^p \quad \text{in } \mathbb{R}^N.$$

The optimal condition on the range of the exponent is $p < \frac{N_\alpha + 2}{N_\alpha - 2}$, where

$$N_\alpha := N_1 + (1+\alpha)N_2$$

is called the homogeneous dimension. The main tool in [12, 15] is the Kelvin transform combined with the moving planes technique. Before that, Dolcetta and Cutrì [3] established the Liouville type theorem for nonnegative super-solutions under the condition $p \le \frac{N_\alpha}{N_\alpha - 2}$ (see also [6]).

In addition, we should mention that problem (1.1) with $\alpha = 0$ and $w \ne 0$ satisfying some additional conditions was studied in [4]. By using Farina's approach, the authors obtained the Liouville property for stable positive classical solutions. We summarize here some results in [4].

Theorem 1.3 ([4, Corollary 2 and Theorem 1.3]). *Let $\alpha = 0$.*

(i) *Suppose that w is bounded together with its gradient. If $s = 0$ and*

$$N < 1 + \frac{2}{p-1}\left(p + \sqrt{p(p-1)}\right)$$

then there is no stable positive classical sub-solution of (1.1).

(ii) *Let* $w = -\log(|x| + |y| + 1)^{\beta}$ *and*

$$N + \beta - 2 < \frac{2(2+s)(p + \sqrt{p(p-1)})}{p-1}.$$

Then there is no stable positive classical sub-solution of (1.1).

Naturally, a question raised from Theorem 1.3 is about the Liouville property for a more general class, for example, the class of stable sign-changing weak solutions. As far as we know, the Liouville type theorem for the problem (1.1) with $\alpha \neq 0$ and $w \neq 0$ has not been studied in the literature. The purpose of this paper is then to establish the Liouville property for the class of stable sign-changing weak solutions of (1.1). In particular, we show that Theorem 1.3 remains valid for this class of solutions and recover Theorems 1.1 and 1.2 in the case $s \geq 0$.

Before stating our main result, we need to make precise several terminologies. Denote by $H^{1,\alpha}(\mathbb{R}^N)$ the space of $u \in L^2(\mathbb{R}^N)$ satisfying $\nabla_G u \in L^2(\mathbb{R}^N)$ endowed with the norm

$$\|u\| = \Big(\|u\|^2_{L^2(\mathbb{R}^N)} + \|\nabla_G u\|^2_{L^2(\mathbb{R}^N)}\Big)^{1/2}.$$

It is easy to see that when $\alpha = 0$, $H^{1,\alpha}(\mathbb{R}^N)$ is the usual Sobolev space $H^1(\mathbb{R}^N)$. Denote also by $H^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ the space of all functions u such that $u\psi \in H^{1,\alpha}(\mathbb{R}^N)$ for all $\psi \in C^1_c(\mathbb{R}^N)$. Here and in what follows, $C^k_c(\mathbb{R}^N)$ is the set of C^k - functions with compact support in $\mathbb{R}^N$.

Definition 1.4. We say that u is a weak solution of the equation (1.1) if $u \in H^{1,\alpha}_{\rm loc}(\mathbb{R}^N) \cap L^p_{\rm loc}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} (\nabla_G u \cdot \nabla_G \psi + \nabla_G w \cdot \nabla_G u \psi - \|\mathbf{x}\|^s |u|^{p-1} u\psi) = 0, \quad \text{for all } \psi \in C^1_c(\mathbb{R}^N). \tag{1.2}$$

Next we recall the stability of solutions. Note that the energy functional corresponding to (1.1) is given by

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla_G u|^2 e^{-w} - \frac{1}{p+1}\int_{\mathbb{R}^N} \|\mathbf{x}\|^s |u|^{p+1} e^{-w}.$$

Roughly speaking, a solution u is stable if the second variation at u of the energy functional is nonnegative (see [7]). Therefore, we say that a weak solution u of the equation (1.1) is stable if

$$\int_{\mathbb{R}^N} (|\nabla_G \psi|^2 - p\|\mathbf{x}\|^s |u|^{p-1}\psi^2) e^{-w} \geq 0, \quad \text{for all } \psi \in C^1_c(\mathbb{R}^N). \tag{1.3}$$

Now we present the main results in this paper. Throughout this paper, we always assume that $p > 1, s \geq 0$.

Theorem 1.5. *Suppose that there is a nonnegative constant* θ *such that*

$$|\nabla_G w| \leq \frac{C}{\|\mathbf{x}\|^{\theta} + 1}.$$

Assume in addition that for $\gamma \in (1, 2p + 2\sqrt{p(p-1)} - 1)$ *we have*

$$\lim_{R\to+\infty} R^{-\frac{(1+\min(\theta;1))(p+\gamma)+s(\gamma+1)}{p-1}} \int_{R<\|\mathbf{x}\|<2R} e^{-w} = 0. \tag{1.4}$$

Then any stable weak solution u *to* (1.1) *must be the trivial one.*

It is easy to see that, when w is bounded from below, one has

$$\int_{R<\|\mathbf{x}\|<2R} e^{-w} \le C R^{N_\alpha}.$$

Thus, the following is a direct consequence of Theorem 1.5.

Corollary 1.6. *Suppose that there is a nonnegative constant θ such that*

$$|\nabla_G w| \le \frac{C}{\|\mathbf{x}\|^\theta + 1}.$$

Assume in addition that w is bounded from below and

$$N_\alpha - 1 - \min(\theta,1) - \frac{2(p+\sqrt{p(p-1)})(1+\min(\theta,1)+s)}{p-1} < 0. \tag{1.5}$$

Then any stable weak solution u to (1.1) *must be the trivial solution.*

Furthermore, we choose $w = -\log(\|\mathbf{x}\|+1)^\beta$ for some $\beta \in \mathbb{R}$, then w is bounded from below if $\beta < 0$ and is unbounded from below if $\beta > 0$. Note that, in this case

$$|\nabla_G w| \le \frac{C}{\|\mathbf{x}\| + 1}.$$

Thus, Theorem 1.5 implies the following.

Corollary 1.7. *If u is a stable weak solution of* (1.1) *with $w = -\log(\|\mathbf{x}\|+1)^\beta$ and*

$$N_\alpha + \beta - 2 < \frac{2(2+s)(p+\sqrt{p(p-1)})}{p-1}, \tag{1.6}$$

then u is the trivial solution.

Remark 1.8. (i) By using the same argument as below, one can show that our main result is still valid for the class of stable positive weak sub-solutions to (1.1). Moreover, our arguments can be applied to study the equation (1.1) where the non-linear term $|u|^{p-1}u$ is replaced by e^u.

(ii) Theorem 1.5 is sharp in the sense that when $\alpha = 0, w \equiv 0$ and (1.4) is not satisfied, one can construct a sequence of stable weak solutions of (1.1), see e.g., [14]. On the other hand, one can see from our main result the explicit effects of α and the advection term on the range of the exponent.

(ii) The first assertion in Theorem 1.3 follows from Corollary 1.6 by choosing $\alpha = 0$ and $\theta = 0$. The second one is a consequence of Corollary 1.7. Theorems 1.1 and 1.2 in the case $s \ge 0$ are also consequences of Corollary 1.6 by choosing $\alpha = 0$ and $w = 0$.

Although this work is motivated by the idea by Farina [8], it should be mentioned that the use of this technique in our case was by no means straightforward and required many nontrivial additional ideas.

- The first difficulty in the study of problem (1.1) is that the principal linear term, the Grushin operator, has nonconstant coefficients. This requires to design appropriate scaled test functions in the integral estimate.
- Secondly, the fact that weak solutions are not locally bounded also leads to another difficulty. We need to construct a sequence of suitable cut-off functions and the estimates become very delicate.
- It seems that the presence of the advection term $w(\mathbf{x})$ makes the problem more challenging. We need to use a suitable weighted integral to treat this term.

Moreover, we use the properties of the Grushin gradient and the associated distance to derive the nonlinear integral estimates. We also note that in the case $\alpha = 0$, $w = 0$, $N \geq 10 + 4s$ and $p \geq p(N, s)$, it is not too complicated to build a radial solution to (1.1) (see [14]). Nevertheless, it seems very difficult to prove the existence of solutions to (1.1). Up to now, there have been two articles [11, 13] dealing with this problem in the case $p = \frac{N_\alpha+2}{N_\alpha-2}$, $w = 0$.

Since Corollaries 1.6 and 1.7 are immediate consequences of Theorem 1.5, the rest of this paper is devoted to proving Theorem 1.5.

2. Proof of Theorem 1.5

In what follows, for the sake of simplicity, we denote by $\int$ the integral $\int_{\mathbb{R}^N} dxdy$. The following proposition plays a crucial role in the proof of our main result.

Proposition 2.1. *Let $p > 1$ and u be a stable weak solution of* (1.1). *Fix a real number $\gamma \in [1, 2p + 2\sqrt{p(p-1)} - 1)$ and an integer $m \geq \frac{p+\gamma}{p-1}$. Then there is a constant $C_{p,m,\gamma} > 0$ depending only on p, m and γ, such that*

$$\begin{aligned}
&\int \Big(|\nabla_x(|u|^{\frac{\gamma-1}{2}}u)|^2 + |x|^{2\alpha}|\nabla_y(|u|^{\frac{\gamma-1}{2}}u)|^2 + \|\mathbf{x}\|^s|u|^{p+\gamma}\Big)\psi^{2m}e^{-w} \\
&\leq C_{p,m,\gamma}\int \|\mathbf{x}\|^{-\frac{(\gamma+1)s}{p-1}}\Big(|\nabla_G\psi|^2 + |\psi|(|\Delta_x\psi| + |x|^{2\alpha}|\Delta_y\psi| \\
&\quad + |\nabla_G\psi\|\nabla_G w|)\Big)^{\frac{p+\gamma}{p-1}}e^{-w},
\end{aligned} \tag{2.1}$$

for all $\psi \in C_c^2(\mathbb{R}^N; [-1; 1])$.

Proof. As mentioned above, the solution u is not necessary locally bounded. Then, we need to construct a sequence of suitable cut-off functions.

Let k be a positive integer. A sequence of cut-off functions is chosen as follows

$$\varphi_k(t) = \begin{cases} -k & \text{if } t < -k \\ t & \text{if } -k \leq t \leq k \\ k & \text{if } t > k. \end{cases}$$

It is easy to see that $\varphi_k'(t) = 1$ for $|t| < k$, $\varphi_k'(t) = 0$ for $|t| > k$ and $|\varphi_k(t)| \leq |t|$ for all $t \in \mathbb{R}$. We shall prove the inequality

$$\begin{aligned}
&\int \Big(\big|\nabla_x(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2 + |x|^{2\alpha}\big|\nabla_y(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2 \\
&+ \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\Big)\psi^{2m}e^{-w} \\
&\leq C_{p,m,\gamma}\int \|\mathbf{x}\|^{-\frac{(\gamma+1)s}{p-1}}|\Big(|\nabla_G\psi|^2 + |\psi|(|\Delta_x\psi| + |x|^{2\alpha}|\Delta_y\psi| \\
&\quad + |\nabla_G\psi\|\nabla_G w|)\Big)^{\frac{p+\gamma}{p-1}}e^{-w},
\end{aligned} \tag{2.2}$$

for all $\psi \in C_c^2(\mathbb{R}^N; [-1; 1])$. Here the constant $C_{p,m,\gamma}$ depends only on p, m, γ.

Suppose that (2.2) is holds. Letting $k \to +\infty$ in (2.2) and using Fatou's Lemma, we obtain (2.1). Hence, it is sufficient to prove (2.2).

Since the proof of (2.2) is quite long and technical, we first give the outline of the proof.

Step 1. By using the definition of weak solutions and the stability condition, we show that

$$\begin{aligned} &\int \big|\nabla_G\big(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u\big)\big|^2\phi^2e^{-w} + \int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2e^{-w} \\ &\le C\int |\varphi_k(u)|^{\gamma-1}u^2(|\nabla_G\phi|^2 + |G_\alpha\phi^2| + |\nabla_G\phi^2\|\nabla_G w|)e^{-w} \end{aligned} \tag{2.3}$$

for all $\phi \in C_c^2(\mathbb{R}^N)$.

Step 2. By choosing $\phi = \psi^m$ where $\psi \in C_c^2(\mathbb{R}^N; [-1;1])$ and employing Hölder's inequality we demonstrate that the right hand side of (2.3) is smaller than or equal to

$$C_{p,m,\gamma}\int \|\mathbf{x}\|^{-\frac{(\gamma+1)s}{p-1}}(|\nabla_G\psi|^2 + |\psi|(|\Delta_x\psi| + |x|^{2\alpha}|\Delta_y\psi| + |\nabla_G\psi\|\nabla_G w|))^{\frac{p+\gamma}{p-1}}e^{-w}.$$

Thus, (2.2) follows from these two steps. □

We now present the proof of (2.2) in detail.

Proof of Step 1 Let u be a weak solution of (1.1). For $\phi \in C_c^2(\mathbb{R}^N)$, using the density argument, (1.2) remains true for the test function $|\varphi_k(u)|^{\gamma-1}u\phi^2e^{-w} \in H^{1,\alpha}(\mathbb{R}^N)$. Consequently, (1.2) and a simple computation gives

$$\int \nabla_G u\cdot\nabla_G(|\varphi_k(u)|^{\gamma-1}u\phi^2)e^{-w} - \int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2e^{-w} = 0. \tag{2.4}$$

Note that

$$\begin{aligned} &\nabla_G(|\varphi_k(u)|^{\gamma-1}u\phi^2) \\ &= ((\gamma-1)\nabla_G\varphi_k(u)|\varphi_k(u)|^{\gamma-1} + |\varphi_k(u)|^{\gamma-1}\nabla_G u)\phi^2 + |\varphi_k(u)|^{\gamma-1}u\nabla_G\phi^2 \end{aligned}$$

and

$$\begin{aligned} \big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2\phi^2 &= \big(\frac{\gamma-1}{2}\big)^2|\nabla_G\varphi_k(u)|^2|\varphi_k(u)|^{\gamma-1}\phi^2 + |\varphi_k(u)|^{\gamma-1}|\nabla_G u|^2\phi^2 \\ &\quad + (\gamma-1)\nabla_G\varphi_k(u)\nabla_G u|\varphi_k(u)|^{\gamma-1}\phi^2. \end{aligned}$$

These computations lead to

$$\begin{aligned} &\int \nabla_G u\cdot\nabla_G(|\varphi_k(u)|^{\gamma-1}u\phi^2)e^{-w} \\ &= \int (\big|\nabla_G\big(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u\big)\big|^2\phi^2 + u|\varphi_k(u)|^{\gamma-1}\nabla_G u\cdot\nabla_G\phi^2)e^{-w} \\ &\quad - \int \big(\frac{\gamma-1}{2}\big)^2|\nabla_G\varphi_k(u)|^2|\varphi_k(u)|^{\gamma-1}\phi^2e^{-w}. \end{aligned} \tag{2.5}$$

Combining (2.4),(2.5), we conclude that

$$\begin{aligned} &\int \big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2\phi^2e^{-w} \\ &= \int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2e^{-w} - \int u|\varphi_k(u)|^{\gamma-1}\nabla_G u\cdot\nabla_G\phi^2e^{-w} \\ &\quad + \big(\frac{\gamma-1}{2}\big)^2\int |\nabla_G\varphi_k(u)|^2|\varphi_k(u)|^{\gamma-1}\phi^2e^{-w}. \end{aligned} \tag{2.6}$$

Notice that

$$2u|\varphi_k(u)|^{\gamma-1}\nabla_G u\cdot\nabla_G\phi^2$$

$$
\begin{aligned}
&= \nabla_G(|\varphi_k(u)|^{\gamma-1}u^2)\nabla_G\phi^2 - (\gamma-1)\nabla_G(|\varphi_k(u)|)|\varphi_k(u)|^{\gamma}\nabla_G\phi^2 \\
&= \nabla_G(|\varphi_k(u)|^{\gamma-1}u^2)\nabla_G\phi^2 - \frac{\gamma-1}{\gamma+1}\nabla_G|\varphi_k(u)|^{\gamma+1}\nabla_G\phi^2.
\end{aligned}
$$

Using this and the integration by parts, we have

$$
\begin{aligned}
&\int u|\varphi_k(u)|^{\gamma-1}\nabla_G u\cdot\nabla_G\phi^2 e^{-w} \\
&= -\frac{1}{2}\int |\varphi_k(u)|^{\gamma-1}u^2(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w} \\
&\quad + \frac{(\gamma-1)}{2(\gamma+1)}\int |\varphi_k(u)|^{\gamma+1}(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w}.
\end{aligned}
\tag{2.7}
$$

Inserting (2.7) in (2.6) we arrive at

$$
\begin{aligned}
&\int \Big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\Big|^2\phi^2 e^{-w} \\
&= \int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w} \\
&\quad + \frac{1}{2}\int |\varphi_k(u)|^{\gamma-1}u^2(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w} \\
&\quad - \frac{(\gamma-1)}{2(\gamma+1)}\int |\varphi_k(u)|^{\gamma+1}(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w} \\
&\quad + (\frac{\gamma-1}{2})^2\int |\nabla_G\varphi_k(u)|^2|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w}.
\end{aligned}
\tag{2.8}
$$

Note also that

$$
\begin{aligned}
&\int |\nabla_G\varphi_k(u)|^2|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w} \\
&= \frac{4}{(\gamma+1)^2}\int \big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}\varphi_k(u))\big|^2\phi^2 e^{-w} \\
&\le \frac{4}{(\gamma+1)^2}\int \big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2\phi^2 e^{-w},
\end{aligned}
\tag{2.9}
$$

where in the last inequality we have used $|\varphi_k(u)| = |k|$ for $|u| > k$. Thus, (2.8) becomes

$$
\begin{aligned}
&\frac{4\gamma}{(\gamma+1)^2}\int \big|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\big|^2\phi^2 e^{-w} \\
&\le \int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w} \\
&\quad + \frac{1}{2}\int |\varphi_k(u)|^{\gamma-1}u^2(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w} \\
&\quad - \frac{(\gamma-1)}{2(\gamma+1)}\int |\varphi_k(u)|^{\gamma+1}(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w}.
\end{aligned}
\tag{2.10}
$$

In the next part, we shall utilize the stability condition. We remark that (1.3) also holds for the test function $|\varphi_k(u)|^{\frac{\gamma-1}{2}}u\phi \in H^{1,\alpha}(\mathbb{R}^N)$ by density argument. From

(1.3), we have

$$\int \left|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u\phi)\right|^2 e^{-w} - p\int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w} \geq 0.$$

By Young's inequality, for any $\delta > 0$,

$$\begin{aligned}
&\int \left|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u\phi)\right|^2 e^{-w}\\
&\leq (1+\delta)\int \left|\nabla_G(|\varphi_k(u)|^{\frac{\gamma-1}{2}}u)\right|^2\phi^2 e^{-w} + (1+\frac{1}{\delta})\int |\nabla_G\phi|^2|\varphi_k(u)|^{\gamma-1}u^2 e^{-w}.
\end{aligned}$$

Then, we obtain

$$\begin{aligned}
&\big(p - \frac{(\gamma+1)^2}{4\gamma}(1+\delta)\big)\int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w}\\
&\leq (1+\frac{1}{\delta})\int |\varphi_k(u)|^{\gamma-1}u^2|\nabla_G\phi|^2 e^{-w}\\
&\quad + \frac{(\gamma+1)^2}{8\gamma}(1+\delta)\int |\varphi_k(u)|^{\gamma-1}u^2(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w}\\
&\quad - \frac{(\gamma^2-1)}{8\gamma}(1+\delta)\int |\varphi_k(u)|^{\gamma+1}(G_\alpha\phi^2 - \nabla_G\phi^2\cdot\nabla_G w)e^{-w}.
\end{aligned} \tag{2.11}$$

Now we choose

$$\delta = \frac{4\gamma}{(\gamma+1)^2}\frac{1}{2}(p - \frac{(\gamma+1)^2}{4\gamma}) > 0.$$

Then

$$\begin{aligned}
&\int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\phi^2 e^{-w}\\
&\leq C\int |\varphi_k(u)|^{\gamma-1}u^2(|\nabla_G\phi|^2 + |G_\alpha\phi^2| + |\nabla_G\phi^2\|\nabla_G w|)e^{-w}.
\end{aligned}$$

This and (2.10) imply (2.3).

Proof of Step 2. Let $m \geq \frac{p+\gamma}{p-1}$ be a fixed integer. For $\psi \in C_c^2(\mathbb{R}^N;[-1;1])$, we set $\phi = \psi^m$. Hence,

$$\begin{gathered}
|\nabla_x\psi^m|^2 = m^2|\nabla_x\psi|^2\psi^{2m-2},\\
|x|^{2\alpha}\,|\nabla_y\psi^m|^2 = m^2|x|^{2\alpha}|\nabla_y\psi|^2\psi^{2m-2}
\end{gathered}$$

and

$$\begin{gathered}
\Delta_x\psi^{2m} = 2m\psi^{2m-2}((2m-1)|\nabla_x\psi|^2 + \psi\Delta_x\psi),\\
|x|^{2\alpha}\Delta_y\psi^{2m} = 2m\psi^{2m-2}((2m-1)|x|^{2\alpha}|\nabla_y\psi|^2 + \psi|x|^{2\alpha}\Delta_y\psi).
\end{gathered}$$

Therefore, the right hand side of the last inequality in (2.3) is less than or equal to

$$\begin{aligned}
&C\int |\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2}\Big(|\nabla_G\psi|^2 + |\psi|(|\Delta_x\psi| + |x|^{2\alpha}|\Delta_y\psi|\\
&+ |\nabla_G\psi\|\nabla_G w|)\Big)e^{-w}.
\end{aligned} \tag{2.12}$$

Applying Hölder's inequality to (2.12), we obtain

$$\begin{aligned}
&\int |\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2}(|\nabla_G\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi| \\
&+|\nabla_G\psi\|\nabla_G w|))e^{-w} \\
&\le \Big[\int (\|\mathbf{x}\|^{s\frac{\gamma+1}{p+\gamma}}|\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2})^{\frac{p+\gamma}{\gamma+1}}e^{-w}\Big]^{\frac{1+\gamma}{p+\gamma}}\Big[\int \|\mathbf{x}\|^{-s\frac{\gamma+1}{p-1}}\Big(|\nabla_G\psi|^2 \\
&\quad +|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|+|\nabla_G\psi\|\nabla_G w|)\Big)^{\frac{p+\gamma}{p-1}}e^{-w}\Big]^{\frac{p-1}{p+\gamma}}.
\end{aligned} \tag{2.13}$$

Moreover, it follows from $m \ge \frac{p+\gamma}{p-1}$ that

$$(2m-2)\frac{p+\gamma}{\gamma+1}-2m \ge 0.$$

By using this with $|\psi|\le 1, |\varphi_k(u)|\le |u|$, we have

$$\begin{aligned}
(|\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2})^{\frac{p+\gamma}{\gamma+1}} &\le |\varphi_k(u)|^{(\gamma-1)\frac{p+\gamma}{\gamma+1}}u^{2\frac{p+\gamma}{\gamma+1}}\psi^{2m} \\
&\le |u|^{p+1}|\varphi_k(u)|^{\gamma-1}\psi^{2m}
\end{aligned} \tag{2.14}$$

which together with (2.13) gives

$$\begin{aligned}
&\int |\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2}(|\nabla_G\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|+|\nabla_G\psi\|\nabla_G w|))e^{-w} \\
&\le C_{p,m,\gamma}\Big[\int \|\mathbf{x}\|^s|u|^{p+1}|\varphi_k(u)|^{\gamma-1}\psi^{2m}e^{-w}\Big]^{\frac{1+\gamma}{p+\gamma}} \\
&\times \Big[\int \|\mathbf{x}\|^{-s\frac{\gamma+1}{p-1}}(|\nabla_G\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|+|\nabla_G\psi\|\nabla_G w|))^{\frac{p+\gamma}{p-1}}e^{-w}\Big]^{\frac{p-1}{p+\gamma}}.
\end{aligned}$$

This inequality and (2.11) with $\phi=\psi^m$ imply

$$\begin{aligned}
&\int |\varphi_k(u)|^{\gamma-1}u^2\psi^{2m-2}(|\nabla_G\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|+|\nabla_G\psi\|\nabla_G w|))e^{-w} \\
&\le C_{p,m,\gamma}^{\frac{p+\gamma}{p-1}}\int \|\mathbf{x}\|^{-s\frac{\gamma+1}{p-1}}(|\nabla_G\psi|^2+|\psi|(|\Delta_x\psi|+|x|^{2\alpha}|\Delta_y\psi|+|\nabla_G\psi\|\nabla_G w|))^{\frac{p+\gamma}{p-1}}e^{-w}.
\end{aligned}$$

The assertion in Step 2 is then proved, and the proof of Proposition 2.1 is complete.

Completion of the proof of Theorem 1.5. Let χ_1 in $C_c^\infty(\mathbb{R}^{N_1};[0,1])$ and χ_2 in $C_c^\infty(\mathbb{R}^{N_2};[0,1])$ be cut-off functions satisfying

$$\begin{aligned}
\chi_1(x)=1 \quad &\text{for } |x|\le 1; \chi_1(x)=0 \text{ for } |x|\ge 2, \\
\chi_2(y)=1 \quad &\text{for } |y|\le 1; \chi_2(y)=0 \text{ for } |y|\ge 2.
\end{aligned}$$

For R large enough, we choose $\psi_R(x,y)=\chi_1(\frac{x}{R})\chi_2(\frac{y}{R^{\alpha+1}})$ which belongs to the space $C_c^\infty(\mathbb{R}^N;[0,1])$. Then, it is easy to see that

$$\begin{aligned}
|\nabla_x\psi_R(x,y)| &= \frac{1}{R}\big|\nabla_x\chi_1(\frac{x}{R})\chi_2(\frac{y}{R^{\alpha+1}})\big| \\
|\nabla_y\psi_R(x,y)| &= \frac{1}{R^{1+\alpha}}\big|\chi_1(\frac{x}{R})\nabla_y\chi_2(\frac{y}{R^{\alpha+1}})\big|, \\
|\Delta_x\psi_R(x,y)| &= \frac{1}{R^2}\big|\Delta_x\chi_1(\frac{x}{R})\chi_2(\frac{y}{R^{\alpha+1}})\big|, \\
|\Delta_y\psi_R(x,y)| &= \frac{1}{R^{2(1+\alpha)}}\big|\chi_1(\frac{x}{R})\Delta_y\chi_2(\frac{y}{R^{\alpha+1}})\big|.
\end{aligned}$$

This and the boundedness of χ_1, χ_2 and the assumption $|\nabla_G w| \le \frac{C}{\|\mathbf{x}\|^\theta + 1}$ imply

$$|\nabla_G \psi_R|^2 + |\psi_R|(|\Delta_x \psi_R| + |x|^{2\alpha}|\Delta_y \psi_R| + |\nabla_G \psi_R\|\nabla_G w|)) \le \frac{C_1}{R^{1+\min(\theta;1)}}, \tag{2.15}$$

$$|\nabla_G \psi_R|^2 + |\psi_R|(|\Delta_x \psi_R| + |x|^{2\alpha}|\Delta_y \psi_R| + |\nabla_G \psi_R\|\nabla_G w|)) = 0 \tag{2.16}$$

outside the annulus $U_R := \{(x,y) \in \mathbb{R}^N; R \le \|\mathbf{x}\| \le 2^{1+\frac{2}{2+2\alpha}} R\}$. Note that the constant C_1 in (2.15) is independent of R.

Thus, (2.1) with $\psi = \psi_R$, (2.15) and (2.16) give

$$\begin{aligned}
&\int \big(\big|\nabla_x(|u|^{\frac{\gamma-1}{2}}u)\big|^2 + |x|^{2\alpha}\big|\nabla_y(|u|^{\frac{\gamma-1}{2}}u)\big|^2 + \|\mathbf{x}\|^s|u|^{p+\gamma}\big)\psi_R^{2m} e^{-w} \\
&\le \int_{U_R} \|\mathbf{x}\|^{-s\frac{\gamma+1}{p-1}} \Big(|\nabla_G \psi_R|^2 + |\psi_R|(|\Delta_x \psi_R| + |x|^{2\alpha}|\Delta_y \psi_R| \\
&\quad + |\nabla_G \psi_R\|\nabla_G w|)\Big)^{\frac{p+\gamma}{p-1}} e^{-w} \\
&\le \frac{C}{R^{(1+\min(\theta;1))(\frac{p+\gamma}{p-1})} R^{\frac{s(\gamma+1)}{(p-1)}}} \int_{U_R} e^{-w} \\
&= C R^{-\frac{(1+\min(\theta;1))(p+\gamma)+s(\gamma+1)}{p-1}} \int_{U_R} e^{-w},
\end{aligned} \tag{2.17}$$

where C is independent of R.

Finally, letting $R \to \infty$ in (2.17) and using (1.4), we obtain $u \equiv 0$ on $\mathbb{R}^N$. The proof of Theorem 1.5 is complete.

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