Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 110, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLUTION TO RANDOM DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Communicated by Mokhtar Kirane

Abstract. We study a family of random differential equations with boundary conditions. Using a random fixed point theorem, we prove an existence theorem that yields a unique random solution.

1. Introduction

Let $C([0,1],\mathbb{R})$ be the set of all continuous real-valued functions on $[0,1]$ endowed with the partial order relation: $x,y \in C([0,1],\mathbb{R})$, $x \precsim y$ if and only if $x(t) \leq y(t)$ for every $t \in [0,1]$. This is a relation which we can extend in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ as follows:

$$(x,y),(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R}), \quad (x,y) \precsim (u,v) \Longleftrightarrow x \precsim u,\ y \precsim v.$$

In this article, we study the following nonlinear boundary value problem for system of random differential equations:

$$\begin{gathered}
x''(\omega,t) = f_1(\omega,t,x(\omega,t),y(\omega,t)), \quad 0<t<1,\ \omega\in\Omega,\\
y''(\omega,t) = f_2(\omega,t,x(\omega,t,),y(\omega,t)), \quad 0<t<1,\ \omega\in\Omega,\\
x(\omega,0)=0, \quad x(\omega,1)=\psi_1\Big(\int_0^1 x(\omega,t)dt\Big), \quad \omega\in\Omega, \quad \psi_1\in C(\mathbb{R},\mathbb{R}),\\
y(\omega,0)=0, \quad y(\omega,1)=\psi_2\Big(\int_0^1 y(\omega,t)dt\Big), \quad \omega\in\Omega, \quad \psi_2\in C(\mathbb{R},\mathbb{R}),
\end{gathered} \tag{1.1}$$

where $f_1, f_2 : \Omega \times [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are two functions with some regularity properties. By a random solution of system (1.1), we mean a couple of measurable functions $(x,y) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ satisfying (1.1). The interest for such a kind of equations is motivated as follows (see also the books of Bharucha-Reid [4] and Skorohod [21]): the mathematical model representation of natural phenomena arising in biology, physics, engineering processes deal with specific parameters which may assume unknown values. If we want to take into account this uncertainty, a

2010 *Mathematics Subject Classification.* 35R60, 47H10.
Key words and phrases. Measurable space; random differential equation; random fixed point; vector-valued metric.

Submitted February 15, 2017. Published April 25, 2017.

way to model it is based on the parameter $\omega \in \Omega$. From (1.1), in absence of ω, we retrieve the system

$$\begin{gathered}
x''(t) = f_1(t, x(t), y(t)), \quad 0 < t < 1, \\
y''(t) = f_2(t, x(t), y(t)), \quad 0 < t < 1, \\
x(0) = 0, \quad x(1) = \psi_1\Big(\int_0^1 x(t)dt\Big), \quad \psi_1 \in C(\mathbb{R}, \mathbb{R}), \\
y(0) = 0, \quad y(1) = \psi_2\Big(\int_0^1 y(t)dt\Big), \quad \psi_2 \in C(\mathbb{R}, \mathbb{R}),
\end{gathered} \tag{1.2}$$

where $f_1, f_2 \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$. So by a solution of system (1.2), we mean a couple of functions $(x, y) \in C([0,1], \mathbb{R}) \times C([0,1], \mathbb{R})$ satisfying (1.2). Precisely, by using Green's function from the literature, the couple of solutions is such that

$$\begin{aligned}
x(t) &= \int_0^1 K(t,s) f_1(s, x(s), y(s)) ds + \psi_1\Big(\int_0^1 x(s)ds\Big)t, \quad 0 < t < 1, \\
y(t) &= \int_0^1 K(t,s) f_2(s, x(s), y(s)) ds + \psi_2\Big(\int_0^1 y(s)ds\Big)t, \quad 0 < t < 1,
\end{aligned}$$

where

$$K(t,s) = \begin{cases} -t(1-s), & 0 \le t \le s \le 1, \\ -s(1-t), & 0 \le s \le t \le 1. \end{cases} \tag{1.3}$$

System (1.2) and its equation version are largely studied by many authors, with different local and nonlocal conditions. Here, we recall some interesting contributions from the existing literature. Multi-point boundary value problems were studied by Moshinsky [11] and Palamides [15]. Existence, localization and multiplicity of solutions for systems of local and nonlocal boundary value problems were proved by Agarwal-O'Regan-Wong [1, 2, 3], Bolojan-Nica-Infante-Precup [5], Henderson-Ntouyas-Purnaras [7], Precup [17, 18, 19]. An interesting way of studying differential equations makes use of the fixed point theory. For instance, Nieto-Rodríguez-López [13, 14] studied ordinary differential equations via fixed point theorems in partially ordered sets. On the other hand, few authors have investigated the case of random differential equations. Here we recall the recently published papers of Li-Duan [9], Nieto-Ouahab-Rodríguez-López [12] (which is the main inspiration of this work) and Sinacer-Nieto-Ouahab [20]. These authors consider the problem of fixed points for random operators and use this problem to study an equivalent problem of solutions for random differential equations. In the references of [12, 20], the reader can find a good list of manuscripts which point out the cornerstones in the development of random fixed point theory and applications; for instance, we refer to Itoh [8] and Papageorgiou [16].

In this paper, using iterative methods from the fixed point theory, together with the theory of measurable spaces and monotone operators, we study problem (1.1). Precisely, first we prove three abstract results which are general random fixed point theorems, then we work with suitable integral operators associated to a large family of random differential equations, finally we deduce the existence of a unique random solution for problem (1.1).

2. Preliminaries

In this section, we collect some basic notions and notation from the literature. By $\mathcal{B}(X)$ we mean the Borel σ-algebra on a metric space X. Given a measurable space (Ω, Σ), by $\Sigma \otimes \mathcal{B}(X)$ we mean the smallest σ-algebra on $\Omega \times X$ containing all the sets $M \times B$ (such that $M \in \Sigma$ and $B \in \mathcal{B}(X)$).

Definition 2.1. Let (Ω, Σ) be a measurable space, X and Y two metric spaces. A mapping $\widehat{h} : \Omega \times X \to Y$ is called Carathéodory if, for all $x \in X$, the mapping $\omega \to \widehat{h}(\omega, x)$ is $(\Sigma, \mathcal{B}(Y))$-measurable (Σ-measurable, for short) and, for all $\omega \in \Omega$, the mapping $x \to \widehat{h}(\omega, x)$ is continuous.

We need the following results from Denkowski-Migórski-Papageorgiou [6].

Theorem 2.2 ([6, Theorem 2.5.22])**.** *If (Ω, Σ) is a measurable space, X is a separable metric space, Y is a metric space and $\widehat{h} : \Omega \times X \to Y$ is a Carathéodory mapping, then $\widehat{h}$ is $\Sigma \otimes \mathcal{B}(X)$-measurable.*

Corollary 2.3 ([6, Corollary 2.5.24])**.** *If (Ω, Σ) is a measurable space, X is a separable metric space, Y is a metric space, $\widehat{h} : \Omega \times X \to Y$ is a Carathéodory mapping and $u : \Omega \to X$ is Σ-measurable, then $\omega \to \widehat{h}(\omega, u(\omega))$ is a Σ-measurable mapping from Ω into Y.*

Let (Ω, Σ) be a measurable space, X a separable metric space and Y a metric space. A mapping $\widetilde{h} : \Omega \times X \to Y$ is said to be superpositionally measurable (sup-measurable, for short), if for all Σ-measurable mapping $u : \Omega \to X$, the mapping $\omega \to \widetilde{h}(\omega, u(\omega))$ is Σ-measurable from Ω into Y. From Corollary 2.3 we deduce that a Carathéodory mapping is sup-measurable. Also every $\Sigma \otimes \mathcal{B}(X)$-measurable mapping is sup-measurable (see Denkowski-Migórski-Papageorgiou [6, Remark 2.5.26]). Moreover, a mapping $f : \Omega \times X \to X$ is called random operator whenever, for any $x \in X$, $\omega \to f(\omega, x)$ is Σ-measurable. So, a random fixed point of f is a Σ-measurable mapping $z : \Omega \to X$ such that $z(\omega) = f(\omega, z(\omega))$ for all $\omega \in \Omega$.

Lemma 2.4. *Let X, Y be two locally compact metric spaces. A mapping $f : \Omega \times X \to Y$ is Carathéodory if and only if the mapping $\omega \to r(\omega)(\cdot) = f(\omega, \cdot)$ is Σ-measurable from Ω to $C(X, Y)$ (i.e., the space of all continuous functions from X into Y endowed with the compact-open topology).*

3. Fixed point theorems

In this section we prove three theorems producing the existence and uniqueness of a random fixed point for a given mapping $f : \Omega \times X \to X$, where Ω and X are two nonempty sets.

Later on, we use the following notation. If (Ω, Σ) is a measurable space and X a metric space, then we denote by X^Ω the family of all mappings from Ω into X and by $\mathcal{M}(\Omega, X)$ the subset of X^Ω containing all Σ-measurable mappings. If X is endowed with a partial order $\precsim$, then the mappings $g, h \in X^\Omega$ are comparable if, for every $\omega \in \Omega$, we have $g(\omega) \precsim h(\omega)$ or $h(\omega) \precsim g(\omega)$. Let $h_0 \in X^\Omega$, if $h_n(\omega) = f(\omega, h_{n-1}(\omega))$ for all $\omega \in \Omega$ and $n \in \mathbb{N}$, then we say that $\{h_n\}$ is a Picard sequence starting at h_0 and $\{h_n(\omega)\}$ is a Picard sequence (associate to ω) starting at $h_0(\omega)$.

The hypotheses on the data of the random fixed point problem are the following:

- (H0) (Ω, Σ) is a measurable space, $(X, d, \precsim)$ is a separable complete ordered metric space, and $f : \Omega \times X \to X$ is a random mapping such that, for each $\omega \in \Omega$, $x \to f(\omega, x)$ is a monotone operator;
- (H1) for each $\omega \in \Omega$, there exists a nondecreasing function $r_\omega : [0, +\infty[\to [0, +\infty[$ such that $\lim_{n\to+\infty} r_\omega^n(t) = 0$ for all $t > 0$ and
$$d(f(\omega, x), f(\omega, y)) \le r_\omega(d(x, y)), \quad \text{for all } x, y \in X,\ x \precsim y;$$
- (H2) there exists a mapping $x_0 \in \mathcal{M}(\Omega, X)$ with "$x_0(\omega) \precsim f(\omega, x_0(\omega))$, for each $\omega \in \Omega$" or "$x_0(\omega) \succsim f(\omega, x_0(\omega))$, for each $\omega \in \Omega$";
- (H3) if $\{x_n\}$ is a monotone sequence in X and $x_n \to x$, then x_n and x are comparable for all $n \in \mathbb{N}$.

Remark 3.1. Hypothesis (H0) characterizes the space setting that we will use here and the monotonic behaviour of the mapping $x \to f(\omega, x)$. Hypothesis (H1) is a contraction condition of Matkowski type (see [10]).

Remark 3.2. Hypothesis (H3) is a regularity condition of the partial order relation, that needs to be satisfied whenever we do not assume that f is Carathéodory.

First we establish our theorem with complete proof in the case that f is a Carathéodory mapping. Then, we state the analogous result without this assumption.

Theorem 3.3. *If* (H0)–(H2) *hold and f is a Carathéodory mapping, then there exists $z \in \mathcal{M}(\Omega, X)$ which is a random fixed point of f. Further, if for all $x, y \in \mathcal{M}(\Omega, X)$, there exists $u \in X^\Omega$ that is comparable to x and y, then z is a unique random fixed point of f.*

Proof. Let x_0 and u_0 be two comparable elements of X^Ω. We consider the Picard sequences $\{x_n\}$ and $\{u_n\}$ starting respectively at x_0 and u_0. We claim that
$$\lim_{n\to+\infty} d(x_n(\omega), u_n(\omega)) = 0, \quad \text{for all } \omega \in \Omega. \tag{3.1}$$
Let $\omega \in \Omega$ be fixed, since $x \to f(\omega, x)$ is a monotone operator, we obtain that $x_n(\omega)$ and $u_n(\omega)$ are comparable for each $n \in \mathbb{N}$. Clearly, (3.1) holds if $x_n(\omega) = u_n(\omega)$ for some $n \in \mathbb{N}$. Thus we assume that $x_n(\omega) \ne u_n(\omega)$ for all $n \in \mathbb{N}$. Then by (H1), we have
$$d(x_n(\omega), u_n(\omega)) \le r_\omega(d(x_{n-1}(\omega), u_{n-1}(\omega))) \le r_\omega^n(d(x_0(\omega), u_0(\omega))) \tag{3.2}$$
for all $n \in \mathbb{N}$. From (3.2), using the property of the function r_ω (see (H1)), if we pass to the limit as $n \to +\infty$, we obtain
$$\lim_{n\to+\infty} d(x_n(\omega), u_n(\omega)) = 0.$$
Clearly, this holds for all $\omega \in \Omega$. Next, let $x_0 \in \mathcal{M}(\Omega, X)$ be a mapping as in (H2). If, for each $\omega \in \Omega$, $f(\omega, x_0(\omega)) = x_0(\omega)$, then x_0 is a random fixed point of f. Suppose that, for some $\omega \in \Omega$, $f(\omega, x_0(\omega)) \ne x_0(\omega)$. From (H2), we have that x_0 and x_1 are two comparable elements of X^Ω. Then, from (3.1), if we choose $u_0 = x_1$, we deduce
$$\lim_{n\to+\infty} d(x_n(\omega), x_{n+1}(\omega)) = 0, \quad \text{for all } \omega \in \Omega. \tag{3.3}$$
Now, we show that $\{x_n(\omega)\}$ is a Cauchy sequence for each $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. First of all, we note that $r_\omega(t) < t$ for all $t > 0$ and $r_\omega(0) = 0$. Given a real

number $\varepsilon > 0$, by (3.3), there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m(\omega), x_{m+1}(\omega)) < \varepsilon - r_\omega(\varepsilon), \quad \text{for all } m \in \mathbb{N},\ m \geq n(\varepsilon).$$

We claim that

$$d(x_m(\omega), x_{n+1}(\omega)) < \varepsilon \tag{3.4}$$

whenever $m \geq n(\varepsilon)$ and $n \geq m$. Clearly, (3.4) holds if $n = m$. Now, we suppose that (3.4) holds for some $n \geq m$ and prove that (3.4) holds also for $n+1$. In fact,

$$\begin{aligned} d(x_m(\omega), x_{n+2}(\omega)) &\leq d(x_m(\omega), x_{m+1}(\omega)) + d(x_{m+1}(\omega), x_{n+2}(\omega)) \\ &\leq d(x_m(\omega), x_{m+1}(\omega)) + r_\omega(d(x_m(\omega), x_{n+1}(\omega)) \\ &< \varepsilon - r_\omega(\varepsilon) + r_\omega(\varepsilon) = \varepsilon. \end{aligned}$$

Thus $\{x_n(\omega)\}$ is a Cauchy sequence for all $\omega \in \Omega$. Then there exists $z \in X^\Omega$ such that

$$z(\omega) = \lim_{n\to+\infty} x_n(\omega), \quad \text{for all } \omega \in \Omega.$$

By Corollary 2.3, we obtain $x_n \in \mathcal{M}(\Omega, X)$ for all $n \in \mathbb{N}$ and hence $z \in \mathcal{M}(\Omega, X)$. We claim that $z(\omega) = f(\omega, z(\omega))$ for each $\omega \in \Omega$. The hypothesis that f is a Carathéodory mapping ensures that

$$d(z(\omega), f(\omega, z(\omega))) = \lim_{n\to+\infty} d(x_n(\omega), f(\omega, x_n(\omega))), \quad \text{for all } \omega \in \Omega.$$

From

$$\begin{aligned} d(x_n(\omega), f(\omega, x_n(\omega))) &= d(f(\omega, x_{n-1}(\omega)), f(\omega, x_n(\omega))) \\ &\leq r_\omega(d(x_{n-1}(\omega), x_n(\omega))) \\ &\leq d(x_{n-1}(\omega), x_n(\omega)), \end{aligned}$$

letting $n \to +\infty$, we obtain $d(z(\omega), f(\omega, z(\omega))) = 0$ for all $\omega \in \Omega$. Thus $z(\omega) = f(\omega, z(\omega))$ for each $\omega \in \Omega$, that is, z is a random fixed point of f.

We have to prove the uniqueness of this fixed point. So, we assume that $v \in \mathcal{M}(\Omega, X)$ is another random fixed point of f. If z and v are comparable, then from (H1), we deduce that $z = v$. Assume that z and v are not comparable, that is $z(\omega)$ is not comparable with $v(\omega)$ for some $\omega \in \Omega$. In this case, let $u \in X^\Omega$ be comparable with z and v and let $\{u_n\}$ be the Picard sequence starting at $u_0 = u$. By (3.1) with $x_0 = z$ and $x_0 = v$, we obtain

$$\lim_{n\to+\infty} d(z(\omega), u_n(\omega)) = \lim_{n\to+\infty} d(v(\omega), u_n(\omega)) = 0. \tag{3.5}$$

From (3.5), we obtain $z = v$ and hence z is a unique random fixed point of f. $\square$

Now, we are ready the theorem that produces the existence of a random fixed point of f, by replacing the Carathéodory assumption with hypothesis (H3) and sup-measurability of f.

Theorem 3.4. *If* (H0)–(H3) *hold and f is a sup-measurable mapping, then there exists a mapping $z \in \mathcal{M}(\Omega, X)$ which is a random fixed point of f.*

Proof. Let $\{x_n\}$ and $z \in \mathcal{M}(\Omega, X)$ as in the proof of Theorem 3.3. We note that the hypothesis that f is sup-measurable ensures that $x_n \in \mathcal{M}(\Omega, X)$ for all $n \in \mathbb{N}$. This implies that $z \in \mathcal{M}(\Omega, X)$. By (H3), $x_n(\omega)$ and $z(\omega)$ are comparable for all $n \in \mathbb{N}$ and $\omega \in \Omega$. Using (H1) we obtain

$$d(z(\omega), f(\omega, z(\omega))) \leq d(z(\omega), f(\omega, x_n(\omega))) + d(f(\omega, x_n(\omega), f(\omega, z(\omega)))$$

$$\begin{aligned}&\preceq d(z(\omega), f(\omega, x_n(\omega))) + r_\omega(d(x_n(\omega), z(\omega)))\\ &\preceq d(z(\omega), x_{n+1}(\omega)) + d(x_n(\omega), z(\omega)).\end{aligned}$$

Letting $n \to +\infty$, we obtain $d(z(\omega), f(\omega, z(\omega)) = 0$ for all $\omega \in \Omega$. This means that z is a random fixed point of f. □

Next, we adapt the previous hypotheses for solving the above random fixed point problem in the setting of generalized metric spaces (see Sinacer-Nieto-Ouahab [20]). Let $\mathbb{R}^k_+ := \{x \in \mathbb{R}^k : x_j \geq 0 \text{ for all } j = 1, \dots, k\}$, where $\mathbb{R}^k$ is equipped with the partial order relation:

$$x = (x_1, \dots, x_k),\ y = (y_1, \dots, y_k) \in \mathbb{R}^k, \quad x \preceq y \Longleftrightarrow x_j \leq y_j \text{ for all } j = 1, \dots, k.$$

Also, $x \prec y$ denote that $x \preceq y$ and $x \neq y$; $x \ll y$ denote that $x_j < y_j$ for all $j = 1, \dots, k$; θ denote the zero vector in $\mathbb{R}^k$. Let $\mathcal{R}_k$ be the family of all nondecreasing functions $r = (r_1, \dots, r_k) : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ such that

(i) $\lim_{n \to +\infty} r^n(t) = \theta$ for all $t \in \mathbb{R}^k_+$ with $\theta \prec t$;
(ii) $r(\theta) = \theta$ and $\theta \prec r(t) \prec t$ for $t \in \mathbb{R}^k_+ \setminus \{\theta\}$;
(iii) $\theta \ll t$ implies $r(t) \ll t$.

Example 3.5. Let $r : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ be defined by

$$r(t) = \Big(\frac{t_1}{1 + t_1}, \dots, \frac{t_k}{1 + t_k}\Big) \quad \text{for all } t = (t_1, \dots, t_k) \in \mathbb{R}^k_+.$$

Then $r \in \mathcal{R}_k$.

Example 3.6. Let $r : \mathbb{R}^k_+ \to \mathbb{R}^k_+$ be defined by

$$r(t) = At^T \quad \text{for all } t = (t_1, \dots, t_k) \in \mathbb{R}^k_+$$

where $A = \operatorname{diag}(a_1, \dots, a_k)$ is a diagonal matrix such that $0 < a_j < 1$ for all $j = 1, \dots, k$. Then $r \in \mathcal{R}_k$.

We consider the following set of hypotheses:

(H4) (Ω, Σ) is a measurable space, $(X, d, \precsim)$ is a separable complete ordered generalized metric space, and $f : \Omega \times X \to X$ is a random operator such that, for each $\omega \in \Omega$, $x \to f(\omega, x)$ is a monotone operator;
(H5) for each $\omega \in \Omega$, there exists a function $r_\omega = (r_{\omega,1}, \dots, r_{\omega,k}) \in \mathcal{R}_k$ such that
$$d(f(\omega, x), f(\omega, y)) \preceq r_\omega(d(x, y)) \quad \text{for all } x, y \in X,\ x \precsim y;$$
(H6) there exists a mapping $x_0 \in \mathcal{M}(\Omega, X)$ with "$x_0(\omega) \precsim f(\omega, x_0(\omega))$, for all $\omega \in \Omega$" or "$x_0(\omega) \succsim f(\omega, x_0(\omega))$, for all $\omega \in \Omega$".

Remark 3.7. For the sake of completeness, we point out that hypotheses (H4) and (H6) sound formally as the previous hypotheses (H0) and (H2), but with the difference that here (X, d) denotes a generalized metric space. Precisely, a generalized metric space (X, d) is a pair, where X is a nonempty set and $d : X \times X \to \mathbb{R}^k_+$ is a vector-valued metric, in the sense of the following definition.

Definition 3.8. Let X be a nonempty set. By a vector-valued metric on X, we mean a mapping $d : X \times X \to \mathbb{R}^k_+$ with the following properties:

(i) if $d(u, v) = \theta$ then $u = v$;
(ii) $d(u, v) = d(v, u)$ for all $u, v \in X$;
(iii) $d(u, v) \preceq d(u, w) + d(w, v)$ for all $u, v, w \in X$.

Let (X,d) be a generalized metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$. The sequence $\{x_n\}$ converges to x if, for every $\varepsilon \in \mathbb{R}^k_+$ with $\theta \ll \varepsilon$, there is an $n(\varepsilon) \in \mathbb{N}$ such that for all $n \geq n(\varepsilon)$ we have $d(x_n, x) \ll \varepsilon$. If, for every $\varepsilon \in \mathbb{R}^k_+$ with $\theta \ll \varepsilon$, there is an $n(\varepsilon) \in \mathbb{N}$ such that $d(x_n, x_m) \ll \varepsilon$ for all $n, m \geq n(\varepsilon)$, then $\{x_n\}$ is a Cauchy sequence. If every Cauchy sequence is convergent in X, then X is called a complete generalized metric space.

On this basis we prove our third theorem producing the existence of a random fixed point of f. This result is analogous to the existence part of Theorem 3.3.

Theorem 3.9. *If* (H4)–(H6) *hold and f is a Carathéodory mapping, then there exists a mapping $z \in \mathcal{M}(\Omega, X)$ which is a random fixed point of f.*

Proof. As in the proof of Theorem 3.3, we consider two comparable elements of X^Ω, say x_0 and u_0, and the corresponding Picard sequences $\{x_n\}$ and $\{u_n\}$ starting respectively at x_0 and u_0. Also in this theorem, the first step of the proof is to claim that

$$\lim_{n\to+\infty} d(x_n(\omega), u_n(\omega)) = \theta \quad \text{for all } \omega \in \Omega. \tag{3.6}$$

Then, we have

$$\begin{aligned}
&x_n(\omega) \text{ and } u_n(\omega) \text{ are comparable for all } n \in \mathbb{N} \ (x \to f(\omega, x) \text{ is monotone}),\\
&\Rightarrow\ d(x_n(\omega), u_n(\omega)) \preceq r_\omega(d(x_{n-1}(\omega), u_{n-1}(\omega))) \quad \text{(by (H5))},\\
&\preceq r^n_\omega(d(x_0(\omega), u_0(\omega)) \quad \text{for each } \omega \in \Omega \text{ and all } n \in \mathbb{N}.
\end{aligned}$$

Letting $n \to +\infty$ in the previous inequalities and by property (i) of the elements of $\mathcal{R}_k$, we deduce that (3.6) holds. Next let $x_0 \in \mathcal{M}(\Omega, X)$ be a mapping as in (H6). If, for each $\omega \in \Omega$, $f(\omega, x_0(\omega)) = x_0(\omega)$, then x_0 is a random fixed point of f. Suppose that, for some $\omega \in \Omega$, $f(\omega, x_0(\omega)) \neq x_0(\omega)$. We consider the Picard sequence $\{x_n\}$ starting at x_0. From hypothesis (H6), we obtain that x_0 and x_1 are two comparable elements of X^Ω. Then, from (3.6), if we choose $u_0 = x_1$, we obtain

$$\lim_{n\to+\infty} d(x_n(\omega), x_{n+1}(\omega)) = \theta, \quad \text{for each } \omega \in \Omega. \tag{3.7}$$

Now, we show that $\{x_n(\omega)\}$ is a Cauchy sequence for all $\omega \in \Omega$. Let $\omega \in \Omega$ be fixed. First of all, we note that $\theta \ll t - r_\omega(t)$ whenever $\theta \ll t$ (by property (iii) of the elements of $\mathcal{R}_k$). Then given $\varepsilon \in \mathbb{R}^k_+$ with $\theta \ll \varepsilon$, by (3.7), there exists $n(\varepsilon) \in \mathbb{N}$ such that

$$d(x_m(\omega), x_{m+1}(\omega)) \ll \varepsilon - r_\omega(\varepsilon), \quad \text{for all } m \in \mathbb{N},\ m \geq n(\varepsilon).$$

We claim that

$$d(x_m(\omega), x_{n+1}(\omega)) \ll \varepsilon \tag{3.8}$$

whenever $m \geq n(\varepsilon)$ and $n \geq m$. Note that $\{x_n(\omega)\}$ is a monotone sequence in virtue of (H4) and (H6). Clearly, (3.8) holds if $n = m$. So, we suppose that (3.8) holds for some $n \geq m$ and prove that (3.8) holds also for $n+1$. In fact,

$$\begin{aligned}
d(x_m(\omega), x_{n+2}(\omega)) &\preceq d(x_m(\omega), x_{m+1}(\omega)) + d(x_{m+1}(\omega), x_{n+2}(\omega))\\
&\preceq d(x_m(\omega), x_{m+1}(\omega)) + r_\omega(d(x_m(\omega), x_{n+1}(\omega)))\\
&\ll \varepsilon - r_\omega(\varepsilon) + r_\omega(\varepsilon) = \varepsilon.
\end{aligned}$$

Thus $\{x_n(\omega)\}$ is a Cauchy sequence for all $\omega \in \Omega$. Then, there exists $z \in X^\Omega$ such that

$$z(\omega) = \lim_{n\to+\infty} x_n(\omega), \quad \text{for all } \omega \in \Omega.$$

By Corollary 2.3, we obtain $x_n \in \mathcal{M}(\Omega, X)$ for all $n \in \mathbb{N}$ and hence $z \in \mathcal{M}(\Omega, X)$. The hypothesis that f is a Carathéodory mapping ensures that

$$d(z(\omega), f(\omega, z(\omega))) = \lim_{n\to+\infty} d(x_n(\omega), f(\omega, x_n(\omega))), \quad \text{for all } \omega \in \Omega.$$

From

$$\begin{aligned} d(x_n(\omega), f(\omega, x_n(\omega))) &= d(f(\omega, x_{n-1}(\omega)), f(\omega, x_n(\omega))) \\ &\preceq r_\omega(d(x_{n-1}(\omega), x_n(\omega))) \\ &\preceq d(x_{n-1}(\omega), x_n(\omega)), \end{aligned}$$

letting $n \to +\infty$, we obtain $d(z(\omega), f(\omega, z(\omega))) = \theta$ for all $\omega \in \Omega$. Thus $z(\omega) = f(\omega, z(\omega))$ for each $\omega \in \Omega$, that is, z is a random fixed point of f. $\square$

Remark 3.10. In respect of Theorem 3.9, one can establish also the uniqueness of random fixed point, by using the additional assumptions in the statement of Theorem 3.3. Here, to avoid repetition, we omit details.

4. Solution of boundary value problem (1.1)

In this section, we prove a theorem producing the existence of a unique random solution of problem (1.1), see also Nieto-Ouahab-Rodríguez-López [12]. Let (Ω, Σ) be a measurable space. Let $f_1, f_2 : \Omega\times[0, 1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be Carathéodory functions, which means that $\omega \to f_i(\omega, t, u, v)$ is measurable for all $(t, u, v) \in [0, 1] \times \mathbb{R} \times \mathbb{R}$ and $(t, u, v) \to f_i(\omega, t, u, v)$ is continuous for all $\omega \in \Omega$, $i = 1, 2$. Denote with $\mathcal{G}$ the family of the functions $g : \Omega \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ such that $g_u : \Omega \times [0, 1] \to \mathbb{R}$ is a Carathéodory function for every $u \in C([0, 1], \mathbb{R})$, where $g_u(\omega, t) = g(\omega, t, u(t))$ for all $(\omega, t) \in \Omega \times [0, 1]$. Then, consider the integral operator $F : \Omega \times C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R}) \to C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$ defined by

$$F(\omega, x, y)(t) = (F_1(\omega, x, y)(t), F_2(\omega, x, y)(t)), \quad x, y \in C([0, 1], \mathbb{R}),\ t \in [0, 1],$$

with

$$F_1(\omega, x, y)(t) = \int_0^1 K(t, s)f_1(\omega, s, x(s), y(s))ds + g_{1,x}(\omega, t), \tag{4.1}$$

$$F_2(\omega, x, y)(t) = \int_0^1 K(t, s)f_2(\omega, s, x(s), y(s))ds + g_{2,y}(\omega, t) \tag{4.2}$$

where $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is a continuous function such that $|K(t, s)| \le 1$ for all $t, s \in \mathbb{R}$ and $g_1, g_2 \in \mathcal{G}$.

Remark 4.1. F is a random operator from $\Omega \times C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$ into $C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$. In fact, given $(x, y) \in C([0, 1], \mathbb{R}) \times C([0, 1], \mathbb{R})$, since f_i and g_i $(i = 1, 2)$ are Carathéodory functions, for $s \in [0, 1]$ fixed, the function $h : \Omega\times[0, 1] \to \mathbb{R}$ defined by $h(\omega, t) = K(t, s)f_i(\omega, s, x(s), y(s))$ is Carathéodory. By Lemma 2.4, the integrals in (4.1) and (4.2) are limit of a finite sum of measurable functions. So, the mappings $\omega \to F_1(\omega, x, y)$ and $\omega \to F_2(\omega, x, y)$ are measurable and hence F is a random operator.

The hypotheses are the following:

(H7) for each $\omega \in \Omega$ there exists a nondecreasing function $\psi_\omega = (\psi_{\omega,1}, \psi_{\omega,2}) : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ such that

$$|f_i(\omega, t, x, y) - f_i(\omega, t, u, v)| \le \psi_{\omega,i}((|x - u|, |y - v|)), \quad i = 1, 2,$$

for each $t \in [0,1]$ and all $x, y, u, v \in \mathbb{R}$ with $(x,y) \preceq (u,v)$;

(H8) for each $\omega \in \Omega$ there exists a function $r_\omega = (r_{\omega,1}, r_{\omega,2}) \in \mathcal{R}_2$ such that

$$\begin{aligned}
&\psi_{\omega,1}((\|x-u\|_\infty, \|y-v\|_\infty)) + |g_{1,x}(\omega,t) - g_{1,u}(\omega,t)| \\
&\le r_{\omega,1}((\|x-u\|_\infty, \|y-v\|_\infty)), \\
&\psi_{\omega,2}((\|x-u\|_\infty, \|y-v\|_\infty)) + |g_{2,y}(\omega,t) - g_{2,v}(\omega,t)| \\
&\le r_{\omega,2}((\|x-u\|_\infty, \|y-v\|_\infty)),
\end{aligned}$$

for each $t \in [0,1]$ and all $x, y, u, v \in C([0,1],\mathbb{R})$;

(H9) for each $\omega \in \Omega$ fixed, $(x,y) \to f_i(\omega,t,x,y)$ and $x \to g_i(\omega,t,x)$, $i = 1,2$, (for every $t \in [0,1]$), are all nondecreasing or all nonincreasing operators;

(H10) one of the following conditions holds:

$$0 \le f_i(\omega,t,0,0), \quad 0 \le g_i(\omega,t,0), \quad \text{for all } t \in [0,1],\ \omega \in \Omega,\ i = 1,2,$$

or

$$0 \ge f_i(\omega,t,0,0), \quad 0 \ge g_i(\omega,t,0), \quad \text{for all } t \in [0,1],\ \omega \in \Omega,\ i = 1,2.$$

Later on, we consider $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ equipped with the generalized metric d given by

$$d((x,y),(u,v)) = (\|x-u\|_\infty, \|y-v\|_\infty),$$

for all $(x,y),(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$. Now, we have the theorem producing a unique random fixed point.

Theorem 4.2. *If the hypotheses* (H7)–(H10) *hold, then the random integral operator F has a unique random fixed point.*

Proof. For $\omega \in \Omega$ fixed, we show that $(x,y) \to F(\omega,x,y)$ is a continuous operator. Indeed, consider a sequence $\{(x_n,y_n)\}$ in $C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ with $(x_n,y_n) \to (x,y) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, as $n \to +\infty$. For $t \in [0,1]$, we have

$$\begin{aligned}
&|F_1(\omega,x_n,y_n)(t) - F_1(\omega,x,y)(t)| \\
&\le \int_0^1 |f_1(\omega,s,x_n(s),y_n(s)) - f_1(\omega,s,x(s),y(s))| ds + |g_{1,x_n}(\omega,t) - g_{1,x}(\omega,t)| \\
&\le \int_0^1 \psi_{\omega,1}((|x_n(s)-x(s)|, |y_n(s)-y(s)|)) ds + |g_{1,x_n}(\omega,t) - g_{1,x}(\omega,t)| \\
&\le \psi_{\omega,1}((\|x_n(s)-x(s)\|_\infty, \|y_n(s)-y(s)\|_\infty)) + |g_{1,x_n}(\omega,t) - g_{1,x}(\omega,t)|
\end{aligned}$$

implies

$$\|F_1(\omega,x_n,y_n) - F_1(\omega,x,y)\|_\infty \le r_{\omega,1}((\|x_n-x\|_\infty, \|y_n-y\|_\infty))$$

by (H8).

By an analogous reasoning one has

$$\|F_2(\omega,x_n,y_n) - F_2(\omega,x,y)\|_\infty \le r_{\omega,2}((\|x_n-x\|_\infty, \|y_n-y\|_\infty)).$$

So $d(F(\omega,x_n,y_n), F(\omega,x,y)) \to (0,0)$, as $n \to +\infty$, implies that $(x,y) \to F(\omega,x,y)$ is a continuous operator, for each fixed $\omega \in \Omega$.

In addition, for each $\omega \in \Omega$, $(x,y) \to F(\omega,x,y)$ is a monotone operator. Indeed, consider $(x,y),(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that $(x,y) \precsim (u,v)$, that is,

$x(t) \le u(t)$, $y(t) \le v(t)$, for all $t \in [0,1]$. For every $t \in [0,1]$, if $(x,y) \to f_i(\omega,t,x,y)$ and $x \to g_i(\omega,t,x)$, $i = 1,2$, are nondecreasing operators, then

$$\begin{aligned} f_i(\omega,t,x(t),y(t)) \le f_i(\omega,t,u(t),v(t)), &\quad \text{for all } t \in [0,1],\ i=1,2,\\ g_1(\omega,t,x(t)) \le g_1(\omega,t,u(t)), &\quad \text{for all } t \in [0,1],\\ g_2(\omega,t,y(t)) \le g_2(\omega,t,v(t)), &\quad \text{for all } t \in [0,1], \end{aligned}$$

implies

$$F_i(\omega,x,y)(t) \le F_i(\omega,u,v)(t), \quad \text{for all } t \in [0,1],\ i=1,2,$$

which implies $F(\omega,x,y) \precsim F(\omega,u,v)$.

In a similar way, for every $t \in [0,1]$, whenever $(x,y) \to f_i(\omega,t,x,y)$ and $x \to g_i(\omega,t,x)$, $i = 1,2$, are nonincreasing operators, then we deduce that $F(\omega,u,v) \precsim F(\omega,x,y)$.

A crucial step of the proof is to show that F satisfies a contraction condition (see hypothesis (H5)). Precisely, for every $\omega \in \Omega$ and all $(x,y),(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ such that $(x,y) \precsim (u,v)$, we have to show that

$$d(F(\omega,x,y),F(\omega,u,v)) \preceq r_\omega(d((x,y),(u,v))).$$

Again, consider $\omega \in \Omega$ fixed. Let $(x,y),(u,v) \in C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$ be such that $(x,y) \precsim (u,v)$, then

$$\begin{aligned} &|F_1(\omega,x,y)(t) - F_1(\omega,u,v)(t)|\\ &\le \int_0^1 |f_1(\omega,s,x(s),y(s)) - f_1(\omega,s,u(s),v(s))|ds + |g_{1,x}(\omega,t) - g_{1,u}(\omega,t)|\\ &\le \int_0^1 \psi_{\omega,1}((|x(s)-u(s)|,|y(s)-v(s)|))ds + |g_{1,x}(\omega,t) - g_{1,u}(\omega,t)|\\ &\le \psi_{\omega,1}((\|x-u\|_\infty,\|y-v\|_\infty)) + |g_{1,x}(\omega,t) - g_{1,u}(\omega,t)| \end{aligned}$$

implies

$$\|F_1(\omega,x,y) - F_1(\omega,x,y)\|_\infty \le r_{\omega,1}((\|x-u\|_\infty,\|y-v\|_\infty)).$$

By an analogous reasoning

$$\begin{aligned} &\|F_2(\omega,x,y) - F_2(\omega,u,v)\|_\infty \le r_{\omega,2}((\|x-u\|_\infty,\|y-v\|_\infty)),\\ &\Rightarrow\ d(F(\omega,x,y),F(\omega,u,v)) \preceq r_\omega(d((x,y),(u,v))). \end{aligned}$$

Now we prove that condition (H6) holds. Precisely, by (H10), we can easily show that

$$0 \le F_1(\omega,\cdot,0,0) \quad \text{and} \quad 0 \le F_2(\omega,\cdot,0,0), \quad \text{for all } \omega \in \Omega,$$

or

$$0 \ge F_1(\omega,\cdot,0,0) \quad \text{and} \quad 0 \ge F_2(\omega,\cdot,0,0), \quad \text{for all } \omega \in \Omega;$$

that is, $F(\omega,0,0) \succeq (0,0)$ for all $\omega \in \Omega$ or $F(\omega,0,0) \preceq (0,0)$ for all $\omega \in \Omega$. This means that, for the couple of null random variables defined as $(0,0) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, by $(0,0)(\omega) = (0,0)$, for all $\omega \in \Omega$, one of the following two conditions holds:

$$F(\omega,(0,0)(\omega)) \succeq (0,0)(\omega), \quad \text{for all } \omega \in \Omega$$

or

$$F(\omega,(0,0)(\omega)) \preceq (0,0)(\omega), \quad \text{for all } \omega \in \Omega.$$

Note that the uniqueness condition also holds. Thus all the hypotheses of Theorem 3.9 are satisfied and so the existence and uniqueness of a fixed point of F is a direct consequence of above Theorem 3.9. $\square$

By particularizing the choice of Carathéodory functions $g_1, g_2 : \Omega\times[0,1]\times\mathbb{R} \to \mathbb{R}$, we can have the theorem producing a unique random solution of problem (1.1). Let $g_i(\omega, t, u(t)) = \psi_i\big(\int_0^1 u(\omega)(s)ds\big)t$, where $\psi_i \in C(\mathbb{R},\mathbb{R})$, for $i = 1, 2$ and consider the random integral operator

$$\widetilde{F}(\omega, x, y)(t) = (\widetilde{F}_1(\omega, x, y)(t), \widetilde{F}_2(\omega, x, y)(t)), \quad x, y \in C([0,1],\mathbb{R}),\ t \in [0,1],$$

with

$$\begin{aligned}
\widetilde{F}_1(\omega, x, y)(t) &= \int_0^1 K(t,s) f_1(\omega, s, x(s), y(s))ds + \psi_1\Big(\int_0^1 x(\omega)(s)ds\Big)t,\\
\widetilde{F}_2(\omega, x, y)(t) &= \int_0^1 K(t,s) f_2(\omega, s, x(s), y(s))ds + \psi_2\Big(\int_0^1 y(\omega)(s)ds\Big)t
\end{aligned}$$

where $K : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is given by (1.3).

Theorem 4.3. *If* (H7)–(H10) *hold, then problem* (1.1) *has a unique random solution.*

Proof. Note that the random fixed points of $\widetilde{F}$ are solutions to (1.1) and conversely. Indeed, given a couple of random variables $(x, y) : \Omega \to C([0,1],\mathbb{R}) \times C([0,1],\mathbb{R})$, we obtain that

$$\widetilde{F}(\omega, x(\omega), y(\omega)) = (x(\omega), y(\omega)), \quad \text{for all } \omega \in \Omega,$$

is equivalent to

$$\begin{aligned}
x(\omega)(t) &= \int_0^1 K(t,s) f_1(\omega, s, x(\omega)(s), y(\omega)(s))ds + \psi_1\Big(\int_0^1 x(\omega)(s)ds\Big)t, \quad 0 < t < 1,\\
y(\omega)(t) &= \int_0^1 K(t,s) f_2(\omega, s, x(\omega)(s), y(\omega)(s))ds + \psi_2\Big(\int_0^1 y(\omega)(s)ds\Big)t, \quad 0 < t < 1,
\end{aligned}$$

so that the corresponding solution of (1.1) is defined as $x(\omega,t) = x(\omega)(t)$, $y(\omega,t) = y(\omega)(t)$, for $t \in [0,1]$ and $\omega \in \Omega$. Then, from Theorem 4.2, there exists a unique random solution to Problem (1.1). $\square$

Acknowledgements. The authors extend their appreciation to the International Scientific Partnership Program ISPP at King Saud University for funding this research work through ISPP#0068.

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