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# APPROXIMATE CONTROLLABILITY OF NONAUTONOMOUS NONLOCAL DELAY DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

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Abstract. The objective of this article is to prove sufficient conditions for the approximate controllability for a class of nonautonomous nonlocal finite delay differential equations with deviating arguments in a Hilbert space. We also establish sufficient conditions for the existence of mild solutions. The results are established using the fixed point theorem of Krasnoselskii and theory of semigroup of bounded linear operators. We discuss an example for the application of the analytical results.

## 1. Introduction

Let $(X, \|\cdot\|)$ be a complex Hilbert space. We study the approximate controllability for the system consisting of nonautonomuos nonlocal finite delay differential equation with deviating arguments in $X$,

$$\begin{gathered}[\frac{d}{dt} + A(t)]x(t) = f(t, x(t), x([h(x(t), t)])) + Bu(t), \quad t \in J = [0, b], \\ x(t) = \phi(t) + g(x)(t), \quad t \in [-a, 0].\end{gathered} \tag{1.1}$$

Here, we assume that $-A(t)$, for each $t \geq 0$ generates a compact analytic semigroup of bounded linear operators on $X$, $u(\cdot)$ is the control function in $L^2(J, U)$ for a Hilbert space $U$, $B$ is a bounded linear operator on $U$ into $X$. The functions $f : J \times X \times X \to X$, $h : X \times J \to J$ and $g : C([-a, b], X) \to C([-a, 0], X)$ satisfy suitable conditions in their arguments stated in Section 2.

Differential equations with deviating arguments are the generalization of differential equations in which the unknown quantity and its derivative appear in different values of their arguments[5, 6, 8, 16, 18, 21, 25]. They arise as models for the important class of physical phenomenon such as self-oscillating systems, the theory of automatic control, the problems of long-term planning in economics, the systems in bio-physics, the study of problems related with combustion in rocket engines, and many other areas of science and technology, and the number is increasing. In some of the models, the information is transferred from the input to the output after finite time. Such systems are called the system with finite delay. The output

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is connected with the state space. Considering the plentiful applications of the differential equation with deviating arguments, many author have studied differential equation with deviating arguments extensively e.g. [5, 6, 8, 13, 16, 17, 18, 21, 25].

The existence of a deviation-delay in time is necessary to avoid the unstable combustion in liquid rocket engines. The delay (in time) in automatic regulator system cause the appearance of a self-exciting oscillation, overregulation, and even of the instability of the system. In this system, the delay is needed to react to the input impulse for the system. Some of the systems in mathematical modeling of many real world phenomena, namely in control theory, population dynamics, biology and epidemiology, electro-mechanical and medical domains can be modelled by nonlocal differential equations with delay. For more details of such systems, we refer to [5, 12, 19, 20, 29].

On the other hand, the concept of controllability is of great importance in mathematical theory of control of finite or infinite dynamical systems [4]. For a nice introduction on control theory of linear systems, we refer to [1, 4]. The main objective of the controllability is to show the existence of a control function, which steers the solution of the system from its initial state to the final state. Exact controllability enables to steer the system to arbitrary final state. However, there are systems where it possible to steer the system to arbitrary small neighbourhood of the final state. This is known as approximate controllability. As far as the applications are concerned, the approximate controllability is more relevant to dynamical systems and the area got much attentions in recent years [5, 6, 7, 11, 12, 14, 15, 19, 20, 22, 23, 24, 29, 31].

It is worth mentioning that the controllability of the systems with nonlocal conditions are better than classical Cauchy condition[2, 9]. So, the approximate controllability of nonlocal systems with delay-deviating arguments have practical importance and studied much in the recent years by many authors in [5, 7, 11, 12, 15, 19, 20, 22, 23, 24, 26, 29, 31].

Das et al. [5] used the Schauder fixed point theorem in their study of approximate controllability for the following system with deviating arguments in a Hilbert space $X$,

$$\begin{gathered} [\frac{d}{dt} - A]x(t) = f(t, x_t, x([h(x(t), t)])) + Bu(t), \quad t \in J = [0, b], \\ x(t) = \phi(t), \quad t \in [-a, 0]. \end{gathered} \tag{1.2}$$

Here, we assume that $-A(t)$, for each $t \geq 0$ generates a strongly continuous semigroup of bounded linear operators on $X$, $u(\cdot)$ is the control function in $L^2(J, U)$ for a Hilbert space $U$, $B$ is a bounded linear operator on $U$ into $X$. The functions $f : J \times X \times X \to X$ and $h : X \times J \to J$ satisfy Lipschitz conditions in their arguments[5].

Very recently, Kamaljeet et al. [20] studied the approximate controllability for the following integro-differential equations with nonlocal condition in a Hilbert space $X$,

$$\begin{gathered} {}^cD^q x(t) + Ax(t) = f(t, x_t) + \int_0^t k(t-s)h(s, x_s)ds + Bu(t), \\ t \in J = [0, b], \\ x_0 = \phi + g(x), \quad \text{on } [-a, 0], \end{gathered} \tag{1.3}$$

where ${}^cD^q$ is the Caputo fractional derivative of order $0 < q < 1$, $A$ generates an analytic semigroup of bounded linear operators on $X$, $u(\cdot)$ is the control function in $L^2(J, U)$ for a Hilbert space $U$, $B$ is a bounded linear operator on $U$ into $X$. The approximate controllability results are established by the fixed point argument for the system (1.3) with appropriate functions $f, g, h$ and the kernel $k$.

However, the approximate controllability for the nonlocal nonautonomous systems with deviating arguments have not studied so far. In this article, we devote our study of the approximate controllability for the nonautonomous systems with deviating arguments for the system (1.1) in an arbitrary infinite dimensional Hilbert space. The results are new and generalize the results in [5, 15].

We organize the article as follows. In Section 2, we provide preliminaries, assumptions and Lemmas that will be needed for proving the main results. We prove the local existence of a solution in Section 3. The approximate controllability results are established in Section 3. Finally, we provide an example to illustrate the application of the abstract results.

## 2. Preliminaries

In this section, we introduce notation, variuos assumptions and Lemmas for the use of the remaining part of the article. We briefly outline the facts concerning evolution family of bounded linear operators, controllability, control function and mild solutions. We refer the book by Bensoussan [1] and Curtain and Zwart [4], Friedman [10], Pazy [27], Tanabe [28] and Yosida[30] for more details.

Let $X$ and $U$ be two complex Hilbert spaces. Let $T \in [0, \infty)$ and $\{A(t) : 0 \le t \le T\}$ be a family of closed linear operators on the Hilbert space $X$. Let $\mathcal{L}(X)$ denote the Banach space of all bounded linear operator on $X$. We assume the following hypothesis.

(H1) For each $0 \le t \le T$, $-A(t)$ is closed linear operators that generates the compact analytic semigroup of bounded linear operator $\mathcal{U} : \Delta \to \mathcal{L}(X)$, where $(t, s) \in \Delta = \{(t, s) \in J \times J : 0 \le s \le t \le T\}$. The domain $D(A)$ of $A(t)$ is dense in $X$ and is independent of $t$.

**Remark 2.1.** The evolution semigroup $\mathcal{U}(t, s)$ is strongly continuous on the compact set $\Delta$, there exists a constant $M > 0$ such that

$$\|\mathcal{U}(t, s)\| \le M \quad \text{for all } (t, s) \in \Delta. \tag{2.1}$$

**Definition 2.2.** An operator $\mathcal{U} : \Delta \to \mathcal{L}(X)$ is said to be a compact evolution family if the following holds,

(a) $\mathcal{U}(s, s) = I$ is the identity operator in $X$ for $s \in J$,
(b) $\mathcal{U}(t, r)U(r, s) = U(t, s)$, $0 \le s \le r \le\le t \le T$,
(c) $\mathcal{U}$ is strongly continuous on $\Delta$,
(d) $\mathcal{U}(t, s)$ satisfies

$$\frac{\partial \mathcal{U}(t, s)}{\partial t} + A(t)\mathcal{U}(t, s) = 0, \quad \frac{\partial \mathcal{U}(t, s)}{\partial s} - \mathcal{U}(t, s)A(s) = 0, \quad (t, s) \in \Delta,$$

(e) $\mathcal{U}(t, s)$ are completely continuous for $(t, s) \in \Delta$.

Let $x(b, \phi, u)$ be the state value of the system (1.1) at terminal time $b$ corresponding to the initial value $\phi$ and the control function $u$. We define the following set

$$R(b, \phi) = \{x(b, \phi, u) : u \in L^2(J, U)\}.$$

The set $R(b,\phi)$ is called the reachable set of the system (1.1) at time $b$.

**Definition 2.3.** (1) A controllability map for the system (1.1) on $J$ is the bounded linear map $\mathcal{B}^b : L^2(J,U) \to X$ which is defined as

$$\mathcal{B}^b u := \int_0^b \mathcal{U}(b,s)Bu(s)ds, \quad \text{for } u \in L^2(J,U). \tag{2.2}$$

(2) The system (1.1) is exactly controllable on $J$ if $R(b,\phi) = X$, that is for all $y_0, y_1 \in X$, there exists $u \in L^2(J,U)$ such that the mild solution to the system (1.1) satisfies $x(0,\phi,u) = y_0$ and $x(b,\phi,u) = y_1$.

(3) The system (1.1) is approximately controllable on $J$ if $\overline{R(b,\phi)} = X$, that is for given $\epsilon > 0$, and $y_0, y_1 \in X$, there exists a control $u \in L^2(J,U)$ steers from the point $x(0,\phi,u) = y_0$ to all points at time $b$ within a distance of $\epsilon$ from $y_1$. More precisely,

$$x(0,\phi,u) = y_0, \ \|x(b,\phi,u) - y_1\| < \epsilon.$$

(4) The controllability Gramian of the system (1.1) on $J$ is defined by

$$\Gamma_0^b := \mathcal{B}^b(\mathcal{B}^b)^*.$$

**Lemma 2.4** ([15])**.** *The following properties hold for the controllability map:*

(a) $(\mathcal{B}^b)^* z(s) = B^*\mathcal{U}^*(b,s)z$, *for* $s \in [0,T]$, $z \in X$.

(b) $\Gamma_0^b = \mathcal{B}^b(\mathcal{B}^b)^* \in \mathcal{L}(X)$ *has the representation*

$$\Gamma_0^b z = \int_0^b \mathcal{U}(b,s)BB^*\mathcal{U}^*(b,s)zds, \quad \textit{for } z \in X \tag{2.3}$$

*and* $\Gamma_0^b \geq 0$, *where* $B^*$ *and* $\mathcal{U}^*$ *denote the adjoint of* $B$ *and* $\mathcal{U}$ *respectively.*

We consider the following control system in $X$,

$$\begin{gathered} [\frac{d}{dt} + A(t)]x(t) = Bu(t), \quad t \in J, \\ x(0) = \phi(0). \end{gathered} \tag{2.4}$$

We define the resolvent operator associated with (2.4) as

$$R(\epsilon, \Gamma_0^b) = (\epsilon I + \Gamma_0^b)^{-1}, \epsilon > 0.$$

We use the assumption

(H2) $\epsilon R(\epsilon, \Gamma_0^b) \to 0$ as $\epsilon \to 0^+$ in the strong operator topology.

**Theorem 2.5** ([22])**.** *Let $H$ be a separable Banach space with dual $H^*$. The following tow statements are equivalent for a symmetric operator $P : H^* \to H$:*

(i) *$P$ is positive,*

(ii) *$x_\epsilon(h) = \epsilon(\epsilon I + PQ)^{-1}(h) \to 0$ as $\epsilon \to 0^+$ in the strong operator topology, where $Q : H \to H^*$ denotes the duality map.*

**Theorem 2.6** ([24])**.** *System* (2.4) *is approximately controllable on $J$ if and only if the condition* (H2) *holds.*

It follows from (H2) that system (2.4) is approximately controllable on $J$ if and only if

$$\langle v, \Gamma_0^b v\rangle = \int_0^b \|B^*\mathcal{U}^*(b,s)v\|^2 ds > 0, \quad \forall v (\neq 0) \in X.$$

We need the following hypotheses for proving the main results.

(H3) For every $t \in J$; $x, y, x', y' \in X$, there exist constants $L_f > 0$ and $M_f > 0$ the nonlinear map $f : J \times X \times X \to X$ satisfies

$$\begin{gathered} \|f(t,x,x') - f(s,y,y')\| \le L_f(\|x-y\| + \|x'-y'\|), \\ \|f(0,x(0),x(h(x(0),0)))\| \le M_f, \quad \forall t,s \in J \\ f(t,\cdot,\cdot) \text{ is continuous.} \end{gathered} \tag{2.5}$$

(H4) There exist constants $L_h > 0$ such that $h : X \times J \to J$ satisfies the condition

$$|h(x,t) - h(y,s)| \le L_h(\|x-y\|), \quad h(\cdot,0) = 0 \tag{2.6}$$

for all $x, y \in X$ and for all $t, s \in J$.

(H5) The function $g : \mathcal{C} \to C([-a,0],X)$ satisfies

$$\begin{gathered} \|g(x) - g(y)\|_{C([-a,0],X)} \le L_g(\|x-y\|_{\mathcal{C}}), \quad \forall x,y \in \mathcal{C}, \\ \|g(x)\|_{C([-a,0],X)} \le L_g(1 + \|x\|_{\mathcal{C}}), \quad \forall x \in \mathcal{C}, \end{gathered} \tag{2.7}$$

where $\mathcal{C} = C([-a,b],X)$.

For $z \in X$ and $\epsilon > 0$, we define the control function $u_\epsilon(t,x)$ for the system (1.1) by

$$\begin{aligned} u_\epsilon(t,x) = B^*\mathcal{U}^*(b,s)R(\epsilon,\Gamma_0^b)\Big\{ & z - U(b,0)[\phi(0) + g(x)(0)] \\ & - \int_0^b \mathcal{U}(b,s)f(s,x(s),x([h(x(s),s)]))ds \Big\}. \end{aligned} \tag{2.8}$$

We also recall the Krasnoselskii's fixed point theorem.

**Theorem 2.7.** *Let $P$ be a map from a closed bounded convex subset $S$ of $X$ into $S$. Suppose that $Px = P_1x + P_2x$ for $x \in S$ and $P_1u + P_2v \in S$ for every pair $u, v \in S$. If $P_1$ is contraction and $P_2$ is compact, then the equation $P_1u + P_2u = u$ has a solution in $S$.*

## 3. Existence of Solution

In this section, we establish the existence and uniqueness of a local solution to the system (1.1) corresponding to a given control function $u_\epsilon$. The proof of the theorem is based on the technique of [15, 20].

We define the

$$C_L(J,X) = \{x \in C(J,X) : \|x(t) - x(s)\| \le L|t-s| \text{ for a constant } L > 0, t,s \in J\}$$

and the space

$$C_{L_0}([-a,b],X) = \{x \in C([-a,b],X) : x \in C_L(J,X)\}.$$

**Definition 3.1.** A function $x \in C_{L_0}([-a,b],X)$ is said to be a mild solution to problem (1.1) if $x(t)$ satisfies

$$\begin{gathered} x(t) = \mathcal{U}(t,0)[\phi(0) + g(x)(0)] + \int_0^t \mathcal{U}(t,s)f(s,x(s),x([h(x(s),s)]))ds \\ + \int_0^t \mathcal{U}(t,s)Bu(s)ds, \quad t \in J = [0,b], \\ x(t) = \phi(t) + g(x)(t), \quad t \in [-a,0]. \end{gathered} \tag{3.1}$$

**Theorem 3.2.** *System* (1.1) *has a unique mild solution in $C_L(J,X)$ for each control $u_\epsilon \in L^2(J,U)$ if the assumptions* (H1)–(H5) *hold and*

$$ML_g + ML_f(2 + LL_h)b < 1.$$

*Proof.* We consider the ball

$$B_r = \{x \in C_{L_0}([-a,b],X) : \|x\|_{C_{L_0}([-a,b],X)} \le r\}.$$

For each $x \in B_r$, we define the map $F_\epsilon$ by

$$F_\epsilon x(t) = \begin{cases} \mathcal{U}(t,0)[\phi(0)+g(x)(0)] + \int_0^t \mathcal{U}(t,s) f(s,x(s),x([h(x(s),s)]))ds \\ + \int_0^t \mathcal{U}(t,s)Bu_\epsilon(s,x)ds, \quad \text{if } t \in J=[0,b], \\ \phi(t)+g(x)(t), \quad \text{if } t \in [-a,0] \end{cases}$$

For simplicity, we denote

$$\begin{gathered} l = \frac{1}{\epsilon}\|BB^*\mathcal{U}^*(b,s)\|\Big\{\|z\| + M[\|\phi(0)\| + L_g(1+r)] \\ + [2M(L_f(1+LL_h)r + M_f M]b\Big\}, \\ K = \frac{1}{\epsilon}\|B\| \sup_{t\in J}\|B^*\mathcal{U}^*(b,t)\|. \end{gathered}$$

For $t \in J$, we have the estimate

$$\begin{aligned} &\|Bu_\epsilon(t,x)\| \\ &\le \frac{1}{\epsilon}\|BB^*\mathcal{U}^*(b,s)\|\Big\{\|z\| + M[\|\phi(0)\| + \|g(x)(0)]\| \\ &\quad + M\int_0^b \Big[\|f(s,x(s),x([h(x(s),s)])) - f(s,0,x([h(x(0),0)])))\| \\ &\quad + \|f(s,0,x([h(x(0),0)])))\|\Big]ds\Big\} \\ &\le K\Big\{\|z\| + M[\|\phi(0)\| + L_g(1+r)] \\ &\quad + M\int_0^b [(L_f(\|x(s)-x(0)\| + LL_h\|x(s)-x(0)\|) + M_f]ds\Big\} \\ &\le K\Big\{\|z\| + M[\|\phi(0)\| + L_g(1+r)] + [2M(L_f(1+LL_h)r + M_f M]b\Big\} \\ &= l. \end{aligned} \tag{3.2}$$

Let $t_1, t_2 \in J$ with $t_1 < t_2$ and $x \in X$. Using [10, Lemmas II.14.1 and 14.4], we obtain

$$\begin{aligned} \|F_\epsilon x(t_1) - F_\epsilon x(t_2)\| &\le \|\mathcal{U}(t_1,0) - \mathcal{U}(t_2,0)\|(\|\phi(0)\| + \|g(x)(0)\|) \\ &\quad + \Big\|\int_0^{t_1}\mathcal{U}(t_1,s)f(s,x(s),x([h(x(s),s)]))ds \\ &\quad - \int_0^{t_2}\mathcal{U}(t_2,s)f(s,x(s),x([h(x(s),s)]))ds\Big\| \\ &\quad + \Big\|\int_0^{t_1}\mathcal{U}(t_1,s)Bu_\epsilon(s,x)ds - \int_0^{t_2}\mathcal{U}(t_2,s)Bu_\epsilon(s,x)ds\Big\| \\ &\le C_1(t_2-t_1) + C_2(M_f + l)(1 + |\log(t_2-t_1)|)(t_2-t_1), \end{aligned}$$

where $C_1 = C(\|\phi(0)\| + \|g(x)(0)\|)$, $C_2$ and $C_3$ are positive constants. Thus $F_\epsilon \in C_L(J,X)$.

Using estimate (3.2), we obtain

$$\|F_\epsilon x(t)\|$$

$$
\begin{aligned}
&\le M[\|\phi(0)\| + L_g(1+r)] + \int_0^t [2M(L_f(1+LL_h)r + M_f]ds + \int_0^t Ml\}ds \\
&\le M[\|\phi(0)\| + L_g(1+r)] + M[2(L_f(1+LL_h)r + M_f]b + Mlb \\
&\le r,
\end{aligned}
$$

provided that

$$
\begin{gathered}
M[\|\phi(0)\| + L_g(1+r)] + M[2(L_f(1+LL_h)r + M_f]b + Mlb \le r, \quad \text{or} \\
M\|\phi\| + ML_g + M\{L_g + 2L_f(1+LL_h)b\}r + M(M_f+l)b \le r, \quad \text{or} \\
M\|\phi\| + ML_g + M(M_f+l)b \le r[1 - M\{L_g + 2L_f(1+LL_h)b\}].
\end{gathered}
$$

This is possible only if $2ML_f(1+LL_h)b \le M\{L_g + 2L_f(1+LL_h)b < 1$. Thus we choose $b$ such that

$$
b < \frac{1}{2ML_f(1+LL_h)}.
$$

So, $F_\epsilon$ maps $B_r$ into itself. We decompose $F_\epsilon$ as $F_\epsilon = F_{\epsilon,1} + F_{\epsilon,2}$, where

$$
F_{\epsilon,1}x(t) = \begin{cases} \mathcal{U}(t,0)[\phi(0) + g(x)(0)] + \int_0^t \mathcal{U}(t,s) f(s, x(s), x([h(x(s), s)]))ds \\ \quad \text{if } t \in J = [0,b], \\ \phi(t) + g(x)(t), \quad \text{if } t \in [-a, 0], \end{cases}
$$

$$
F_{\epsilon,2}x(t) = \begin{cases} \int_0^t \mathcal{U}(t,s) Bu_\epsilon(s)ds, \quad \text{if } t \in J = [0,b], \\ 0, \quad \text{if } t \in [-a, 0]. \end{cases}
$$

We begin by showing that $F_{\epsilon,1}$ is a contraction on $B_r$. For $v_1, v_2 \in B_r$ and $t \in J$, we have

$$
\begin{aligned}
&\|F_{\epsilon,1}v_1(t) - F_{\epsilon,1}v_2(t)\| \\
&\le \|U(t,0)[g(v_1)(0) - g(v_2)(0)]\| \\
&\quad + \int_0^t \mathcal{U}(t,s)\Big[f(s, v_1(s), v_1([h(v_1(s), s)])) - f(s, v_2(s), v_2([h(v_2(s), s)]))\Big]ds \\
&\le ML_g\|v_1 - v_2\|_{\mathcal{C}} + bML_f(2 + LL_h)\|v_1 - v_2\|_{\mathcal{C}} \\
&\le [ML_g + bML_f(2 + LL_h)]\|v_1 - v_2\|_{\mathcal{C}}.
\end{aligned}
$$

Also for $t \in [-a, 0]$, we have

$$
\|F_{\epsilon,1}v_1(t) - F_{\epsilon,1}v_2(t)\| \le L_g\|v_1 - v_2\|_{\mathcal{C}}.
$$

Thus we conclude that

$$
\|F_{\epsilon,1}v_1 - F_{\epsilon,1}v_2\|_{\mathcal{C}} \le \|v_1 - v_2\|_{\mathcal{C}}.
$$

Hence $F_{\epsilon,1}$ is contraction on $B_r$. We next show that the map $F_{\epsilon,2}$ is completely continuous.

**Step I:** Let $\{v_n\}$ be a sequence in $B_r$ such that $v_n \to v \in B_r$ as $n \to \infty$. It follows from $(H3) - (H5)$ that

(a) $\|Bu_\epsilon(s, v_n) - Bu_\epsilon(s, v)\| \to 0$ as $n \to \infty$,
(b) $\|Bu_\epsilon(s, v_n) - Bu_\epsilon(s, v)\| \le 2l$.

Using the dominated convergence theorem, we obtain that

$$
\|F_{\epsilon,2}v_n(t) - F_{\epsilon,2}v(t)\| \le \int_0^t \|U(t,s)[Bu_\epsilon(s, v_n) - Bu_\epsilon(s, v)]\|ds
$$

$$\leq M \int_0^t \|Bu_\epsilon(s, v_n) - Bu_\epsilon(s, v)\| ds \to 0 \quad \text{as } n \to \infty.$$

**Step II:** Let $t_1, t_2 \in J$ such that $t_1 < t_2$ and $v \in B_r$. It follows from [10, Lemma,II. 14.1, 14.4] that

$$\|F_{\epsilon,2}v(t_2) - F_{\epsilon,2}v(t_1)\| \leq C_4(t_2 - t_1)^\beta,$$

for some constants $0 \leq \beta \leq 1$ and $C_4 > 0$. Thus $\{F_{\epsilon,2}(B_r)\}$ is equicontinous on $J$.

**Step III:** We show that $\{F_{\epsilon,2}v(t) : v \in B_r\}$ is relatively compact in $X$. For $t \in [-a, 0]$,

$$\{F_{\epsilon,2}v(t) : v \in B_r\} = \{0\}.$$

If $0 < \eta < t$, then we have

$$\begin{aligned} F^\eta_{\epsilon,2}v(t) &= \int_0^{t-\eta} \mathcal{U}(t, s)Bu_\epsilon(s, v)ds \\ &= \mathcal{U}(t, t-\eta) \int_0^{t-\eta} \mathcal{U}(t-\eta, s)Bu_\epsilon(s, v)ds \\ &= \mathcal{U}(t, t-\eta)I(t, \eta), \end{aligned}$$

where $I(t, \eta) = \int_0^{t-\eta} \mathcal{U}(t-\eta, s)Bu_\epsilon(s, v)ds$. We note that $I(t, \eta)$ is bounded on $B_r$. As $\mathcal{U}(t, s)$ is compact in $X$, so for each $t \in (0, b]$, the set $\{F^\eta_{\epsilon,2}v(t) : v \in B_r\}$ is relatively compact in $X$. Indeed, we have

$$\begin{aligned} \|F_{\epsilon,2}v(t) - F^\eta_{\epsilon,2}v(t)\| &\leq \int_{t-\eta}^t \|\mathcal{U}(t, s)Bu_\epsilon(s, v)\| ds \\ &\leq Ml\eta \to 0 \quad \text{as } \eta \to 0^+. \end{aligned}$$

Thus the set $\{F_{\epsilon,2}v(t) : v \in B_r\}$ is arbitrarily close to the relatively compact set $\{F^\eta_{\epsilon,2}v(t) : v \in B_r\}$ for each $t \in J$. Hence, for all $t \in [-a, b]$ the set $\{F_{\epsilon,2}v(t) : v \in B_r\}$ is relatively compact in $X$.

By Ascoli-Arzela theorem, the set $\{F_{\epsilon,2}v : v \in B_r\}$ is relatively compact in $C([-a, b], X)$. Thus the map $F_{\epsilon,2}$ is completely continuous from $B_r$ to $B_r$.

Thus the map $F_\epsilon$ has fixed point on $B_r$ by Krasnoselskii's fixed point theorem. Hence for each $\epsilon > 0$, the system (1.1) has a mild solution in $B_r$ corresponding to each control $u_\epsilon(s, x)$. □

## 4. Approximate Controllability

We prove the following theorem of approximate controllability for the system (1.1).

**Theorem 4.1.** *Let the assumptions* (H1)–(H5) *hold. Let the functions* $f : J \times X \times X \to X$, $h : X \times J \to J$ *and* $g : \mathcal{C} \to C([-a, 0], X)$ *be uniformly bounded. Then the system* (1.1) *is approximately controllable on* $J$.

*Proof.* From Theorem 3.2, $F_\epsilon$ has fixed point $x_\epsilon$ in $B_r \subset C_{L_0}([-a, b], X)$. That is, $x_\epsilon$ is a mild solution for the control

$$u_\epsilon(t, x_\epsilon) = B^*\mathcal{U}^*(b, t)R(\epsilon, \Gamma^b_0)p(x_\epsilon),$$

where,

$$p(x_\epsilon) = z - \mathcal{U}(b, 0)[\phi(0) + g(x_\epsilon)(0)]$$

$$- \int_0^b \mathcal{U}(b,s) f(s, x_\epsilon(s), x_\epsilon([h(x_\epsilon(s), s)]))ds.$$

Further, we have

$$\begin{aligned} x_\epsilon(b) &= \mathcal{U}(b,0)[\phi(0) + g(x)(0)] + \int_0^b \mathcal{U}(b,s) f(s, x_\epsilon(s), x_\epsilon([h(x_\epsilon(s), s)]))ds \\ &\quad + \int_0^b \mathcal{U}(b,s) B u_\epsilon(s, x_\epsilon) ds, \quad t \in J = [0,b], \\ &= z - p(x_\epsilon) + \Gamma_0^b R(\epsilon, \Gamma_0^b) p(x_\epsilon) \\ &= z - \epsilon R(\epsilon, \Gamma_0^b) p(x_\epsilon). \end{aligned} \tag{4.1}$$

Since $f : J \times X \times X \to X$ and $h : X \times J \to J$ are uniformly bounded, it follows that $f(s, x_\epsilon(s), x_\epsilon([h(x_\epsilon(s), s)])))$ is bounded in $L^2(J, X)$. Thus there exists a subsequence denoted by $f(s, x_\epsilon(s), x_\epsilon([h(x_\epsilon(s), s)])))$ that converges to $f(s)$ say. It follows from the compactness of $\mathcal{U}(b, 0)$ and the boundedness of $g$ that $\mathcal{U}(b, 0)g(x_\epsilon)(0)$ is relatively compact. So, there exists a subsequence denoted by itself and converges to $\widetilde{g}$ say. We define

$$\alpha = z - \mathcal{U}(b,0)\phi(0) - \widetilde{g} - \int_0^b \mathcal{U}(b,s) f(s) ds.$$

By the compactness of $\mathcal{U}(t, s)$ and Arzela-Ascoli theorem, we have

$$\begin{aligned} &\|p(x_\epsilon) - \alpha\| \\ &\le M\|g(x_\epsilon)(0) - \widetilde{g}\| + M \int_0^b \|f(s, x_\epsilon(s), x_\epsilon([h(x_\epsilon(s), s)])) - f(s)\| ds \\ &\to 0 \quad \text{as } \epsilon \to 0+. \end{aligned} \tag{4.2}$$

Again from (4.1), we have

$$\begin{aligned} \|x_\epsilon(b) - z\| &\le \|\epsilon(\epsilon, \Gamma_0^b)(\alpha)\| + \|\epsilon(\epsilon, \Gamma_0^b)(\alpha)\| \|\alpha - p(x_\epsilon)\| \\ &\le \|\epsilon(\epsilon, \Gamma_0^b)(\alpha)\| + \|p(x_\epsilon) - \alpha\|. \end{aligned}$$

By assumption (H2) and (4.2), we have

$$\|x_\epsilon(b) - z\| \to 0 \quad \text{as } \epsilon \to 0^+.$$

This completes the proof. □

## 5. Application

Let $X = L^2([0, \pi] \times [0, b]; \mathbb{R})$. We consider the following system with deviating arguments in $X$,

$$\begin{gathered} \frac{\partial w(x,t)}{\partial t} + [\kappa(x,t) + \frac{\partial^2}{\partial x^2}] w(x,t) \\ = Bu(x,t) + f(x,t,w(x,t), w(x, h(w(x,t),t))), \quad b > t > 0,\ x \in [0,\pi], \\ w(0,t) = 0 = w(\pi, t),\ 0 \le t \le b, \\ w(x,\tau) = \psi(x,\tau) + \int_0^b H(s,\tau)\cos(w(s,x))ds, \quad x \in [0,\pi],\ \tau \in [-a, 0], \end{gathered} \tag{5.1}$$

where

$$f(x,t,w(x,t),w(x,h(w(x,t),t))) = \int_0^\pi \beta(y,x)w(y,\chi(t)|w(y,t)|)dy$$

for all $(x,t) \in [0,\pi] \times [0,b]$, $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Hölder continuous in $t$ with $\chi(0) = 0$ and $\beta \in C^1([0,\pi] \times [0,\pi];\mathbb{R})$, $H(s,\tau)$ is $C^1([0,b] \times [-a,0],\mathbb{R})$, $\kappa(x,t)$ are $C^1([0,\pi] \times [0,b],\mathbb{R})$.

We write $w(t)(x) = w(x,\tau)$

$$f(t,w(t),w(h(w(t),t)))(x) = f(x,t,w(x,t),w(x,h(w(x,t),t))),$$

$\psi(t)(x) = \psi(x,t)$. With this notation, system (5.1) can be put in the form of (1.1).

We define

$$A(t)v(x) = [\kappa(x,t) + \frac{\partial^2}{\partial x^2}]v(x,t),$$

where $\frac{\partial^2}{\partial x^2}$ is the distributional derivative of $u$. Then $D(A(t)) = H^2(0,\pi) \cap H_0^1(0,\pi)$. It is known that that $-A(t)$ generates a compact analytic evolution semigroup of bounded operators $\mathcal{U}(t,s)$ on $L^2[0,\pi]$ [10] and is given by

$$\mathcal{U}(t,s)v = T(t-s)e^{\int_s^t \kappa(\tau)d\tau}v, \quad v \in D(A(t)).$$

Here

$$T(t)v(\tau) = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle v, e_n\rangle_{L^2} e_n(\tau)$$

with $e_n(\tau) = \sqrt{2}\sin n\tau$, $n = 1,2,3,\dots$, and $\|T(t)\| \le e^{-\pi^2 t}, t \ge 0$. We can show that assumptions (H3) and (H4) are satisfied for the functions $f$ and $h$ respectively. We also note that $g$ satisfies assumption (H5).

We define an infinite dimensional control space

$$U = \{u : u = \sum_0^\infty u_n e_n(x), \sum_0^\infty |u_n|^2 < \infty\},$$

endowed with the norm $\|u\| = (\sum_0^\infty |u_n|^2)^{1/2}$. We define $B : U \to X$ by

$$Bu = 3u_2 e_1(x) + \sum_{n=2}^{\infty} u_n e_n(x).$$

Then $B$ is a bounded linear map and the adjoint is

$$B^*v = (3v_2 + 2v_2)e_2(x) + \sum_{n=3}^{\infty} u_n e_n(x).$$

If we assume that $B^*U*(t,s)v = 0$, then $v = 0$. Thus system (5.1) is approximately controllable on $[0,b]$.

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