Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 113, pp. 1–10.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF POSITIVE SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS INVOLVING GRADIENT TERM AND REACTION POTENTIAL

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Communicated by Jesus Ildefonso Diaz

Abstract. In this note we study the elliptic system

$$\begin{gathered} -\Delta u = z^p + f(x) \quad \text{in } \Omega, \\ -\Delta z = |\nabla u|^q + g(x) \quad \text{in } \Omega, \\ z, u > 0 \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $p > 0$, $0 < q \leq 2$ with $pq < 1$ and f, g are two nonnegative measurable functions. The main result of this work is to analyze the interaction between the potential and the gradient terms in order to get the existence of a positive solution.

1. Introduction

In this work we study the elliptic system

$$\begin{gathered} -\Delta u = z^p + f(x) \quad \text{in } \Omega, \\ -\Delta z = |\nabla u|^q + g(x) \quad \text{in } \Omega, \\ z, u > 0 \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $p > 0$, $0 < q \leq 2$ and f, g are non negative measurable functions. Our goal is to prove the existence of a positive solution under some suitable hypotheses on the data.

Elliptic systems with gradient appear when dealing of the modeling of an electrochemical engineering problem, see [11]. We refer also to [10] and [7] for other applications of these class of systems.

Recently, in [5], the authors consider the system

$$\begin{gathered} -\operatorname{div}(b(x,z)\nabla u) = f(x) \quad \text{in } \Omega, \\ -\operatorname{div}(a(x,z)\nabla z) = b(x,z)|\nabla u|^2 \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

2010 *Mathematics Subject Classification.* 35K15, 35K55, 35K65, 35B05, 35B40.
Key words and phrases. Elliptic system; Schauder fixed point theorem; gradient dependance.

Submitted July 6, 2016. Published April 26, 2017.

where $a(x,s), b(x,s)$ are positive and coercive Caratheodory functions. Under the hypothesis that $f \in L^m(\Omega)$ with $m \geq \frac{2N}{N+2}$, they proved the existence and regularity of a positive solution.

When the gradient appears as an absorption term, the system becomes

$$\begin{gathered} -\operatorname{div}(a(x,z)\nabla u) = f \quad \text{in } \Omega, \\ -\operatorname{div}(b(x,z)\nabla z) + K(x,z)|\nabla u|^2 = g \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

This system was studied in [6]. It is clear that in this case a priori estimate can be obtained easily and existence is allowed for L^1 data.

In [1] the authors deal with the so-quoted "elliptic system with triangular structure", namely they consider the system

$$\begin{gathered} -\Delta u_i = f_i(x,u,\nabla u) + F_i(x) \quad \text{in } \Omega, \\ u_i = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where $\sum_{1\leq j\leq i} f_j \leq 0$ and $1 \leq i \leq m$. It is clear that under the above condition on $\{f_i\}_i$, the gradient terms in (1.2) have an absorption effect and then a priori estimates can be inferred directly. We refer also to [13] where a variation of the system (1.2) is studied in a radial domain with blow-up boundary conditions.

In the case where $p=1$, $q=2$, $g=f=0$ and $\Omega = B_R(0)$, the system is reduced to

$$-\Delta u = z, -\Delta z = |\nabla u|^2 \quad \text{in } B_R(0). \tag{1.3}$$

Using the radial structure of the previous system, the authors in [7] were able to reduce (1.3) to the study of first-order ODEs and then they proved existence and uniqueness of a nonnegative large radial solution to (1.3).

The parabolic version of problem (1.3) is studied as a modification of the classical Boussinesq approximation for buoyancy-driven flows of viscous incompressible fluids, see [8, 9] for more details in this direction.

In our case the situation is quite different and we need to analyze the approximated system to get a priori estimates. This note is organized as follows, in Section 2 we introduce some preliminaries results, like the functional setting and some other useful tools. Section 3 is dedicated to prove our main existence result. Notice that, as a consequence of the existence results we will be able to show an existence result for the Bi-Laplacian operator with gradient term $|\nabla u|^q$ under suitable hypothesis on q.

In Section 4 we give some optimal conditions and we collet some open problems. In the first subsection we prove non existence results, that, in some sense, justify the conditions imposed on p and q to get the existence of positive solution for all $f, g \in L^2(\Omega)$. Some interesting open problems related to (1.1) are given in the last subsection.

2. Preliminaries

In this section, we begin by recalling some useful results. Since we are considering problems with general datum, we will use the concept of weak solution.

Definition 2.1. Let $f, g \in L^1(\Omega)$ be nonnegative functions. Assume that $p > 0$ and $0 < q \leq 2$, we say that $(u,z) \in L^1(\Omega) \times L^1(\Omega)$ is a weak solution of (1.1), if

$|\nabla u|^q \in L^1(\Omega)$, $z^p \in L^1(\Omega)$ and for all $\varphi \in \mathcal{C}_0^\infty(\Omega)$, we have

$$\int_\Omega (-\Delta\varphi)u = \int_\Omega z^p\varphi + \int_\Omega f\varphi, \quad \text{and} \int_\Omega (-\Delta\varphi)z = \int_\Omega |\nabla u|^q\varphi + \int_\Omega g\varphi. \tag{2.1}$$

Notice that, since $(z^p + f) \in L^1(\Omega)$, then we can see u as a weak solution of the problem

$$-\Delta u = z^p + f(x) \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

Therefore, by a result in [3] we know that $u \in W_0^{1,\sigma}(\Omega)$ for all $\sigma < \frac{N}{N-1}$, more precisely we will use the following result proved in the appendix of [2].

Lemma 2.2. *Assume that $u \in L^1_{\rm loc}(\Omega)$ is such that $\Delta u \in L^1_{\rm loc}(\Omega)$, then for all $p \in [0, \frac{N}{N-1})$, and for any open sets $\Omega_1 \subset \Omega_2 \subset \overline{\Omega}_2 \subset \Omega$, there exists a positive constant $C \equiv C(p, \Omega_1, \Omega_2, N)$ such that*

$$\|u\|_{W^{1,p}(\Omega_1)} \le C \int_{\Omega_2} (|u| + |\Delta u|)\, dx. \tag{2.2}$$

Moreover if $u \in L^1(\Omega)$ and $\Delta u \in L^1(\Omega)$, then the above estimate holds globally in the domain Ω.

To prove the main existence result, we use the next Schauder fixed point Theorem.

Theorem 2.3. *Let T be a continuous and compact mapping of a Banach space into itself, such that the set*

$$\{x \in X : x = \lambda Tx \text{ for some } 0 \le \lambda \le 1\}$$

is bounded. Then T has a fixed point.

3. Existence results

We begin by considering an approximating problem with regular data. More precisely we have the next existence result.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and suppose that $f, g \in L^\infty(\Omega)$ are nonnegative functions. Then for all $p > 0$, $0 < q \le 2$ and for all $\varepsilon > 0$, the system*

$$\begin{gathered} -\Delta u = \frac{z^p}{1+\varepsilon z^p} + f(x) \quad \text{in } \Omega, \\ -\Delta z = \frac{|\nabla u|^q}{1+\varepsilon|\nabla u|^q} + g(x) \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

has a positive solution $(u, z) \in (W_0^{1,2}(\Omega))^2 \cap (L^\infty(\Omega))^2$.

Proof. We will use a fixed point argument. Let $u \in L^1(\Omega)$ be fixed and define (φ, z) to be the unique solution of the system

$$\begin{gathered} -\Delta\varphi = h_\varepsilon(x,u) = \frac{u_+^p}{1+\varepsilon u_+^p} + f(x) \quad \text{in } \Omega, \\ -\Delta z = \frac{|\nabla\varphi|^q}{1+\varepsilon|\nabla\varphi|^q} + g(x) \quad \text{in } \Omega, \\ \varphi = z = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

It is clear that $h_\varepsilon \in L^\infty(\Omega)$, thus $\varphi \in \mathbb{X}(\Omega) \equiv \mathcal{C}^{1,\sigma}(\Omega) \cap L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$. Thus z is well defined and $z \in \mathbb{X}(\Omega)$. Hence we can define the operator $T : L^1(\Omega) \to L^1(\Omega)$, $T(u) = z$. We claim that T satisfies the conditions of Schauder fixed point Theorem. The proof of the claim will be done in several steps.

Step I: T is continuous. Let $\{u_n\}_n \subset L^1(\Omega)$ be such that $u_n \to u$ strongly in $L^1(\Omega)$. We set $z_n = T(u_n)$ and $z = T(u)$, then (φ_n, z_n) and (φ, z) satisfy

$$
\begin{gathered}
-\Delta\varphi_n = h_\varepsilon(x, u_n) \quad \text{in } \Omega,\\
-\Delta z_n = \frac{|\nabla\varphi_n|^q}{1+\varepsilon|\nabla\varphi_n|^q} + g(x) \quad \text{in } \Omega,\\
\varphi_n = z_n = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.3}
$$

and

$$
\begin{gathered}
-\Delta\varphi = h_\varepsilon(x, u) \quad \text{in } \Omega,\\
-\Delta z = \frac{|\nabla\varphi|^q}{1+\varepsilon|\nabla\varphi|^q} + g(x) \quad \text{in } \Omega,\\
\varphi = z = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{3.4}
$$

Notice that

$$
-\Delta(\varphi_n - \varphi) = h_\varepsilon(x, u_n) - h_\varepsilon(x, u).
$$

Taking into account that $|h_\varepsilon(x, s)| \le C(\varepsilon)$ and since $u_n \to u$ strongly in $L^1(\Omega)$, it holds $h_\varepsilon(., u_n) \to h_\varepsilon(., u)$ strongly in $L^a(\Omega)$ for all $a > 0$. Moreover, using Hölder and Poincaré inequalities, it follows that

$$
\int_\Omega |\nabla(\varphi_n - \varphi)|^2\,dx \le \int_\Omega (h_\varepsilon(x, u_n) - h_\varepsilon(x, u))^2\,dx \to 0 \quad \text{as } n \to \infty.
$$

Hence $\varphi_n \to \varphi$ strongly in $W_0^{1,2}(\Omega)$. Now going back to the problems of z_n and z and since

$$
\frac{|\nabla\varphi_n|^q}{1+\varepsilon|\nabla\varphi_n|^q} \to \frac{|\nabla\varphi|^q}{1+\varepsilon|\nabla\varphi|^q} \quad \text{strongly in } L^a(\Omega) \text{ for all } a > 1,
$$

it follows that $z_n \to z$ strongly in $W_0^{1,2}(\Omega)$. Hence $z_n \to z$ strongly in $L^1(\Omega)$. Then T is continuous.

Step II: T is compact. Consider now a sequence $\{u_n\}_n$ such that $\|u_n\|_{L^1(\Omega)} \le C$. As above we set $z_n = T(u_n)$ and define φ_n as the unique solution of the first problem in (3.3). It is clear that $\{\varphi_n\}_n$ is bounded in $L^\infty(\Omega) \cap W_0^{1,2}(\Omega)$. Then up to a subsequence not relabeled, $\varphi_n \rightharpoonup \varphi$ weakly in $W_0^{1,2}(\Omega)$ and strongly in $L^a(\Omega)$ for all $a < 2^*$. Thus $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. Using $(\varphi_n - \varphi)$ as a test function in the equation of φ_n, there results that

$$
\int_\Omega |\nabla(\varphi_n - \varphi)|^2\,dx \le \int_\Omega \nabla\varphi\nabla(\varphi - \varphi_n)\,dx + o(1).
$$

Since $\varphi_n \rightharpoonup \varphi$ weakly in $W_0^{1,2}(\Omega)$, it follows that $\varphi_n \to \varphi$ strongly in $W_0^{1,2}(\Omega)$.

Hence up to a subsequence, we reach that $z_n \to z$ strongly in $W_0^{1,2}(\Omega)$ and in particular in $L^1(\Omega)$. Hence T is a compact operator.

To complete the proof of the claim we just have to show that $T(B_R(0)) \subset B_R(0)$ for some ball $B_R(0) \subset L^1(\Omega)$. Notice that by using φ_n as test function in the first equation of (3.3), it follows that $\|\varphi_n\|_{W_0^{1,2}(\Omega)} < C(\varepsilon, \Omega)$. On the other hand, using z_n as a test function in the second equation of (3.3), we obtain $\|z_n\|_{W_0^{1,2}(\Omega)} <$

$C_1(\varepsilon,\Omega)$. Thus $\|z_n\|_{L^1(\Omega)} < C_2(\varepsilon,\Omega)$. Hence choosing $R > C_2(\varepsilon,\Omega)$, we conclude that if $\|u\|_{L^1(\Omega)} \le R$, then $\|z\|_{L^1(\Omega)} \le R$. Hence the claim follows.

Therefore, by Schauder fixed point Theorem, we obtain the existence of u such that $T(u) = u$. It is clear that $u > 0$ in Ω, hence (u,z) solves the system (3.1). Now, by classical regularity results and the previous a priori estimates we obtain easily that $(u,z) \in (W_0^{1,2}(\Omega))^2 \cap (L^\infty(\Omega))^2$. □

Now, we are able to state the main result in this note.

Theorem 3.2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $p > 0$, $0 < q < 2$ with $pq < 1$, then for all $f,g \in L^2(\Omega)$, then system (1.1) has a positive solution (u,z) such that $(u, z^{\frac{\alpha+1}{2}}) \in W_0^{1,2}(\Omega)$ times$W_0^{1,2}(\Omega)$ where $\alpha > 0$ satisfies $p < \frac{\alpha+1}{2} < \frac{1}{q}$.*

Proof. We proceed by approximation. Let $\{f_n\}_n, \{g_n\}_n \subset L^\infty(\Omega)$ be such that $f_n \uparrow f$ and $g_n \uparrow g$ strongly in $L^2(\Omega)$. Let $(u_n,z_n) \in [W_0^{1,2}(\Omega) \cap L^\infty(\Omega)]^2$ be the unique positive solution to the approximate system

$$\begin{gathered} -\Delta z_n = \frac{|\nabla u_n|^q}{1+\frac{1}{n}|\nabla u_n|^q} + g_n(x) \quad \text{in } \Omega, \\ z_n = 0 \quad \text{on } \partial\Omega; \end{gathered} \tag{3.5}$$

$$\begin{gathered} -\Delta u_n = \frac{z_n^p}{1+z_n^p} + f_n(x) \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.6}$$

Notice that the existence of (u_n,z_n) follows by using Theorem 3.1.

Fix $\alpha > 0$ such that the above condition on α holds. Using z_n^α as a test function in (3.5), it follows that

$$-\int_\Omega \Delta z_n z_n^\alpha\,dx = \int_\Omega \frac{|\nabla u_n|^q}{1+\frac{1}{n}|\nabla u_n|^q} z_n^\alpha\,dx + \int_\Omega g_n z_n^\alpha\,dx.$$

Thus by Young and Hölder inequalities we obtain

$$\begin{aligned} &\frac{4\alpha}{(\alpha+1)^2}\int_\Omega \big|\nabla z_n^{\frac{\alpha+1}{2}}\big|^2\,dx \\ &\le \frac{q}{2}\int_\Omega |\nabla u_n|^2\,dx + \frac{2-q}{2}\int_\Omega z_n^{\alpha\frac{2}{2-q}}\,dx + \|g_n\|_{L^2(\Omega)}\Big(\int_\Omega z_n^{2\alpha}\ dx\Big)^{1/2}. \end{aligned}$$

Let us estimate each term in the left hand side of the previous inequality.

Using Sobolev and Young inequalities we easily reach

$$\Big(\int_\Omega z_n^{2\alpha}\ dx\Big)^{1/2} \le \varepsilon\int_\Omega |\nabla z_n^{\frac{\alpha+1}{2}}|^2\ dx + c(\varepsilon).$$

Furthermore, by the fact that $\alpha < \frac{2-q}{q}$, it follows that $\frac{2\alpha}{2-q} < 2^*\frac{\alpha+1}{2}$. Hence

$$\int_\Omega z_n^{\alpha\frac{2}{2-q}}\,dx \le C(\Omega)\Big(\int_\Omega z_n^{2^*\frac{\alpha+1}{2}}\,dx\Big)^{1/\beta} \le C(\Omega)\Big(\int_\Omega |\nabla z_n^{\frac{\alpha+1}{2}}|^2\,dx\Big)^{\frac{2^*}{2\beta}},$$

where $\beta = \frac{2^*(\alpha+1)(2-q)}{4\alpha}$. It is clear that $\frac{2^*}{2\beta} < 1$. Therefore, combining the above estimates we have

$$\int_\Omega |\nabla z_n^{\frac{\alpha+1}{2}}|^2\,dx \le C_1\int_\Omega |\nabla u_n|^2\,dx + C_2. \tag{3.7}$$

Thus

$$\|z_n^{\frac{\alpha+1}{2}}\|^2_{L^{2^*}(\Omega)} \leq C_1\|u_n\|^2_{W_0^{1,2}(\Omega)} + C_3. \tag{3.8}$$

Let us choose u_n as a test function in (3.6), we obtain

$$\int_\Omega |\nabla u_n|^2\,dx = \int_\Omega (z_n^p + f_n)u_n\,dx. \tag{3.9}$$

It is clear that

$$\int_\Omega f_n u_n\,dx \leq C\|f\|_{L^2(\Omega)}\|u_n\|_{W_0^{1,2}(\Omega)}.$$

Now, using Hölder inequality and taking into consideration the estimate (3.8), we obtain

$$\begin{aligned}\int_\Omega z_n^p u_n\,dx &\leq \Big(\int_\Omega z_n^{2^*\frac{\alpha+1}{2}}\,dx\Big)^{\frac{2p}{2^*(\alpha+1)}}\Big(\int_\Omega u_n^{\frac{2^*(\alpha+1)}{2^*(\alpha+1)-2p}}\,dx\Big)^{\frac{2^*(\alpha+1)-2p}{2^*(\alpha+1)}}\\ &\leq C_2\Big(\int_\Omega |\nabla u_n|^2\,dx\Big)^{\frac{p}{\alpha+1}}\Big(\int_\Omega u_n^{2^*}\,dx\Big)^{\frac{1}{2^*}},\end{aligned}$$

this is true because $\frac{2^*(\alpha+1)}{2^*(\alpha+1)-2p} \leq 2^*$. Hence

$$\int_\Omega z_n^p u_n\,dx \leq C\Big(\int_\Omega |\nabla u_n|^2\,dx\Big)^{\frac{p}{\alpha+1}+\frac{1}{2}} + C_1(\Omega).$$

Going back to (3.9) and taking into consideration that $\frac{p}{\alpha+1} + \frac{1}{2} < 1$. We conclude that

$$\int_\Omega |\nabla u_n|^2\,dx \leq C \quad \text{for all } n.$$

Hence we obtain the existence of $u \in W_0^{1,2}(\Omega)$ such that, up to subsequences not relabeled, $u_n \rightharpoonup u$ weakly in $W_0^{1,2}(\Omega)$ and $u_n \to u$ strongly in $L^\sigma(\Omega)$ for all $\sigma < 2^*$.

Now, by (3.7) we conclude that

$$\|z_n^{\frac{\alpha+1}{2}}\|_{W_0^{1,2}(\Omega)} \leq C_1 \quad \text{for all } n.$$

Hence we obtain the existence of a measurable function z such that $z^{\frac{\alpha+1}{2}} \in W_0^{1,2}(\Omega)$ and, up to subsequences not relabeled, $z_n^{\frac{\alpha+1}{2}} \rightharpoonup z^{\frac{\alpha+1}{2}}$ weakly in $W_0^{1,2}(\Omega)$ and $z_n \to z$ strongly in $L^\sigma(\Omega)$ for all $\sigma < \frac{2^*(\alpha+1)}{2}$.

Since $p < \frac{\alpha+1}{2}$, then $\frac{2^*p}{2^*-1} < \frac{2^*(\alpha+1)}{2}$. Thus $z_n^p \to z^p$ strongly in $L^{\frac{2^*p}{2^*-1}}(\Omega)$. Hence classical results for elliptic problem allows us to conclude that

$$u_n \to u \quad \text{strongly in } W_0^{1,2}(\Omega).$$

As a conclusion we obtain that (u, z) is a solution to system (1.1) in the sense of Definition 2.1 with $(u, z^{\frac{\alpha+1}{2}}) \in (W_0^{1,2}(\Omega))^2$. □

As a direct application of the Theorem 3.2, we obtain the next existence result for the Bi-Laplacian problem with gradient term.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. Suppose that $q < 1$ and $g \in L^2(\Omega)$, then the problem*

$$\begin{gathered}\Delta^2 u = |\nabla u|^q + g(x) \quad \text{in } \Omega,\\ \Delta u = u = 0 \quad \text{on } \partial\Omega,\end{gathered} \tag{3.10}$$

has a positive solution u such that $u \in W_0^{1,2}(\Omega)$ and $|\Delta u|^{\frac{\alpha+1}{2}} \in W_0^{1,2}(\Omega)$ where α satisfies $1 < \frac{\alpha+1}{2} < \frac{1}{q}$.

Proof. Taking into consideration the result of Theorem 3.2 with $f \equiv 0$ and $p = 1$, it follows that the system

$$\begin{gathered} -\Delta u = z \quad \text{in } \Omega, \\ -\Delta z = |\nabla u|^q + g(x) \quad \text{in } \Omega, \\ z = u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.11}$$

has a solution (u, z) with $(u, z^{\frac{\alpha+1}{2}}) \in W_0^{1,2}(\Omega)$
$times W_0^{1,2}(\Omega)$ and $1 < \frac{\alpha+1}{2} < \frac{1}{q}$. Hence

$$\Delta^2 u = |\nabla u|^q + g(x) \text{ in } \Omega,$$

and the result follows. □

Remark 3.4. (1) Following closely the above arguments, we can prove that the existence result holds for all $f \in L^{\frac{2^*}{2^*-1}}(\Omega)$ and $g \in L^{\frac{2^*}{2^*-(2-q)}}(\Omega)$.

(2) The same arguments can be used to treat the quasi-linear system

$$\begin{gathered} -\Delta_p u = v^r + f(x) \quad \text{in } \Omega, \\ -\Delta_p v = |\nabla u|^q + g(x) \quad \text{in } \Omega, \\ v, u > 0 \quad \text{in } \Omega, \\ v = u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.12}$$

In this case, we have the next existence result.

Theorem 3.5. *Assume that $r > 0, 0 < q < p$ with $rq < (p-1)^2$ then for all $f, g \in L^{p'}(\Omega)$, then system (3.12) has a positive solution (u, v) such that $(u, v^{\frac{\gamma+p-1}{p}}) \in W_0^{1,p}(\Omega) \times W_0^{1,p}(\Omega)$ where $\gamma > 0$ satisfies $r < \frac{p-1}{p}(p+\gamma-1) < \frac{(p-1)^2}{q}$.*

4. Optimal results and open problems

4.1. **Optimality of the obtained results.**

Theorem 4.1. *Assume that $N > 4$ and that $q > \frac{2N}{N-2} = 2^*$, then there exist $f, g \in L^2(\Omega)$ such that the system (1.1) has no positive solution.*

Proof. We set $f(x) = \frac{1}{|x-x_0|^{2+\sigma}}$ where $x_0 \in \Omega$ and $\sigma > 0$ to be chosen later. Since $N > 4$ and $q > \frac{2N}{N-2}$, then the interval $(\frac{N-q}{q}, \frac{N-4}{2})$ is not empty. Hence we choose $\sigma \in (\frac{N-q}{q}, \frac{N-4}{2})$. It is clear that $f \in L^2(\Omega)$. Now, we argue by contradiction. Assume that the system (1.1) has a positive solution (u, z) such that $|\nabla u|^q \in L^1(\Omega)$ and $(z^p + f) \in L^1(\Omega)$. Then $u \in W_0^{1,q}(\Omega)$. Recall that

$$-\Delta u = z^p + f \geq \frac{1}{|x-x_0|^{2+\sigma}} \quad \text{in } \Omega.$$

Using a simple comparison argument it holds that $u(x) \geq \frac{1}{|x-x_0|^{\sigma}}$ in a small ball $B_r(x_0) \subset\subset \Omega$. Since $u \in W_0^{1,q}(\Omega)$, using Sobolev inequality we conclude that $u \in L^{q^*}(\Omega)$. Thus $u \in L^{q^*}(B_r(x_0))$. As a consequence we reach that $\frac{1}{|x-x_0|^{\sigma}} \in L^{q^*}(B_r(x_0))$. Hence $\sigma q^* < N$; which is a contradiction with the choice of σ. □

Let us begin by showing the optimality of the condition $pq < 1$. More precisely we have the next non existence result.

Theorem 4.2. *Assume that $q = 2$, then for all $p > 1$, there exist $f, g \in L^2(\Omega)$ such that the system (1.1) has no positive solution.*

Proof. Without loss of generality we can assume that $f = \lambda f_1$ and $g = \mu g_1$ with $f_1, g_1 \in L^\infty(\Omega)$. We argue by contradiction. Suppose that the system (1.1) has a positive solution (u, z) such that $|\nabla u|^q \in L^1(\Omega)$ and $z^p \in L^1(\Omega)$. Let $\phi \in \mathcal{C}_0^\infty(\Omega)$, using ϕ^2 as a test function in the equation of u in the system (1.1), it follows that

$$\int_\Omega z^p \phi^2\,dx + \lambda \int_\Omega f_1 \phi^2\,dx = 2\int_\Omega \phi \nabla\phi \nabla u\,dx.$$

Now by Young inequality, it holds

$$\int_\Omega z^p \phi^2\,dx + \lambda \int_\Omega f_1\,\phi^2\,dx \le \int_\Omega \phi^2 |\nabla u|^2\,dx + \int_\Omega |\nabla\phi|^2\,dx. \tag{4.1}$$

From the second equation in the system (1.1) we reach that $|\nabla u|^2 \le -\Delta z$, thus,

$$\int_\Omega \phi^2 |\nabla u|^2\,dx \le \int_\Omega \phi^2(-\Delta z)\,dx = \int_\Omega z(-\Delta\phi)\,dx \le 2\int_\Omega z\phi(-\Delta\phi)\,dx,$$

where the last estimate follows using Kato inequality.

Since $p > 1$, using Young inequality, we conclude that

$$\int_\Omega \phi^2 |\nabla u|^2\,dx \le \varepsilon \int_\Omega \phi^2 z^p\,dx + C(\varepsilon)\int_\Omega |\phi|^{\frac{p-2}{p-1}} |\Delta\phi|^{p'}\,dx.$$

Choosing ε small and going back to (4.1), we obtain that

$$\lambda \int_\Omega f_1\,\phi^2\,dx \le C(\varepsilon)\int_\Omega |\phi|^{\frac{p-2}{p-1}} |\Delta\phi|^{p'}\,dx + \int_\Omega |\nabla\phi|^2\,dx.$$

Thus

$$\lambda \le \frac{C(\varepsilon)\int_\Omega |\phi|^{\frac{p-2}{p-1}} |\Delta\phi|^{p'}\,dx + \int_\Omega |\nabla\phi|^2\,dx}{\int_\Omega f_1\,\phi^2\,dx}.$$

Setting

$$M \equiv \inf_{\phi \in \mathcal{C}_0^\infty(\Omega)} \frac{C(\varepsilon)\int_\Omega |\phi|^{\frac{p-2}{p-1}} |\Delta\phi|^{p'}\,dx + \int_\Omega |\nabla\phi|^2\,dx}{\int_\Omega f_1\,\phi^2\,dx},$$

then if $\lambda > M$, then system (1.1) has no positive solution and we have the conclusion. ☐

4.2. **Some open problems.** In this subsection we collect some interesting open problems.

(1) The case $pq \ge 1$ and $q \le 2$: the arguments used to treat the case $pq < 1$ can not be adapted to the new situation $pq \ge 1$ and $q \le 2$. Hence new arguments are needed to deal with this last case.

(2) If $p = 1$, problem (1.1) takes the form

$$\begin{gathered}
-\Delta u = z + f(x) \quad \text{in } \Omega,\\
-\Delta z = |\nabla u|^q + g(x) \quad \text{in } \Omega,\\
v, u > 0 \quad \text{in } \Omega,\\
v = u = 0 \quad \text{on } \partial\Omega,
\end{gathered}$$

Now, by computing $\Delta^2 u$, we reach that

$$\begin{gathered} \Delta^2 u = |\nabla u|^q + \lambda h(x) \quad \text{in } \Omega, \\ u = \Delta u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{4.2}$$

where $\lambda h = -\Delta f + g$. The existence of solution for (4.2) is interesting for itself since, in the case where Bi-Laplacian operator is substituted by the Laplacian operator, an approach based on the classical elliptic capacity $\mathrm{Cap}_{1,q'}$ gives a necessary and sufficient condition to obtain the existence of a positive solution, see for instance the nice paper [14]. For Bi-Laplacian operator, some particular cases were studied in [7] with radial structure. It seems to be very interesting to get some similar approach in the case of Bi-Laplacian operator with gradient term if $q > 1$.

Acknowledgements. The authors would like to express their gratitude to the anonymous referees for their comments and suggestions that improve the last version of the manuscript.

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