EXISTENCE OF SOLUTIONS FOR DEGENERATE KIRCHHOFF TYPE PROBLEMS WITH FRACTIONAL p-LAPLACIAN

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Abstract. In this article, by using the Fountain theorem and Mountain pass theorem in critical point theory without Palais-Smale (PS) condition, we show the existence and multiplicity of solutions to the degenerate Kirchhoff type problem with the fractional p-Laplacian

\[
(a + b \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy)(-\Delta)^s_p u = f(x, u) \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

where \((-\Delta)^s_p\) is the fractional p-Laplace operator with \(0 < s < 1 < p < \infty\), \(\Omega\) is a smooth bounded domain of \(\mathbb{R}^N\), \(N > 2s\), \(a, b > 0\) are constants and \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a continuous function.

1. Introduction and statement of main results

The aim of this article is to establish the existence of solutions to the Kirchhoff nonlocal problem

\[
(a + b \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N + ps}} \, dx \, dy)(-\Delta)^s_p u = f(x, u) \quad \text{in } \Omega,
\]

\[
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,
\]

where \(\Omega\) is an open bounded subset of \(\mathbb{R}^N\) with Lipschitz boundary, \(N > 2s\) with \(s \in (0, 1)\), \(a, b > 0\) are constants, \(f : \Omega \times \mathbb{R} \to \mathbb{R}\) is a continuous function and \((-\Delta)^s_p\) is the fractional p-Laplacian operator which, up to normalization factors, may be defined as

\[
(-\Delta)^s_p u(x) = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N + ps}} \, dy
\]

for \(x \in \mathbb{R}^N\), where \(B_\varepsilon(x) := \{ y \in \mathbb{R}^N : |x - y| < \varepsilon\}\). As for some recent results on the fractional p-Laplacian, we refer to for example \([22, 21, 24]\) and the references therein.

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When \( a \equiv 1, b \equiv 0 \) and \( p = 2 \), problem (1.1) becomes the fractional Laplacian problem
\[
(-\Delta)^s u = f(x, u) \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega.
\] (1.2)

For the basic properties of fractional Sobolev spaces and the functional framework that takes into account the problem (1.2), we refer the readers to [36]. In [37, 38], Servadei and Valdinoci considered the existence of nontrivial weak solutions of the problem (1.2) by using variational methods. For other recent results in (1.2), the reader is referred, for example, to [4, 39, 40].

Fractional and nonlocal operators and on their applications is very interesting, we refer the readers to [5, 13, 15, 18, 19, 25, 26, 27, 28, 30, 32, 33, 35, 47] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [12, 27]. In [30], Molica Bisci and Vilasi studied a class of Kirchhoff nonlocal fractional equation in a bounded domain \( \Omega \) and obtained three solutions by using three critical point theorem. Pucci and Saldi [32] established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in \( \mathbb{R}^N \) involving a critical nonlinearity and the nonlocal fractional Laplacian. We refer also to [16, 17, 27, 29] for related problems.

Notice that when \( a \equiv 1 \) and \( b \equiv 0 \), as \( s \to 1^- \), problem (1.1) reduces to the problem
\[
-\Delta_p u = f(x, u) \quad \text{in } \Omega,
\] (1.3)
where \( \Omega \subset \mathbb{R}^N \) is a smooth domain.

For the case of a bounded domain, there are several articles considering the system
\[
-\left( a + b \int_\Omega |\nabla u|^p \right) \Delta_p u = g(x, u) \quad \text{in } \Omega
\]
where \( \Omega \subset \mathbb{R}^N \) is a smooth domain, which is related to the stationary analogue of the Kirchhoff equation
\[
u_{tt} - \left( a + b \int_\Omega |\nabla u|^p \right) \Delta_p u = g(x, u),
\]
which was proposed by Kirchhoff [23] as an extension of the classical D’Alembert’s wave equation for free vibrations of elastic string. In recent years, many authors are interesting in Kirchhoff type problems, see for example [2, 3, 8, 9, 10, 11, 33, 42, 43, 44, 45, 46] and references therein.

Motivated by the above works and [7, 31, 36, 37, 38, 41], we study the existence and multiplicity of solutions for Kirchhoff type problem (1.1).

Before proving our main results, some preliminary material on function spaces and norms is needed. In the following, we briefly recall the definition of the functional space \( X_0 \), introduced in [36], and we give some notation. We denote \( Q = \mathbb{R}^{2N} \setminus \mathcal{O} \), where \( \mathcal{O} = \mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega \). We denote
\[
X = \left\{ u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^p(\Omega), \int \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy < \infty \right\},
\]
where \( u|_\Omega \) represents the restriction to \( \Omega \) of function \( u(x) \). Also, we define the following linear subspace of \( X \),
\[
X_0 = \left\{ g \in X : g = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega \right\}.
\]
The linear space $X$ is endowed with the norm
\[ \|u\|_X := \|u\|_{L^2(\Omega)} + \left( \int_Q \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p}. \]
It is easily seen that $\| \cdot \|_X$ is a norm on $X$ and $C_0^\infty(\Omega) \subseteq X_0$ (see [45, Lemma 2.1]). Also, we know that $X_0$, endowed with the norm
\[ \|v\|_{X_0} = \left( \int_Q \frac{|v(x) - v(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{1/p} \]
is a uniformly convex Banach space and a reflexive Banach space [45, Remark 2.1 and Lemma 2.4].

We consider the nonlinear eigenvalue problem
\[ \|u\|_{X_0}^p (-\Delta)_p^s u = \lambda |u|^{2p-2} u \quad \text{in } \Omega, \]
\[ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \]
whose eigenvalues are the critical values of the functional
\[ J_p(u) = \|u\|_{X_0}^{2p}, \quad u \in M = \left\{ u \in X_0 : \int_{\Omega} |u|^{2p} \, dx = 1 \right\}. \]
We know that the first eigenvalue $\lambda_1 := \inf_{u \in M} J_p(u) > 0$. The first eigenfunction is denoted by $\varphi_1$ (see [44] for the case $\theta = 2$).

We denote the usual $L^q(\Omega)$-norm by $\| \cdot \|_q$. Since $\Omega$ is a bounded domain, it is well known that $X_0 \hookrightarrow L^q(\Omega)$ continuously for $p \in [1, p_s^*]$ (see [45, Lemma 2.3]) and compactly for $q \in [1, p_s^*]$, where $p_s^* := \frac{Np}{N-2p}$. Moreover there exists $C_q > 0$ such that
\[ \|u\|_q \leq C_q \|u\|_{X_0}, \quad u \in X_0. \]
We consider the functional $J : X_0 \to \mathbb{R}$ defined by
\[ J(u) = \frac{a}{p} \|u\|_{X_0}^p + \frac{b}{2p} \|u\|_{X_0}^{2p} - \int_{\Omega} F(x, u(x)) \, dx \]
and set
\[ \Psi(u) = \int_{\Omega} F(x, u(x)) \, dx, \]
where $F(x, u) = \int_0^u f(x, s) \, ds$. Obviously, the functional $J$ is well-defined, it is of class $C^1(X_0, \mathbb{R})$ and
\[ \langle J'(u), v \rangle = (a + b \|u\|_{X_0}^p) \int_Q \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|y|^{N+ps}} (v(x) - v(y)) \, dx \, dy \]
\[ - \int_{\Omega} f(x, u(x)) v(x) \, dx, \quad \text{for all } u, v \in X_0, \]
Moreover, the critical points of $J$ are the solutions of problem (1.1). Let
\[ E_j := \oplus_{i \in J} \ker((-\Delta)_p^s - \mu_i), \]
where $0 < \mu_1 \leq \mu_2 \leq \ldots, \mu_i \leq \ldots$, are the eigenvalue of $((-\Delta)_p^s, X_0)$ (see [22, 24, 20]).

**Definition 1.1.** We say that $J$ satisfies the Palais-Smale (PS) condition if any sequence $(u_n) \in X$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.
Also, we need the following definition, which is a weak version of the (PS) condition, due to Cerami [6].

**Definition 1.2.** Let $J \in C^1(X, \mathbb{R})$, we say that $J$ satisfies the Cerami condition at the level $c \in \mathbb{R}$ ((Ce)$_c$ for short), if any sequence $(u_n) \in X$ with

$$J(u_n) \to c, \quad (1 + \|u_n\|)|J'(u_n)| \to 0 \quad \text{as } n \to \infty,$$

possesses a convergent subsequence in $X$; $J$ satisfies the (Ce) condition if $J$ satisfies the (Ce)$_c$ for all $c \in \mathbb{R}$.

The assumptions on the function $f$ are stated as follows:

(A1) There exists a positive constant $C > 0$ such that $|f(x, t)| \leq C(|t|^{p-1} + 1)$, for some $2p < r < p^*_s$, $x \in \Omega$ and all $t \in \mathbb{R}$;

(A2) $\lim_{|t| \to \infty} \frac{1}{2p} f(x, t)t - F(x, t) + \frac{\mu t}{p} t^p = +\infty$ uniformly in $x \in \Omega$;

(A3) there exists $\mu > \mu_1$ such that $F(x, t) \geq \frac{\mu t}{p} t^p$ for $|t|$ small;

(A4) $\lim_{|t| \to \infty} \frac{\mu t}{p} t^p + \frac{\mu t}{2p} 2^p - F(x, t) = +\infty$ uniformly in $x \in \Omega$.

Now we state our main results.

**Theorem 1.3.** Assume that $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, (A1)–(A4) hold. Then (1.1) has at least one nontrivial solution.

In the next theorem we use the assumptions:

(A5) $\lim_{|t| \to \infty} \frac{F(x, t)}{|t|^m} \to \infty$ uniformly in $x \in \Omega$, and there exists $L_1 \geq 0$ such that $F(x, t) \geq 0$ for all $(x, t) \in \Omega \times \mathbb{R}$ and $|t| \geq L_1$;

(A6) there exists $\theta_0 > 0$ such that $F(x, t) \leq \frac{1}{2p} f(x, t)t + \theta_0 |t|^p$, $\forall (x, t) \in \Omega \times \mathbb{R}$;

(A7) $f(x, -t) = -f(x, t)$ for all $(x, t) \in \Omega \times \mathbb{R}$;

**Theorem 1.4.** Assume that (A1), (A5)–(A7) are satisfied. Then problem (1.1) possesses infinitely many nontrivial solutions \{uk\} such that $J(uk) \to +\infty$.

Now, we study the existence of infinitely many solutions of the following problem, which it is a special case of problem (1.1),

$$\left( a + b \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}} \, dx \right) (-\Delta)^s_x u = g(x, u(x)) + H(x)|u|^{p-2} u \quad \text{in } \Omega,$$

$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \quad (1.10)$$

with the following conditions:

(A8) There exists a positive constant $C_G$ such that $|G(x, t)| \leq C_G(|t|^{r-1} + 1)$ for some $2p < r < p^*$, $x \in \Omega$ and all $t \in \mathbb{R}$, where $G(x, t) = \int_0^t f(x, s) \, ds$;

(A9) $\lim_{|t| \to \infty} \frac{1}{2p} g(x, t)t - G(x, t) + \mu q|t|^p + m|t|^q = +\infty$ uniformly in $x \in \Omega$ where $q \leq \frac{1}{p} \left( \frac{1}{p^*} - \frac{1}{r} \right) \mu_1$, $1 < q < p^*$, $r_0 > 2p$ and $m$ is an arbitrary positive constant;

(A10) $G(x, t) \geq 0$, for all $x \in \Omega$, $t \in \mathbb{R}$;

(A11) the function $H$ is a nonnegative and satisfies $0 < m \leq H \leq M$;

(A12) $\lim_{|t| \to 0} \frac{h(x, t)}{|t|^{p-1}} = 0$, uniformly in $x \in \Omega$;

(A13) $G(x, 0) = 0$ for all $x \in \Omega$ and $G(x, -t) = G(x, t)$, for all $x \in \Omega$, $t \in \mathbb{R}$. 

Then problem (1.10) has a sequence of solutions \( \{u_k\} \) such that \( I(u_k) \to +\infty \).

The proofs of our main results are fully based on some theorems that we recalled here for the reader’s convenience.

**Theorem 1.6** (Mountain Pass Theorem [14]). Let \( X \) be a real Banach space and \( J \in C^1(X, \mathbb{R}) \) satisfying the (Ce) condition. Suppose \( J(0) = 0 \),

(i) there are constants \( \rho, \beta > 0 \) such that \( J|_{\partial B_\rho} \geq \beta \)

\[ B_\rho = \{ u \in X : \|u\| \leq \rho \}; \]

(ii) there is \( u_1 \in X \) and \( \|u_1\| > \rho \) such that \( J(u_1) < 0 \).

Then \( J \) possesses a critical value \( c \geq \beta \). Moreover \( c \) can be characterized as

\[ c = \inf_{g \in \Gamma} \max_{u \in \gamma([0,1])} J(u), \quad \Gamma = \{ g \in C([0,1]) : g(0) = 0, g(1) = u_1 \}. \]

**Theorem 1.7** ([14]). Let \( X \) be an infinite dimensional Banach space, \( X = Y \oplus Z \), where \( Y \) is finite dimensional. If \( J \in C^1(X, \mathbb{R}) \) satisfies (Ce)_c-condition for all \( c > 0 \), and

(i) \( J(0) = 0, J(-u) = J(u) \) for all \( u \in X \);

(ii) there exist constants \( \rho, \alpha > 0 \) such that \( J|_{\partial B_\rho} \geq \alpha \);

(iii) for any finite dimensional subspace \( \tilde{X} \subset X \), there is \( R = R(\tilde{X}) > 0 \) such that \( J(u) \leq 0 \) on \( \tilde{X} \setminus B_\rho \);

then \( J \) possesses an unbounded sequence of critical values.

**Theorem 1.8** (Fountain theorem). Let \( X_0 \) be a Banach space with the norm \( \| \cdot \| \) let \( X_i \) be a sequence of subspace of \( X \) with \( \dim X_i < \infty \) for each \( i \in \mathbb{N} \). Further, set

\[ X = \bigoplus_{i=1}^{\infty} X_i, \quad Y_k = \bigoplus_{i=1}^{k} X_i, \quad Z_k = \bigoplus_{i=k}^{\infty} X_i \]

Consider an even functional \( \Phi \in C^1(X, \mathbb{R}) \). Assume that for each \( k \in \mathbb{N} \), there exists \( \rho_k > \gamma_k > 0 \) such that

1. \( \phi_k := \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0, \)
2. \( \beta_k := \inf_{u \in Z_k, \|u\| = \gamma_k} \Phi(u) \to +\infty, \quad k \to +\infty, \)
3. \( \Phi \) satisfies the (PS)_c condition for every \( c > 0 \).

Then \( \phi \) has an unbounded sequence of critical values.

Now, we need the following lemma about the (Ce) condition which will play an important role in the proof of our main results.

**Lemma 1.9.** Assume that (A1) and (A2) hold. Then the functional \( J : X_0 \to \mathbb{R} \) satisfies the (Ce) condition.

**Proof.** Let \( \{u_n\} \) be a (Ce)_c sequence for \( c \in \mathbb{R} \),

\[ J(u_n) \to c, \quad (1 + \|u_n\|_{X_0})J'(u_n) \to 0 \quad \text{as} \ n \to \infty. \quad (1.11) \]

We first show that \( \{u_n\} \) is a bounded sequence. In view of (1.8), (1.9) and (1.11), one has

\[ 1 + c \geq J(u_n) - \frac{1}{2p} J'(u_n) u_n \]

\[ = \frac{a}{2p} \|u_n\|_{X_0}^p + \int_{\Omega} \left( \frac{1}{2p} f(x, u_n(x)) u_n(x) - F(x, u_n(x)) \right) dx. \quad (1.12) \]
From (A2), there exists $\theta > 0$ such that
\[
- \theta \leq \frac{1}{2p} f(x,t) t - F(x,t) + \frac{\alpha \mu_1}{2p} |t|^p, \quad \forall x \in \Omega, \ t \in \mathbb{R}.
\] (1.13)

Now, We define $u_n = \varphi_n + v_n$, where $\varphi_n \in E_1$ and $v_n \in E_1^\perp$. By (1.12) and (1.13), we have
\[
1 + c \geq \frac{a}{2p} \lVert u_n \rVert_{X_0}^p - \frac{\alpha \mu_1}{2p} \lVert u_n \rVert_{L^p}^p + \int_\Omega \left( \frac{1}{2p} f(x,u_n(x)) u_n(x) - F(x,u_n(x)) + \frac{\alpha \mu_1}{2p} |u_n(x)|^p \right) dx
\geq \frac{a}{2p} \left(1 - \frac{\mu_1}{\mu_2}\right) \lVert u_n \rVert_{X_0}^p - \theta |\Omega|,
\] (1.14)
which implies that $\lVert v_n \rVert_{X_0}$ is bounded. Now, we assume that $\{u_n\}$ is an unbounded sequence, so there is a subsequence $\{u_{n_k}\}$ (to simplify the notation) of $\{u_n\}$ satisfying $\lVert u_{n_k} \rVert_{X_0} \to +\infty$ as $n \to +\infty$. Hence we have $\frac{v_{n_k}}{\lVert u_{n_k} \rVert_{X_0}} \to 0 \in X_0$. Since $\frac{\varphi_n}{\lVert u_n \rVert_{X_0}}$ is bounded in finite dimensional $E_1$, one can get $\frac{\varphi_n}{\lVert u_n \rVert_{X_0}} \to w$ in $E_1$. Using
\[
w_{n_k} := \frac{u_{n_k}}{\lVert u_{n_k} \rVert_{X_0}} = \frac{\varphi_n + v_{n_k}}{\lVert u_{n_k} \rVert_{X_0}} = \frac{\varphi_n}{\lVert u_{n_k} \rVert_{X_0}} + \frac{v_{n_k}}{\lVert u_{n_k} \rVert_{X_0}} \to w,
\]
in $E_1$, yields
\[
\frac{u_{n_k}(x)}{\lVert u_{n_k} \rVert_{X_0}} \to w(x) \quad \text{a.e. in } \Omega.
\] (1.15)
So, by this fact $\lVert w \rVert_{X_0} = 1$ (we know that $\lVert w \rVert_{X_0} = 1$, $w \in E_1$ and (1.15), we have
\[
|u_{n_k}(x)| \to +\infty \quad \text{as } n \to +\infty.
\] (1.16)
In view of (A2), (1.14), (1.16) and Fatou’s lemma, one has
\[
1 + c \geq J(u_{n_k}) - \frac{1}{2p} J'(u_{n_k}) u_{n_k}
= \frac{a}{2p} \lVert u_{n_k} \rVert_{X_0}^p + \int_\Omega \left( \frac{1}{2p} f(x,u_{n_k}(x)) u_{n_k}(x) - F(x,u_{n_k}(x)) \right) dx
\geq \int_\Omega \left( \frac{1}{2p} f(x,u_{n_k}(x)) u_{n_k}(x) - F(x,u_{n_k}(x)) + \frac{\alpha \mu_1}{2p} |u_{n_k}(x)|^p \right) dx
\to +\infty \quad \text{as } n \to +\infty,
\] (1.17)
which is a contradiction. Then we get that $\{u_n\}$ is bounded in $X_0$. By (A1), we can easily obtain that $\{u_n\}$ has a convergence subsequence. Therefore, the functional $J$ satisfies the (Ce) condition.

**Proof of Theorem 1.3.** By Lemma 1.9, we know that the functional $J : X_0 \to \mathbb{R}$ satisfies the (Ce) condition. Hence, it is sufficient to show that $J$ satisfies (i) and (ii) of Theorem 1.3.

First, we claim that there are constant $\beta, \rho > 0$ such that $J(u) \geq \beta$ for all $\lVert u \rVert_{X_0} = \rho$. By (A1) and (A4), we can get
\[
F(x,t) \leq \frac{\alpha \mu_1}{p} |t|^p + \frac{b(\lambda_1 - \varepsilon)}{2p} |t|^{2p} + C|t|^r,
\] (1.18)
for all $\varepsilon$ small enough, $t \in \mathbb{R}$ and $x \in \Omega$. Then, from (1.6)-(1.8) and (1.18), we have
\[
J(u) = \frac{a}{p} \lVert u \rVert_{X_0}^p + \frac{b}{2p} \lVert u \rVert_{X_0}^{2p} - \int_\Omega F(x,u(x)) dx
\]
There are only two cases need to be consider:

Proof. Let \( \|u\|_X \to 0 \) for all \( X \to \infty \).

Also, in view of (A1) and (A3) that

\[
J(u) = \frac{a}{p} \|u\|_X^p + \frac{b}{2p} \|u\|_X^{2p} - \frac{a\mu_1}{p} \|u\|_P^p + \frac{b(\lambda_1 - \varepsilon)}{2p} \|u\|_{L^2p}^2 - C \int_\Omega |u(x)|^r \, dx
\]

Since \( 2p < r < p^* \) then for \( \varepsilon \) small enough, there exists \( \beta > 0 \) such that \( J(u) \geq \beta \) for all \( \|u\|_X = \rho \), where \( \rho > 0 \) small enough.

Next, we will show that there exists \( u_1 \in X_0 \) and \( \|u_1\|_X > \rho \) such that \( J(u_1) < 0 \). By the definition of \( \lambda_1 \), for small enough \( \varepsilon > 0 \), we can choose \( u \in M \) satisfying

\[
\lambda_1 + \frac{\varepsilon}{p} \geq \|u\|_{X_0}^2.
\]

Also, in view of (A1) and (A3) that

\[
F(x,t) > \frac{b(\lambda_1 + \varepsilon)}{2p} t^{2p} - C.
\]

So, from (1.19) and (1.20), one can get

\[
J(tu) = \frac{a}{p} t^p \|u\|_X^p + \frac{b}{2p} t^{2p} \|u\|_X^{2p} - \int_\Omega F(x,tu(x)) \, dx
\]

\[
\leq \frac{a}{p} t^p \|u\|_X^p + \frac{b}{2p} t^{2p} \|u\|_X^{2p} - \frac{b}{2p} (\lambda_1 + \varepsilon) + C |\Omega|
\]

\[
\leq \frac{a}{p} t^p \|u\|_X^p + \frac{b}{2p} t^{2p} \lambda_1 + \frac{b\varepsilon}{2p^2} t^{2p} - \frac{b}{2p} (\lambda_1 + \varepsilon) + C |\Omega|
\]

\[
= \frac{a}{p} t^p \|u\|_X^p - \frac{b\varepsilon}{2p^2} t^{2p} + C |\Omega|.
\]

Then, \( J(tu) \to -\infty \) as \( t \to \infty \). Therefore, there exists \( u_1 \in X_0 \) and \( \|u_1\|_X > \rho \) such that \( J(u_1) < 0 \).

To prove of Theorem 1.4, we need the following lemmas.

Lemma 1.10. Assume that (A1), (A6) and (A7) hold. Then the functional \( J : X_0 \to \mathbb{R} \) satisfies the (Ce) condition.

Proof. Let \( \{u_n\} \subset X_0 \) is a (Ce)\(_c\) sequence for \( c \in \mathbb{R} \),

\[
J(u_n) \to c, \quad (1 + \|u_n\|_X)J'(u_n) \to 0 \quad \text{as} \ n \to \infty.
\]

We first claim that \( \{u_n\} \) is a bounded sequence. Suppose to the contrary that \( \|u_n\|_X \to \infty \). We consider \( w_n := \frac{u_n}{\|u_n\|_X} \), then \( \|w_n\|_X = 1 \). Going if necessary to a subsequence, we may assume that

\[
w_n \to w, \quad \text{weakly in} \ X_0,
\]

\[
w_n \to w, \quad \text{strongly in} \ L^q(\Omega) \ 1 \leq q < p_2^*
\]

\[
w_n \to w, \quad \text{a.e.} \ x \in \Omega.
\]

There are only two cases need to be consider: \( w = 0 \) or \( w \neq 0 \). We firs consider the case \( w = 0 \). By (A6) and (1.21), one obtains

\[
\frac{1}{\|u_n\|_X^p} \left( J(u_n) - \frac{1}{2p} J'(u_n) u_n \right)
\]

\[
\geq \frac{a}{2p} + \frac{1}{\|u_n\|_X^p} \int_\Omega \left[ \frac{1}{2p} f(x,u_n(x))u_n(x) - F(x,u_n(x)) \right] dx
\]
Using (1.21), (1.23) and (1.24), we obtain

\[ \lim_{n \to \infty} \frac{F(x, u_n(x))}{|u_n(x)|^{2p}} |w_n(x)|^{2p} \to \infty. \]

So, using Fatou’s Lemma, we can get

\[ \lim_{n \to \infty} \int_{\Omega} \frac{F(x, u_n(x))}{|u_n(x)|^{2p}} |w_n(x)|^{2p} \, dx \to \infty. \tag{1.23} \]

From (A1), it follows that

\[ |F(x, t)| \leq M|t|, \quad \forall x \in \Omega, \quad |t| \leq L_1. \]

Combining this with (A5), we obtain

\[ F(x, t) \geq -M|t|, \quad \forall (x, t) \in \Omega \times \mathbb{R}. \]

So, by (1.7), we obtain

\[ \int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{|u_n(x)|^{2p} \chi_0} \, dx \geq -M \int_{\Omega \cap \Omega_1} |u_n|^{2p} \chi_0 \geq -M \frac{\|u_n\|_1^{2p}}{\|u_n(x)\|^{2p} \chi_0} \geq -MC_1, \]

which implies

\[ \lim_{n \to \infty} \inf \int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{|u_n(x)|^{2p} \chi_0} \, dx \geq 0, \tag{1.24} \]

Using (1.21), (1.23) and (1.24), we obtain

\[ 0 = \lim_{n \to \infty} \frac{c + o(1)}{\|u_n(x)\|^{2p} \chi_0} = \lim_{n \to \infty} \frac{J(u_n)}{\|u_n(x)\|^{2p} \chi_0} \]

\[ = \lim_{n \to \infty} \frac{1}{\|u_n(x)\|^{2p} \chi_0} \left( \frac{a}{p} \|u_n\|^{p} \chi_0 + \frac{b}{2p} \|u_n\|^{2p} \chi_0 - \int_{\Omega} F(x, u_n(x)) \, dx \right) \]

\[ = \lim_{n \to \infty} \frac{1}{\|u_n(x)\|^{2p} \chi_0} \left( \frac{a}{p} \|u_n\|^{p} \chi_0 + \frac{b}{2p} \|u_n\|^{2p} \chi_0 - \int_{\Omega} F(x, u_n(x)) \, dx \right) \]

\[ - \int_{\Omega \setminus \Omega_1} F(x, u_n(x)) \, dx \tag{1.25} \]

\[ \leq \frac{b}{2p} + \lim_{n \to \infty} \frac{a}{p} \|u_n(x)\|^{p} \chi_0 - \lim_{n \to \infty} \int_{\Omega_1} \frac{F(x, u_n)}{\|u_n(x)\|^{2p} \chi_0} \, dx \]

\[ - \lim_{n \to \infty} \inf \int_{\Omega \setminus \Omega_1} \frac{F(x, u_n)}{\|u_n(x)\|^{2p} \chi_0} \, dx \]

\[ \leq \frac{b}{2p} - \lim_{n \to \infty} \int_{\Omega_1} \frac{F(x, u_n)}{\|u_n(x)\|^{2p} \chi_0} \, dx = -\infty, \]

which is a contradiction. Then we \( \{u_n\} \) is bounded in \( X_0 \). By (A1), we can easily obtain that \( \{u_n\} \) has a convergence subsequence. Therefore, the functional \( J \) satisfies the (Ce) condition. \( \square \)
Proof of Theorem 1.4. Let \( \{e_j\} \) is an orthonormal basis of \( X_0 \) and define \( X_j = \mathbb{R} e_j \),
\[
Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{Z}
\]
and \( Y_k \) is finite-dimensional. Set \( X = X_0 \), \( Y = Y_k \) and \( Z = Z_k \). Clearly, \( J(0) = 0 \) and (A7) implies \( J \) is even and from Lemma 1.10, \( J \) satisfies the (Ce) condition. conditions (i) of Theorem 1.7 is satisfied. So, we only need to verify (ii) and (ii) of Theorem 1.7. Set
\[
\beta_k(r) := \sup_{u \in Z_k, \|u\|_{X_0} = 1} \|u\|.
\]
By a direct calculation, we have \( \beta_k \to 0 \) as \( k \to \infty \) for all \( 1 \leq r < p^*_a \), choose
\[
\rho := \min \left\{ \left( \frac{a}{4 p C \beta_k(1)} \right)^{\frac{1}{p}}, \left( \frac{a r}{4 p C \beta_k(1)} \right)^{\frac{1}{p}} \right\}
\]
Then, by (A1) and (1.26), for \( u \in Z_k \) and \( \|u\|_{X_0} = \rho \), we have
\[
J(u) = \frac{a}{p} \|u\|_{X_0}^p + \frac{b}{2p} \|u\|_{X_0}^{2p} - \int_{\Omega} F(x, u(x)) \, dx \geq \frac{a}{p} \|u\|_{X_0}^p - C ||u||_1 - C \|u\|_{r}^r \\
\geq \frac{a}{p} \|u\|_{X_0}^p - C \beta_k(1) ||u||_{X_0} - \frac{r}{r} \beta_k^r(r) ||u||_{X_0} \\
\geq \frac{a}{2p} \rho^p := \alpha > 0.
\]
Thus condition (ii) of Theorem 1.7 is satisfied.

Since all norms are equivalent in a finite dimensional space, there is a constant \( \Upsilon > 0 \) such that
\[
\|u\|_{2p} \geq \Upsilon \|u\|_{X_0}, \quad \forall u \in Y.
\]
In view of (A5), for any \( M_1 > \frac{b}{2 p \Upsilon r} \), there is a constant \( \Gamma_0 > 0 \) such that
\[
F(x, t) \geq M_1 t^{2p}, \quad \forall x \in \Omega, \; |t| \geq \Gamma_0.
\]
By (A1), we have
\[
|F(x, t)| \leq C(1 + \Gamma_0^{-1}) |t|, \quad \forall x \in \Omega, \; |t| \leq \Gamma_0,
\]
which implies
\[
F(x, t) \geq M_2 t^{2p} - C' |t|, \quad \forall (x, t) \in \Omega \times \mathbb{R}, \quad (1.28)
\]
where \( C' \) is a positive constant. Hence from (1.7), (1.27) and (1.28), one can get
\[
J(u) \leq \frac{a}{p} \|u\|_{X_0}^p + \frac{b}{2p} \|u\|_{X_0}^{2p} - M_1 \|u\|_{2p} + C' \|u\|_1 \\
\leq \frac{a}{p} \|u\|_{X_0}^p - (M_1 \Upsilon^{2p} - \frac{b}{2p}) \|u\|_{X_0}^{2p} + C' \|u\|_{X_0}, \quad \forall u \in Y.
\]
Consequently, there is a large \( R = R(\tilde{X}) > 0 \) such that \( J(u) \leq 0 \) on \( Y \setminus B_R \). Thus the condition (iii) of Theorem 1.7 is satisfied. Then all conditions of Theorem 1.7 are satisfied. Therefore, problem (1.1) possesses infinitely many nontrivial solutions.

To proof Theorem 1.5, we need the following lemmas.

**Lemma 1.11.** Assume that (A8)-(A10) hold. Then the functional \( J : X_0 \to \mathbb{R} \) satisfies the \( (PS)_c \) condition.
Proof. Assume that \( \{u_n\} \subset X_0 \) such that

\[
J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in} \quad X_0^*.
\]

So we first prove that \( \{u_n\} \) is bounded in \( X_0 \).

By (A9), there exists \( \theta_0 \) such that

\[
\frac{1}{r_0} g(x, t) t - G(x, t) + a \varrho |t|^p + m |t|^q > -\theta_0.
\]

So, by (1.8), (1.9) and (1.29), we have

\[
C + 1 \geq J(u_n) - \frac{1}{n} J'(u_n, u_n)
\]

\[
= a \left( \frac{1}{p} - \frac{1}{r_0} \right) \| u_n \|^p_{X_0} + b \left( \frac{1}{2p} - \frac{1}{r_0} \right) \| u_n \|^{2p}_{X_0} + \int_\Omega \left[ \frac{1}{r_0} g(x, u_n) - G(x, u_n) \right] \, dx
\]

\[
\geq a \left( \frac{1}{p} - \frac{1}{r_0} \right) \| u_n \|^p_{X_0} + b \left( \frac{1}{2p} - \frac{1}{r_0} \right) \| u_n \|^{2p}_{X_0} - a \varrho \int_\Omega \| u_n \|^p \, dx
\]

\[
- m \int_\Omega \| u_n \|^q \, dx - \theta_0 |\Omega|
\]

\[
= a \left( \frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1} \right) \| u_n \|^p_{X_0} + b \left( \frac{1}{2p} - \frac{1}{r_0} \right) \| u_n \|^{2p}_{X_0} - a \varrho \| u_n \|^p - m \| u_n \|^q - \theta_0 |\Omega|
\]

\[
\geq a \left( \frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1} \right) \| u_n \|^p_{X_0} - m C_q \| u_n \|^q_{X_0} - \theta_0 |\Omega|
\]

This implies

\[
a \left( \frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1} \right) \| u_n \|^p_{X_0} \leq C + 1 + m C_q \| u_n \|^q_{X_0} + \theta_0 |\Omega|.
\]

Since \( 1 < q < p' \) and \( \varrho < \left( \frac{1}{p} - \frac{1}{r_0} \right) \mu_1 \), it follows that \( \{u_n\} \) in \( X_0 \) is bounded. By condition (A8), we can easily obtain that \( \{u_n\} \) has a convergence subsequence. Therefore, \( J \) satisfies the (PS)_c condition. \( \square \)

Proof of Theorem 1.3. From Lemma 1.11, conditions (3) of Theorem 1.8 is satisfied. So, we only need to verify (1) and (2) of Theorem 1.8. By (A10) and (A11), we can get

\[
J(u) = a \left( \frac{1}{p} || u \|^p_{X_0} + \frac{b}{2p} || u \|^{2p}_{X_0} - \int_\omega G(x, u) \, dx - \frac{1}{r_0} \int_\omega H |u|_0^r \, dx
\]

\[
\leq a \left( \frac{1}{p} || u \|^p_{X_0} + \frac{b}{2p} || u \|^{2p}_{X_0} - \frac{1}{r_0} m || u \|_{r_0}^r
\]

since \( r_0 > 2p \) and all norms are equivalent on a finite dimensional space, there exists large \( \rho_k > 0 \) such that

\[
ak := \max_{u \in Y_k, || u \|_{X_0} = \rho_k} J(u) < 0.
\]

Then, condition (1) of Theorem 1.8 is satisfied. Set

\[
\beta_k := \max \left\{ \sup_{u \in Z_k, || u \|_{X_0} = 1} || u \|_r, \sup_{u \in Z_k, || u \|_{X_0} = 1} || u \|_{r_0} \right\}.
\]
In view of \( Z_{k+1} \subset Z_k \), one has \( 0 < \beta_{k+1} \leq \beta_k \) and by a direct calculation, we have \( \beta_k \to 0 \) as \( k \to \infty \). By (A8) and (A12), for any \( \epsilon > 0 \) there exists \( \delta = \delta(\epsilon) > 0 \) such that a.e. \( x \in \Omega \) and for any \( t \in \mathbb{R} \)
\[
|G(x,t)| \leq \epsilon|t|^p + r\delta(\epsilon)|u|^p.
\]
Then, by (A11),
\[
J(u) = \frac{a}{p} \|u\|_{X_0}^p + \frac{b}{2p} \|u\|_{X_0}^{2p} - \int_{\Omega} G(x,u) \, dx - \frac{1}{r_0} \int_{\Omega} H|u|^{r_0} \, dx
\geq \frac{a}{p} \|u\|_{X_0}^p - \epsilon \|u\|_{X_0}^{p} - r\delta(\epsilon)\|u\|_r^{r} - \frac{M}{r_0} \|u\|_{r_0}^{r_0}
\geq \frac{a}{p} \|u\|_{X_0}^p - \frac{\epsilon}{\mu_1} \|u\|_{X_0}^{p} - r\delta(\epsilon)\|u\|_r^{r} - \frac{M}{r_0} \|u\|_{r_0}^{r_0}
\geq \left( \frac{a}{p} - \frac{\epsilon}{\mu_1} \right) \|u\|_{X_0}^{p} - r\delta(\epsilon)\beta_k^{p} \|u\|_{X_0}^{r} - \frac{M}{r_0} \beta_k^{r_0} \|u\|_{X_0}^{r_0}.
\]
For every \( \epsilon \) with \( 0 < \epsilon < \frac{a\mu_1}{p} \), choose
\[
\|u\|_{X_0} = \gamma_k = \min \left\{ \left( \frac{a\mu_1 - \epsilon p}{3r\delta(\epsilon)p\mu_1\beta_k^p} \right)^{\frac{1}{p}}, \left( \frac{(a\mu_1 - \epsilon p)r_0}{3p\mu_1 M\beta_k^{r_0}} \right)^{\frac{1}{r_0}} \right\}.
\]
Since \( \beta_k \to 0 \) as \( k \to \infty \), we have \( \|u\| = \gamma_k \to +\infty \) as \( k \to \infty \). Hence
\[
b_k := \inf_{u \in Z_k, \|u\|_{X_0} = \gamma_k} J(u)
\geq \left( \frac{a}{p} - \frac{\epsilon}{\mu_1} \right) \gamma_k^{p} - r\delta(\epsilon)\left( \frac{a\mu_1 - \epsilon p}{3r\delta(\epsilon)p\mu_1} \right) \gamma_k^{p} - \frac{M}{r_0} \left( \frac{(a\mu_1 - \epsilon p)r_0}{3p\mu_1 M} \right) \gamma_k^{p}
= \frac{1}{3} \left( \frac{a\mu_1 - \epsilon p}{p\mu_1} \right) \gamma_k^{p} \to +\infty, \quad \text{as} \quad k \to \infty.
\]
Then, condition (2) of Theorem \ref{thm:existence} is satisfied.

So, its follows that the conditions of Theorem \ref{thm:existence} was satisfied and we have unbounded sequence which yields that \( I(u_k) \to +\infty \) then the proof is complete. \( \square \)

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