Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 115, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS FOR DEGENERATE KIRCHHOFF TYPE PROBLEMS WITH FRACTIONAL p-LAPLACIAN

NEMAT NYAMORADI, LAHIB IBRAHIM ZAIDAN

Communicated by Paul H. Rabinowitz

Abstract. In this article, by using the Fountain theorem and Mountain pass theorem in critical point theory without Palais-Smale (PS) condition, we show the existence and multiplicity of solutions to the degenerate Kirchhoff type problem with the fractional p-Laplacian

$$\Big(a + b\int\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)(-\Delta)_p^s u = f(x,u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,$$

where $(-\Delta)_p^s$ is the fractional p-Laplace operator with $0 < s < 1 < p < \infty$, Ω is a smooth bounded domain of $\mathbb{R}^N$, $N > 2s$, $a, b > 0$ are constants and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

1. Introduction and statement of main results

The aim of this article is to establish the existence of solutions to the Kirchhoff nonlocal problem

$$\begin{gathered}\Big(a + b\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)(-\Delta)_p^s u = f(x,u) \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega,\end{gathered} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, $N > 2s$ with $s \in (0,1)$, $a, b > 0$ are constants, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $(-\Delta)_p^s$ is the fractional p-Laplacian operator which, up to normalization factors, may be defined as

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+ps}}\,dy$$

for $x \in \mathbb{R}^N$, where $B_\varepsilon(x) := \{y \in \mathbb{R}^N : |x-y| < \varepsilon\}$. As for some recent results on the fractional p-Laplacian, we refer to for example [22, 21, 24] and the references therein.

2010 *Mathematics Subject Classification.* 34B27, 35J60, 35B05.
Key words and phrases. Kirchhoff nonlocal operators; fractional differential equations; fountain theorem; mountain pass theorem; critical point theory.

Submitted February 23, 2017. Published April 27, 2017.

When $a \equiv 1$, $b \equiv 0$ and $p = 2$, problem (1.1) becomes the fractional Laplacian problem

$$\begin{gathered} (-\Delta)^s u = f(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{1.2}$$

For the basic properties of fractional Sobolev spaces and the functional framework that takes into account the problem (1.2), we refer the readers to [36]. In [37, 38], Servadei and Valdinoci considered the existence of nontrivial weak solutions of the problem (1.2) by using variational methods. For other recent results in (1.2), the reader is referred, for example, to [4, 39, 40].

Fractional and nonlocal operators and on their applications is very interesting, we refer the readers to [5, 13, 15, 18, 19, 25, 26, 27, 28, 30, 32, 33, 35, 47] and the references therein. For the basic properties of fractional Sobolev spaces, we refer the readers to [12, 27]. In [30], Molica Bisci and Vilasi studied a class of Kirchhoff nonlocal fractional equation in a bounded domain Ω and obtained three solutions by using three critical point theorem. Pucci and Saldi [32] established the existence and multiplicity of nontrivial solutions for a Kirchhoff type eigenvalue problem in $\mathbb{R}^N$ involving a critical nonlinearity and the nonlocal fractional Laplacian. We refer also to [16, 17, 27, 29] for related problems.

Notice that when $a \equiv 1$ and $b \equiv 0$, as $s \to 1^-$, problem (1.1) reduces to the problem

$$-\Delta_p u = f(x,u) \quad \text{in } \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain.

For the case of a bounded domain, there are several articles considering the system

$$-\Big(a + b\int_\Omega |\nabla u|^p\Big)\Delta_p u = g(x,u) \quad \text{in } \Omega$$

where $\Omega \subset \mathbb{R}^N$ is a smooth domain, which is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \Big(a + b\int_\Omega |\nabla u|^p\Big)\Delta_p u = g(x,u),$$

which was proposed by Kirchhoff [23] as an extension of the classical D'Alembert's wave equation for free vibrations of elastic string. In recent years, many authors are interesting in Kirchhoff type problems, see for example [2, 3, 8, 9, 10, 11, 33, 42, 43, 44, 45, 46] and references therein.

Motivated by the above works and [7, 31, 36, 37, 38, 41], we study the existence and multiplicity of solutions for Kirchhoff type problem (1.1).

Before proving our main results, some preliminary material on function spaces and norms is needed. In the following, we briefly recall the definition of the functional space X_0, introduced in [36], and we give some notation. We denote $\mathrm{Q} = \mathbb{R}^{2N} \setminus \mathcal{O}$, where $\mathcal{O} = \mathbb{R}^N \setminus \Omega \times \mathbb{R}^N \setminus \Omega$. We denote

$$X = \Big\{u : \mathbb{R}^N \to \mathbb{R} : u|_\Omega \in L^p(\Omega),\ \int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy < \infty\Big\},$$

where $u|_\Omega$ represents the restriction to Ω of function $u(x)$. Also, we define the following linear subspace of X,

$$X_0 = \big\{g \in X : g = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\big\}.$$

The linear space X is endowed with the norm

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \Big(\int\int_Q \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p}.$$

It is easily seen that $\|\cdot\|_X$ is a norm on X and $C_0^\infty(\Omega) \subseteq X_0$ (see [45, Lemma 2.1]). Also, we know that X_0, endowed with the norm

$$\|v\|_{X_0} = \Big(\int\int_Q \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\,dx\,dy\Big)^{1/p} \quad \text{for all } v \in X_0, \tag{1.4}$$

is a uniformly convex Banach space and a reflexive Banach space [45, Remark 2.1 and Lemma 2.4].

We consider the nonlinear eigenvalue problem

$$\begin{gathered} \|u\|_{X_0}^p(-\Delta)_p^s u = \lambda |u|^{2p-2}u \quad \text{in } \Omega,\\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \end{gathered} \tag{1.5}$$

whose eigenvalues are the critical values of the functional

$$J_p(u) = \|u\|_{X_0}^{2p}, \quad u \in \mathcal{M} = \Big\{u \in X_0 : \int_\Omega |u|^{2p}\,dx = 1\Big\}. \tag{1.6}$$

We know that the first eigenvalue $\lambda_1 := \inf_{u\in\mathcal{M}} J_p(u) > 0$. The first eigenfunction is denoted by φ_1 (see [44] for the case $\theta = 2$).

We denote the usual $L^p(\Omega)$-norm by $\|\cdot\|_p$. Since Ω is a bounded domain, it is well known that $X_0 \hookrightarrow L^p(\Omega)$ continuously for $p \in [1, p_s^*]$, (see [45, Lemma 2.3]) and compactly for $q \in [1, p_s^*)$, where $p_s^* := \frac{Np}{N-sp}$. Moreover there exists $C_q > 0$ such that

$$\|u\|_q \le C_q \|u\|_{X_0}, \quad u \in X_0. \tag{1.7}$$

We consider the functional $J : X_0 \to \mathbb{R}$ defined by

$$J(u) = \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\Omega F(x, u(x))\,dx \tag{1.8}$$

and set

$$\Psi(u) = \int_\Omega F(x, u(x))\,dx,$$

where $F(x,u) = \int_0^u f(x,s)\,dx$. Obviously, the functional J is well-defined, it is of class $C^1(X_0, \mathbb{R})$ and

$$\begin{aligned} &\langle J'(u), v\rangle \\ &= (a + b\|u\|_{X_0}^p) \int\int_Q \frac{|u(x)-u(y)|^{p-2}\big(u(x)-u(y)\big)}{|y|^{N+ps}}(v(x)-v(y))\,dx\,dy \\ &\quad - \int_\Omega f(x,u(x))v(x)\,dx, \quad \text{for all } u, v \in X_0, \end{aligned} \tag{1.9}$$

Moreover, the critical points of J are the solutions of problem (1.1). Let

$$E_j := \oplus_{i\le j} \ker((-\Delta)_p^s - \mu_i),$$

where $0 < \mu_1 \le \mu_2 \le \dots, \mu_i \le \dots$, are the eigenvalue of $((-\Delta)_p^s, X_0)$ (see [22, 24, 20]).

Definition 1.1. We say that J satisfies the Palais-Smale (PS) condition if any sequence $(u_n) \in X$ for which $J(u_n)$ is bounded and $J'(u_n) \to 0$ as $n \to \infty$ possesses a convergent subsequence.

Also, we need the following definition, which is a weak version of the (PS) condition, due by Cerami [6].

Definition 1.2. Let $J \in C^1(X, \mathbb{R})$, we say that J satisfies the Cerami condition at the level $c \in \mathbb{R}$ ($(\mathrm{Ce})_c$ for short), if any sequence $(u_n) \in X$ with

$$J(u_n) \to c, \quad (1 + \|u_n\|)J'(u_n) \to 0 \quad \text{as } n \to \infty,$$

possesses a convergent subsequence in X; J satisfies the (Ce) condition if J satisfies the $(\mathrm{Ce})_c$ for all $c \in \mathbb{R}$.

The assumptions on the function f are stated as follows:

(A1) There exists a positive constant $C > 0$ such that $|f(x,t))| \le C(|t|^{r-1} + 1)$, for some $2p < r < p_s^*$, $x \in \Omega$ and all $t \in \mathbb{R}$;
(A2) $\lim_{|t|\to\infty} \big(\frac{1}{2p} f(x,t)t - F(x,t) + \frac{a\mu_1}{p} t^p\big) = +\infty$ uniformly in $x \in \Omega$;
(A3) there exists $\mu > \mu_1$ such that $F(x,t) \ge \frac{a\mu}{p} t^p$ for $|t|$ small;
(A4) $\lim_{|t|\to\infty} \big(\frac{a\mu_1}{p} t^p + \frac{b\mu_1}{2p} t^{2p} - F(x,t)\big) = +\infty$ uniformly in $x \in \Omega$.

Now we state our main results.

Theorem 1.3. *Assume that* $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, (A1)–(A4) *hold. Then* (1.1) *has at least one nontrivial solution.*

In the next theorem we use the assumptions:

(A5) $\lim_{|t|\to\infty} \frac{F(x,t)}{|t|^{2p}} \to \infty$ uniformly in $x \in \Omega$, and there exists $L_1 \ge 0$ such that $F(x,t) \ge 0$ for all $(x,t) \in \Omega \times \mathbb{R}$ and $|t| \ge L_1$;
(A6) there exists $\theta_0 > 0$ such that

$$F(x,t) \le \frac{1}{2p} f(x,t)t + \theta_0 |t|^p, \quad \forall (x,t) \in \Omega \times \mathbb{R};$$

(A7) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$;

Theorem 1.4. *Assume that* (A1), (A5)–(A7) *are satisfied. Then problem* (1.1) *possesses infinitely many nontrivial solutions* $\{u_k\}$ *such that* $J(u_k) \to +\infty$.

Now, we study the existence of infinitely many solutions of the following problem, which it is a special case of problem (1.1),

$$\begin{gathered} \Big(a + b \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dxdy\Big)(-\Delta)_p^s u \\ = g(x, u(x)) + H(x)|u|^{r_0 - 2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{1.10}$$

with the following conditions:

(A8) There exists a positive constant C_G such that $|G(x,t)| \le C_G(|t|^{r-1} + 1)$ for some $2p < r < p^*$, $x \in \Omega$ and all $t \in \mathbb{R}$, where $G(x,t) = \int_0^t f(x,s)\, ds$;
(A9) $\lim_{|t|\to\infty} \big(\frac{1}{r_0} g(x,t)t - G(x,t) + a\varrho |t|^p + m|t|^q\big) = +\infty$ uniformly in $x \in \Omega$ where $\varrho < (\frac{1}{p} - \frac{1}{r_0})\mu_1$, $1 < q < p < p^*$, $r_0 > 2p$ and m is a arbitrary positive constant;
(A10) $G(x,t) \ge 0$, for all $x \in \Omega$, $t \in \mathbb{R}$;
(A11) the function H is a nonnegative and satisfies $0 < m \le H \le M$;
(A12) $\lim_{|t|\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$, uniformly in $x \in \Omega$;
(A13) $G(x,0) = 0$ for all $x \in \Omega$ and $G(x,-t) = G(x,t)$, for all $x \in \Omega$, $t \in \mathbb{R}$.

Theorem 1.5. *Assume that $g \in C(\Omega \times \mathbb{R}, \mathbb{R})$, $H \in C(\Omega, \mathbb{R})$, and* (A8)–(A13) *hold. Then problem* (1.10) *has a sequence of solutions $\{u_k\}$ such that $I(u_k) \to +\infty$.*

The proofs of the our main results are fully based on some theorems that we recalled here for the reader's convenience.

Theorem 1.6 (Mountain Pass Theorem [1, 14])**.** *Let X be a real Banach space and $J \in C^1(X, \mathbb{R})$ satisfying the* (Ce) *condition. Suppose $J(0) = 0$,*

(i) *there are constants $\rho, \beta > 0$ such that $J|_{\partial B_\rho} \geq \beta$ where*
$$B_\rho = \{u \in X : \|u\| \leq \rho\};$$
(ii) *there is $u_1 \in X$ and $\|u_1\| > \rho$ such that $J(u_1) < 0$.*

Then J possesses a critical value $c \geq \beta$. Moreover c can be characterized as
$$c = \inf_{g \in \Gamma} \max_{u \in g([0,1])} J(u), \quad \Gamma = \{g \in C([0,1]) : g(0) = 0, \ g(1) = u_1\}.$$

Theorem 1.7 ([34])**.** *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $J \in C^1(X, \mathbb{R})$ satisfies $(Ce)_{\rm c}$-condition for all $c > 0$, and*

(i) *$J(0) = 0$, $J(-u) = J(u)$ for all $u \in X$;*
(ii) *there exist constants $\rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$;*
(iii) *for any finite dimensional subspace $\widetilde{X} \subset X$, there is $R = R(\widetilde{X}) > 0$ such that $J(u) \leq 0$ on $\widetilde{X} \setminus B_\rho$;*

then J possesses an unbounded sequence of critical values.

Theorem 1.8 (Fountain theorem)**.** *Let X_0 be a Banach space with the norm $\|\cdot\|$ let X_i be a sequence of subspace of X with $dim X_i < \infty$ for each $i \in N$. Further, set*
$$X = \overline{\oplus_{\infty}^{i=1} X_i}, \quad Y_k = \oplus_{i=1}^{k} X_i, \quad Z_k = \overline{\oplus_{i=k}^{\infty} X_i}$$
Consider an even functional $\Phi \in C^1(X, \mathbb{R})$. Assume that for each $k \in \mathbb{N}$, there exists $\rho_k > \gamma_k > 0$ such that

(1) *$a_k := \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$,*
(2) *$b_k := \inf_{u \in Z_k, \|u\| = \gamma_k} \Phi(u) \to +\infty$, $k \to +\infty$,*
(3) *Φ satisfies the $(PS)_c$ condition for every $c > 0$.*

Then ϕ has an unbounded sequence of critical values.

Now, we need the following lemma about the (Ce) condition which will play an important role in the proof of our main results.

Lemma 1.9. *Assume that* (A1) *and* (A2) *hold. Then the functional $J : X_0 \to \mathbb{R}$ satisfies the* (Ce) *condition.*

Proof. Let $\{u_n\}$ be a $(\mathrm{Ce})_{\rm c}$ sequence for $c \in \mathbb{R}$,
$$J(u_n) \to c, \quad (1 + \|u_n\|_{X_0}) J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{1.11}$$
We first show that $\{u_n\}$ is a bounded sequence. In view of (1.8), (1.9) and (1.11), one has
$$\begin{aligned} 1 + c &\geq J(u_n) - \frac{1}{2p} J'(u_n) u_n \\ &= \frac{a}{2p} \|u_n\|_{X_0}^p + \int_\Omega \Big(\frac{1}{2p} f(x, u_n(x)) u_n(x) - F(x, u_n(x)) \Big) \, dx. \end{aligned} \tag{1.12}$$

From (A2), there exists $\theta > 0$ such that
$$-\theta \leq \frac{1}{2p} f(x,t)t - F(x,t) + \frac{a\mu_1}{2p}|t|^p, \quad \forall x \in \Omega,\ t \in \mathbb{R}. \tag{1.13}$$
Now, We define $u_n = \varphi_n + v_n$, where $\varphi_n \in E_1$ and $v_n \in E_1^{\perp}$. By (1.12) and (1.13), we have
$$\begin{aligned} 1 + c &\geq \frac{a}{2p}\|u_n\|_{X_0}^p - \frac{a\mu_1}{2p}\|u_n\|_{L^p} \\ &\quad + \int_\Omega \Big(\frac{1}{2p} f(x,u_n(x))u_n(x) - F(x,u_n(x) + \frac{a\mu_1}{2p}|u_n(x)|^p)\Big)\,dx \\ &\geq \frac{a}{2p}\big(1 - \frac{\mu_1}{\mu_2}\big)\|v_n\|_{X_0}^p - \theta|\Omega|, \end{aligned} \tag{1.14}$$
which implies that $\|v_n\|_{X_0}$ is bounded. Now, we assume that $\{u_n\}$ is unbounded sequence, so there is a subsequence $\{u_n\}$ (to simplify the notation) of $\{u_n\}$ satisfying $\|u_n\|_{X_0} \to +\infty$ as $n \to +\infty$. Hence we have $\frac{v_n}{\|u_n\|_{X_0}} \to 0 \in X_0$. Since $\frac{\varphi_n}{\|u_n\|_{X_0}}$ is bounded in finite dimensional E_1, one can get $\frac{\varphi_n}{\|u_n\|_{X_0}} \to w$ in E_1. Using
$$w_n := \frac{u_n}{\|u_n\|_{X_0}} = \frac{\varphi_n + v_n}{\|u_n\|_{X_0}} = \frac{\varphi_n}{\|u_n\|_{X_0}} + \frac{v_n}{\|u_n\|_{X_0}} \to w,$$
in E_1, yields
$$\frac{u_n(x)}{\|u_n\|_{X_0}} \to w(x) \quad \text{a.e. in } \Omega. \tag{1.15}$$
So, by this fact $\|w\|_{X_0} = 1$ (we know that $\|w_n\|_{X_0} = 1$), $w \in E_1$ and (1.15), we have
$$|u_n(x)| \to +\infty \quad \text{as } n \to +\infty. \tag{1.16}$$
In view of (A2), (1.14), (1.16) and Fatou's lemma, one has
$$\begin{aligned} 1 + c &\geq J(u_n) - \frac{1}{2p}J'(u_n)u_n \\ &= \frac{a}{2p}\|u_n\|_{X_0}^p + \int_\Omega \Big(\frac{1}{2p} f(x,u_n(x))u_n(x) - F(x,u_n(x))\Big)\,dx \\ &\geq \int_\Omega \Big(\frac{1}{2p} f(x,u_n(x))u_n(x) - F(x,u_n(x) + \frac{a\mu_1}{2p}|u_n(x)|^p)\Big)\,dx \\ &\to +\infty \quad \text{as } n \to +\infty, \end{aligned} \tag{1.17}$$
which is a contradiction. Then we get that $\{u_n\}$ is bounded in X_0. By (A1), we can easily obtain that $\{u_n\}$ has a convergence subsequence. Therefore, the functional J satisfies the (Ce) condition. □

Proof of Theorem 1.3. By Lemma 1.9, we know that the functional $J : X_0 \to \mathbb{R}$ satisfies the (Ce) condition. Hence, it is sufficient to show that J satisfies (i) and (ii) of Theorem 1.3.

First, we claim that there are constant $\beta, \rho > 0$ such that $J(u) \geq \beta$ for all $\|u\|_{X_0} = \rho$. By (A1) and (A4), we can get
$$F(x,t) \leq \frac{a\mu_1}{p}|t|^p + \frac{b(\lambda_1 - \varepsilon)}{2p}|t|^{2p} + C|t|^r, \tag{1.18}$$
for all ε small enough, $t \in \mathbb{R}$ and $x \in \Omega$. Then, from (1.6)-(1.8) and (1.18), we have
$$J(u) = \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\Omega F(x,u(x))\,dx$$

$$
\begin{aligned}
&\geq \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \frac{a\mu_1}{p}\|u\|_{L^p}^p + \frac{b(\lambda_1-\varepsilon)}{2p}\|u\|_{L^{2p}}^{2p} - C\int_\Omega |u(x)|^r\,dx \\
&\geq \frac{b}{2p}\Big(1-\frac{\lambda_1-\varepsilon}{\lambda_1}\Big)\|u\|_{X_0}^{2p} - CC_r\|u\|_{X_0}^r.
\end{aligned}
$$

Since $2p < r < p^*$ then for ε small enough, there exists $\beta > 0$ such that $J(u) \geq \beta$ for all $\|u\|_{X_0} = \rho$, where $\rho > 0$ small enough.

Next, we will show that there exists $u_1 \in X_0$ and $\|u_1\|_{X_0} > \rho$ such that $J(u_1) < 0$. By the definition of λ_1, for small enough $\varepsilon > 0$, we can choose $u \in \mathcal{M}$ satisfying

$$
\lambda_1 + \frac{\varepsilon}{p} \geq \|u\|_{X_0}^{2p}. \tag{1.19}
$$

Also, in view of (A1) and (A3) that

$$
F(x,t) > \frac{b(\lambda_1+\varepsilon)}{2p}t^{2p} - C. \tag{1.20}
$$

So, From (1.19) and (1.20), one can get

$$
\begin{aligned}
J(tu) &= \frac{a}{p}t^p\|u\|_{X_0}^p + t^{2p}\frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\Omega F(x,tu(x))\,dx \\
&\leq \frac{a}{p}t^p\|u\|_{X_0}^p + \frac{b}{2p}t^{2p}\|u\|_{X_0}^{2p} - \frac{b}{2p}t^{2p}(\lambda_1+\varepsilon) + C|\Omega| \\
&\leq \frac{a}{p}t^p\|u\|_{X_0}^p + \frac{b}{2p}t^{2p}\lambda_1 + \frac{b\varepsilon}{2p^2}t^{2p} - \frac{b}{2p}t^{2p}(\lambda_1+\varepsilon) + C|\Omega| \\
&= \frac{a}{p}t^p\|u\|_{X_0}^p - \frac{b\varepsilon}{2p^2}t^{2p} + C|\Omega|.
\end{aligned}
$$

Then, $J(tu) \to -\infty$ as $t \to \infty$. Therefore, there exists $u_1 \in X_0$ and $\|u_1\|_{X_0} > \rho$ such that $J(u_1) < 0$. $\square$

To prove of Theorem 1.4, we need the following lemmas.

Lemma 1.10. *Assume that* (A1), (A6) and (A7) *hold. Then the functional* $J : X_0 \to \mathbb{R}$ *satisfies the* (Ce) *condition.*

Proof. Let $\{u_n\} \subset X_0$ is a $(\mathrm{Ce})_\mathrm{c}$ sequence for $c \in \mathbb{R}$,

$$
J(u_n) \to c, \quad (1+\|u_n\|_{X_0})J'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{1.21}
$$

We first claim that $\{u_n\}$ is a bounded sequence. Suppose to the contrary that $\|u_n\|_{X_0} \to \infty$. We consider $w_n := \frac{u_n}{\|u_n\|_{X_0}}$, then $\|w_n\|_{X_0} = 1$. Going if necessary to a subsequence, we may assume that

$$
\begin{gathered}
w_n \rightharpoonup w, \quad \text{weakly in } X_0, \\
w_n \to w, \quad \text{strongly in } L^q(\Omega)\ 1 \leq q < p_s^*) \\
w_n \to w, \quad \text{a.e. } x \in \Omega.
\end{gathered} \tag{1.22}
$$

There are only two cases need to be consider: $w = 0$ or $w \neq 0$. We firs consider the case $w = 0$. By (A6) and (1.21), one obtains

$$
\begin{aligned}
&\frac{1}{\|u_n\|_{X_0}^p}\Big(J(u_n) - \frac{1}{2p}J'(u_n)u_n\Big) \\
&\geq \frac{a}{2p} + \frac{1}{\|u_n\|_{X_0}^p}\int_\Omega \Big(\frac{1}{2p}f(x,u_n(x))u_n(x) - F(x,u_n(x))\Big)dx
\end{aligned}
$$

$$\geq \frac{a}{2p} - \theta_0 \int_\Omega |w_n|^p dx,$$

which implies $0 \geq a/(2p)$. This is a contradiction.

If $w \neq 0$, setting $\Omega_1 := \{x \in \Omega : w(x) \neq 0\}$, obviously $|\Omega_1| > 0$ where $|\Omega_1|$ is Lebesgue measure of Ω_1. For $x \in \Omega_1$, we have $|u_n(x)| \to \infty$ as $n \to \infty$. In view of (A5), one has

$$\lim_{n\to\infty} \frac{F(x, u_n(x))}{|u_n(x)|^{2p}} |w_n(x)|^{2p} \to \infty.$$

So, using Fatou's Lemma, we can get

$$\lim_{n\to\infty} \int_\Omega \frac{F(x, u_n(x))}{|u_n(x)|^{2p}} |w_n(x)|^{2p} dx \to \infty. \tag{1.23}$$

From (A1), it follows that

$$|F(x,t)| \leq M|t|, \quad \forall x \in \Omega,\ |t| \leq L_1.$$

Combining this with (A5), we obtain

$$F(x,t) \geq -M|t|, \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

So, by (1.7), we obtain

$$\int_{\Omega\setminus\Omega_1} \frac{F(x,u_n)}{\|u_n(x)\|_{X_0}^{2p}}\, dx \geq -\frac{M\int_{\Omega\setminus\Omega_1} |u_n| dx}{\|u_n(x)\|_{X_0}^{2p}} \geq -\frac{M\|u_n\|_1}{\|u_n(x)\|_{X_0}^{2p}} \geq -\frac{MC_1}{\|u_n(x)\|_{X_0}^{2p-1}},$$

which implies

$$\liminf_{n\to\infty} \int_{\Omega\setminus\Omega_1} \frac{F(x,u_n)}{\|u_n(x)\|_{X_0}^{2p}}\, dx \geq 0, \tag{1.24}$$

Using (1.21), (1.23) and (1.24), we obtain

$$\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|u_n(x)\|_{X_0}^{2p}} = \lim_{n\to\infty} \frac{J(u_n)}{\|u_n(x)\|_{X_0}^{2p}} \\
&= \lim_{n\to\infty} \frac{1}{\|u_n(x)\|_{X_0}^{2p}} \Big(\frac{a}{p}\|u_n\|_{X_0}^p + \frac{b}{2p}\|u_n\|_{X_0}^{2p} - \int_\Omega F(x,u_n(x))\, dx \Big) \\
&= \lim_{n\to\infty} \frac{1}{\|u_n(x)\|_{X_0}^{2p}} \Big(\frac{a}{p}\|u_n\|_{X_0}^p + \frac{b}{2p}\|u_n\|_{X_0}^{2p} - \int_{\Omega_1} F(x,u_n(x))\, dx \\
&\quad - \int_{\Omega\setminus\Omega_1} F(x,u_n(x))\, dx \Big) \qquad (1.25) \\
&\leq \frac{b}{2p} + \lim_{n\to\infty} \frac{a}{p\|u_n(x)\|_{X_0}^p} - \lim_{n\to\infty} \int_{\Omega_1} \frac{F(x,u_n)}{\|u_n(x)\|_{X_0}^{2p}}\, dx \\
&\quad - \liminf_{n\to\infty} \int_{\Omega\setminus\Omega_1} \frac{F(x,u_n)}{\|u_n(x)\|_{X_0}^{2p}}\, dx \\
&\leq \frac{b}{2p} - \lim_{n\to\infty} \int_{\Omega_1} \frac{F(x,u_n)}{\|u_n(x)\|_{X_0}^{2p}}\, dx = -\infty,
\end{aligned}$$

which is a contradiction. Then we $\{u_n\}$ is bounded in X_0. By (A1), we can easily obtain that $\{u_n\}$ has a convergence subsequence. Therefore, the functional J satisfies the (Ce) condition. $\square$

Proof of Theorem 1.4. Let $\{e_j\}$ is an orthonormal basis of X_0 and define $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^k X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{Z}$$

and Y_k is finite-dimensional. Set $X = X_0$, $Y = Y_k$ and $Z = Z_k$. Clearly, $J(0) = 0$ and (A7) implies J is even and from Lemma 1.10, J satisfies the (Ce) condition. conditions (i) of Theorem 1.7 is satisfied. So, we only need to verify (ii) and (ii) of Theorem 1.7. Set

$$\beta_k(r) := \sup_{u\in Z_k, \|u\|_{X_0=1}} \|u\|_r. \tag{1.26}$$

By a direct calculation, we have $\beta_k \to 0$ as $k \to \infty$ for all $1 \le r < p_s^*$. choose

$$\rho := \min\Big\{\Big(\frac{a}{4pC\beta_k(1)}\Big)^{\frac{1}{1-p}}, \Big(\frac{ar}{4pC\beta_k^r(r)}\Big)^{\frac{1}{r-p}}\Big\}$$

Then, by (A1) and (1.26), for $u \in Z_k$ and $\|u\|_{X_0} = \rho$, we have

$$\begin{aligned} J(u) &= \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\Omega F(x, u(x))\,dx \\ &\ge \frac{a}{p}\|u\|_{X_0}^p - C\|u\|_1 - \frac{C}{r}\|u\|_r^r \\ &\ge \frac{a}{p}\|u\|_{X_0}^p - C\beta_k(1)\|u\|_{X_0} - \frac{C}{r}\beta_k^k(r)\|u\|_{X_0}^r \\ &\ge \frac{a}{2p}\rho^p := \alpha > 0. \end{aligned}$$

Thus condition (ii) of Theorem 1.7 is satisfied.

Since all norms are equivalent in a finite dimensional space, there is a constant $\Upsilon > 0$ such that

$$\|u\|_{2p} \ge \Upsilon\|u\|_{X_0}, \quad \forall u \in Y. \tag{1.27}$$

In view of (A5), for any $M_1 > \frac{b}{2p\Upsilon^{2p}}$, there is a constant $\Gamma_0 > 0$ such that

$$F(x,t) \ge M_1 t^{2p}, \quad \forall x \in \Omega,\ |t| \ge \Gamma_0.$$

By (A1), we have

$$|F(x,t)| \le C(1 + \Gamma_0^{r-1})|t|, \quad \forall x \in \Omega,\ |t| \le \Gamma_0,$$

which implies

$$F(x,t) \ge M_1 t^{2p} - C'|t|, \quad \forall (x,t) \in \Omega\times\mathbb{R}, \tag{1.28}$$

where C' is a positive constant. Hence from (1.7), (1.27) and (1.28), one can get

$$\begin{aligned} J(u) &\le \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - M_1\|u\|_{2p}^{2p} + C'\|u\|_1 \\ &\le \frac{a}{p}\|u\|_{X_0}^p - \big(M_1\Upsilon^{2p} - \frac{b}{2p}\big)\|u\|_{X_0}^{2p} + C'C\|u\|_{X_0}, \quad \forall u \in Y. \end{aligned}$$

Consequently, there is a large $R = R(\widetilde{X}) > 0$ such that $J(u) \le 0$ on $Y \setminus B_\rho$. Thus the condition (iii) of Theorem 1.7 is satisfied. Then all conditions of Theorem 1.7 are satisfied. Therefore, problem (1.1) possesses infinitely many nontrivial solutions. □

To proof Theorem 1.5, wee need the following lemmas.

Lemma 1.11. *Assume that* (A8)-(A10) *hold. Then the functional* $J : X_0 \to \mathbb{R}$ *satisfies the* $(\mathrm{PS})_c$. *condition.*

Proof. Assume that $\{u_n\} \subset X_0$ such that

$$J(u_n) \to c \quad \text{and } J'(u_n) \to 0 \quad \text{in } X_0^*.$$

So we first prove that $\{u_n\}$ is bounded in X_0.

By (A9), there exists θ_0 such that

$$\frac{1}{r_0} g(x,t)t - G(x,t) + a\varrho |t|^p + m|t|^q > -\theta_0. \tag{1.29}$$

So, by (1.8), (1.9) and (1.29), we have

$$\begin{aligned}
C + 1 &\geq J(u_n) - \frac{1}{n} \langle J'(u_n), u_n \rangle \\
&= a\big(\frac{1}{p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^p + b\big(\frac{1}{2p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^{2p} + \int_\Omega \big[\frac{1}{r_0} g(x,u_n) - G(x,u_n)\big]\,dx \\
&\geq a\big(\frac{1}{p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^p + b\big(\frac{1}{2p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^{2p} - a\varrho \int_\Omega |u_n|^p\,dx \\
&\quad - m\int_\Omega |u_n|^q\,dx - \theta_0|\Omega| \\
&= a\big(\frac{1}{p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^p + b\big(\frac{1}{2p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^{2p} - a\varrho\|u_n\|_p^p - m\|u_n\|_q^q - \theta|\Omega| \\
&\geq a\big(\frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1}\big)\|u_n\|_{X_0}^p + b\big(\frac{1}{2p} - \frac{1}{r_0}\big)\|u_n\|_{X_0}^{2p} - mC_q^q\|u_n\|_{X_0}^q - \theta_0|\Omega| \\
&\geq a\big(\frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1}\big)\|u_n\|_{X_0}^p - mC_q^q\|u_n\|_{X_0}^q - \theta_0|\Omega|.
\end{aligned}$$

This implies

$$a\big(\frac{1}{p} - \frac{1}{r_0} - \frac{\varrho}{\mu_1}\big)\|u_n\|_{X_0}^p \leq C + 1 + mC_q^q\|u_n\|_{X_0}^q + \theta_0|\Omega|.$$

Since $1 < q < p < p^*$ and $\varrho < (\frac{1}{p} - \frac{1}{r_0})\mu_1$, it follows that $\{u_n\}$ in X_0 is bounded. By condition (A8), we can easily obtain that $\{u_n\}$ has a convergence subsequence. Therefore, J satisfies the $(PS)_c$ condition. $\square$

Proof of Theorem 1.5. From Lemma 1.11, conditions (3) of Theorem 1.8 is satisfied. So, we only need to verify (1) and (2) of Theorem 1.8. By (A10) and (A11), we can get

$$\begin{aligned}
J(u) &= \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\omega G(x,u)\,dx - \frac{1}{r_0}\int_\omega H|u|_0^r\,dx \\
&\leq \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \frac{1}{r_0} m\|u\|_{r_0}^{r_0},
\end{aligned}$$

since $r_0 > 2p$ and all norms are equivalent on a finite dimensional space, there exists large $\rho_k > 0$ such that

$$a_k := \max_{a \in Y_k, \|u\|_{X_0} = \rho_k} J(u) < 0.$$

Then, condition (1) of Theorem 1.8 is satisfied. Set

$$\beta_k := \max\Big\{ \sup_{u \in Z_k, \|u\|_{X_0} = 1} \|u\|_r, \sup_{u \in Z_k, \|u\|_{X_0} = 1} \|u\|_{r_0} \Big\}.$$

In view of $Z_{k+1} \subset Z_k$, one has $0 < \beta_{k+1} \leq \beta_k$ and by a direct calculation, we have $\beta_k \to 0$ as $k \to \infty$. By (A8) and (A12), for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that a.e. $x \in \Omega$ and for any $t \in \mathbb{R}$

$$|G(x,t)| \leq \epsilon |t|^p + r\delta(\epsilon)|u|^r.$$

Then, by (A11),

$$\begin{aligned}
J(u) &= \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \int_\Omega G(x,u)\,dx - \frac{1}{r_0}\int_\Omega H|u|^{r_0}\,dx \\
&\geq \frac{a}{p}\|u\|_{X_0}^p - \epsilon\|u\|_p^p - r\delta(\epsilon)\|u\|_r^r - \frac{M}{r_0}\|u\|_{r_0}^{r_0} \\
&\geq \frac{a}{p}\|u\|_{X_0}^p + \frac{b}{2p}\|u\|_{X_0}^{2p} - \frac{M}{r_0}\|u\|_{r_0}^{r_0} \\
&\geq \frac{a}{p}\|u\|_{X_0}^p - \frac{\epsilon}{\mu_1}\|u\|_{X_0}^p - r\delta(\epsilon)\|u\|_r^r - \frac{M}{r_0}\|u\|_{r_0}^{r_0} \\
&\geq (\frac{a}{p} - \frac{\epsilon}{\mu_1})\|u\|_{X_0}^p - r\delta(\epsilon)\beta_k^r\|u\|_{X_0}^r - \frac{M}{r_0}\beta_k^{r_0}\|u\|_{X_0}^{r_0}.
\end{aligned}$$

For every ϵ with $0 < \epsilon < \frac{a\mu_1}{p}$, choose

$$\|u\|_{X_0} = \gamma_k = \min\Big\{\Big(\frac{a\mu_1 - \epsilon p}{3r\delta(\epsilon)p\mu_1\beta_k^r}\Big)^{\frac{1}{r-p}}, \Big(\frac{(a\mu_1 - \epsilon p)r_0}{3p\mu_1 M\beta_k^{r_0}}\Big)^{\frac{1}{r_0-p}}\Big\}.$$

Since $\beta_k \to 0$ as $k \to \infty$, we have $\|u\| = \gamma_k \to +\infty$ as $k \to \infty$. Hence

$$\begin{aligned}
b_k &:= \inf_{u \in Z_k, \|u\|_{X_0} = \gamma_k} J(u) \\
&\geq \Big(\frac{a}{p} - \frac{\epsilon}{\mu_1}\Big)\gamma_k^p - r\delta(\epsilon)\Big(\frac{a\mu_1 - \epsilon p}{3r\delta(\epsilon)p\mu_1}\Big)\gamma_k^p - \frac{M}{r_0}\Big(\frac{(a\mu_1 - \epsilon p)r_0}{3p\mu_1 M}\Big)\gamma_k^p \\
&= \frac{1}{3}\Big(\frac{a\mu_1 - \epsilon p}{p\mu_1}\Big)\gamma_k^p \to +\infty, \quad \text{as } k \to \infty.
\end{aligned}$$

Then, condition (2) of Theorem 1.8 is satisfied.

So, its follows that the conditions of Theorem 1.8 was satisfied and we have unbounded sequence which yields that $I(u_k) \to +\infty$ then the proof is complete. □

Acknowledgements. The authors would like to thank the anonymous referees for their valuable suggestions and comments.

References

[1] A. Ambrosetti, P. H. Rabinowitz; *Dual variational methods in critical point theory and applications*, J. Funct. Anal., 14(4) (1973), 349-381.

[2] G. A. Afrouzi, M. Mirzapour, N. T. Chung; *Existence and multiplicity of solutions for a $p(x)$-Kirchhoff type equation*, Rend. Semin. Mat. Univ. Padova, 136 (2016), 95-109.

[3] E. Cabanillas Lapa, F. León Barboza, J. B. Bernui Barros, B. Godoy Torres Benigno; *Existence results for a nonlinear elliptic transmission problem of $p(x)$-Kirchhoff type*, Electron. J. Qual. Theory Differ. Equ., 105 (2016), 1-14.

[4] L. Caffarelli, L. Silvestre; *An extension problem related to the fractional Laplacian*, Commun. PDE, 32 (2007), 1245-1260.

[5] M. Caponi, P. Pucci; *Existence theorems for entire solutions of stationary Kirchhoff fractional p-Laplacian equations*, Ann. Mat. Pura Appl., 195 (6) (2016), 20992129.

[6] G. Cerami; *Un criterio di esistenza per i punti critici su varietá illimitate*, Rend. Ist. Lomb. Sci. Lett., 112 (1978), 332-336.

[7] P. Chen, X. H. Tang; *Existence and multiplicity results for infinitely many solutions for Kirchhoff-type problems in* $\mathbb{R}^N$, Mathematical methods in the Applied Sciences, (2013), to appear.
[8] B. Cheng; *A new result on multiplicity of nontrivial solutions for the nonhomogenous Schrdinger-Kirchhoff type problem in* $\mathbb{R}^N$, Mediterr. J. Math., 13 (3) (2016), 1099-1116.
[9] N. T. Chung; *Multiple solutions for a* $p(x)$*-Kirchhoff-type equation with sign-changing nonlinearities*, Complex Variables and Elliptic Equations, 58(12) (2013), 1637-1646.
[10] N. T. Chung; *Multiplicity results for a class of* $p(x)$*-Kirchhoff type equations with combined nonlinearities*, E. J. Qualitative Theory of Diff. Equ., 42 (2012) 1-13.
[11] G. Dai, R. Hao; *Existence of solutions for a* $p(x)$*-Kirchhoff-type equation*. J. Math. Anal. Appl., 359 (2009), 275-284.
[12] E. Di Nezza, G. Palatucci, E. Valdinoci; *Hitchhiker's guide to the fractional Sobolev spaces*, Bull. Sci. Math., 136 (2012), 521573.
[13] S. Dipierro, G. Palatucci, E. Valdinoci; *Existence and symmetry results for a Schrödinger type problem involving the fractional Laplacian*, Le Math., 68 (2013), 201-216.
[14] I. Ekeland; *Convexity methods in Hamiltonian mechanics. Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* [Results in Mathematics and Related Areas (3)], 19. Springer-Verlag, Berlin, 1990, PP 247.
[15] P. Felmer, A. Quaas, J.Tan; *Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian*, Proc. R. Soc. Edinb. A, 142 (2012), 1237-1262.
[16] G. M. Figueiredo, G. Molica Bisci, R. Servadei; *On a fractional Kirchhoff-type equation via Krasnoselskii's genus*, Asymptot. Anal., 94 (2015), 347-361.
[17] A. Fiscella, E. Valdinoci; *A critical Kirchhoff type problem involving a nonlocal operator*, Nonlinear Anal., 94 (2014), 156-170.
[18] A. Fiscella, P. Pucci; *On certain nonlocal Hardy-Sobolev critical elliptic Dirichlet problems*, Adv. Differ. Equat., 2 1(5-6) (2016), 571599.
[19] A. Fiscella, P. Pucci; *p-fractional Kirchhoff equations involving critical nonlinearities*, Nonlinear Anal. (RWA), 35 (2017) 350378.
[20] G. Franzina, G. Palatucci; *Fractional p-eigenvalues*, Riv. Mat. Univ. Parma. to appear.
[21] A. Iannizzotto, S. Liu, K. Perera, M. Squassina; *Existence results for fractional p-Laplacian problems via Morse theory*, Advan. Calcu. Variat., 9(2) (2014), 101-125.
[22] A. Iannizzotto, M. Squassina; *Weyl-type laws for fractional p-eigenvalue problems*, Asymptotic Anal., 2013;88(4): Doi: 10.3233/ASY-141223.
[23] G. Kirchhoff; *Mechanik*, Teubner, Leipzig, Germany, 1883.
[24] E. Lindgren E, P. Lindqvist; *Fractional eigenvalues*, Calc.Var. Part. Differ. Equat., 49 (2014), 795826.
[25] X. Mingqi, G. Molica Bisci, G. Tian, B. Zhang; *Infinitely many solutions for the stationary Kirchhoff problems involving the fractional p-Laplacian*, Nonlinearity, 29 (2016), 357-374.
[26] G. Molica Bisci, V. Radulescu; *Ground state solutions of scalar field fractional Schroedinger equations*, Calc. Var. Partial Differential Equations, 54 (2015), 2985-3008.
[27] G. Molica Bisci, V. Radulescu, R. Servadei; *Variational Methods for Nonlocal Fractional Problems*. Encyclopedia of Mathematics and its Applications, 162 Cambride University Press - ISBN: 9781107111943, 2016.
[28] G. Molica Bisci, D. Repovš; *On doubly nonlocal fractional elliptic equations*, Rend. Lincei Mat. Appl. 26 (2015) 161-176.
[29] G. Molica Bisci, F. Tulone; *An existence result for fractional Kirchhoff-type equations*, Z. Anal. Anwendungen, 35 (2016), 181-197.
[30] G. Molica Bisci, L. Vilasi; *On a fractional degenerate Kirchhoff-type problem*, Commun. Contemp. Math., 19 (2017), no. 1, 1550088, 23 pp.
[31] N. Nyamoradi; *Existence of three solutions for Kirchhoff nonlocal operators of elliptic type*, Math. Commun., 19 (2014), 11-24.
[32] P. Pucci, S. Saldi; *Critical stationary Kirchhoff equations in* $\mathbb{R}^N$ *involving nonlocal operators*, Rev. Mat. Iberoam., 29 (2013), 1091-1126.
[33] P. Pucci, M. Q. Xiang, B. L. Zhang; *Multiple solutions for nonhomogeneous Schrödinger-Kirchhoff type equations involving the fractional p-Laplacian in* $\mathbb{R}^N$, Calcul. Var. Part. Differ. Equat.. 54(3) (2015), 2785-2806.
[34] P. Rabinowitz; *Minimax method in critical point theory with applications to differential equations*, CBMS Amer. Math. Soc., No 65, 1986.

[35] S. Secchi; *Ground state solutions for the fractional Schrdinger in* $\mathbb{R}^N$, J. Math. Phys. 54 (2013), 031501.
[36] R. Servadei, E. Valdinoci; *Lewy-Stampacchia type estimates for variational inequalities deriven by (non)local operators*, Rev. Mat. Iberoam 29 (2013), 887-898.
[37] R. Servadei E. Valdinoci; *Mountain Pass solutions for non-local elliptic operators*, J. Math. Anal. Appl., 389 (2012), 887-898.
[38] R. Servadei E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33(5) (2013), 2105-2137.
[39] R. Servadei, E. Valdinoci; *Variational methods for non-local operators of elliptic type*, Discrete Contin. Dyn. Syst., 33 (2013), 2105-2137.
[40] R. Servadei, E. Valdinoci; *On the spectrum of two different fractional operators*, Proc. R. Soc. Edinburgh A, 144 (2014), 831-855.
[41] K. Teng; *Two nontrivial solutions for hemivariational inequalities driven by nonlocal elliptic operators*, Nonlinear Anal. (RWA), 14 (2013) 867-874.
[42] L. Vilasi; *Eigenvalue estimates for stationary* $p(x)$*-Kirchhoff problems*, Electron. J. Differ. Equat., 186 (2016), 1-9.
[43] Y. Wu, Y. Huang, Z. Liu; *Zeng Kirchhoff type problems with potential well and indefinite potential*, Electron. J. Differ. Equat., 178 (2016), 1-13.
[44] M. Q. Xiang, B. L. Zhang; *Degenerate Kirchhoff problems involving the fractional p-Laplacian without the (AR) condition*, Complex Variab. Elliptic Equat., 60(9) (2015), 1-11.
[45] M. Q. Xiang, B. L. Zhang, M. Ferrara; *Existence of solutions for Kirchhoff type problem involving the non-local fractional p-Laplacian*, J. Math. Anal. Appl., 424 (2015), 1021-1041.
[46] Q. L. Xie, X. P. Wu, C. L. Tang; *Existence of solutions for Kirchhoff type equations*, Electronic J. Differ. Equat., 47 (2015), 18.
[47] B. Zhang, G. Molica Bisci, M. Xiang; *Multiplicity results for nonlocal fractional p-Kirchhoff equations via Morse theory*, Topol. Meth. Nonlinear Anal. (Preprint).

Nemat Nyamoradi (corresponding author)
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran
E-mail address: nyamoradi@razi.ac.ir, neamat80@yahoo.com

Lahib Ibrahim Zaidan
Department of Mathematics, Faculty of Sciences, Razi University, 67149 Kermanshah, Iran.
Education College, University Of Babylon, Babil, Iraq
E-mail address: lahibzaidan75@yahoo.com; lahibzaidan@uobabylon.edu.iq