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OSCILLATION CRITERION FOR FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS WITH SIGN-CHANGING COEFFICIENTS

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Abstract. We establish conditions so that all solutions to a first-order linear delay differential equation become oscillatory. The coefficients of this equation are allowed to have negative and positive values, which is an expansion of the results in the references. To illustrate our results we present two examples.

1. Introduction

In this article, we study the oscillation of solutions to the first-order linear delay differential equation

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $p(t)$ and $\tau(t)$ are continuous on $[t_0, \infty)$, the delay argument $\tau(t)$ is strictly increasing, $\tau(t) < t$, and $\lim_{t\to\infty} \tau(t) = \infty$.

As is known, solutions to (1.1) are obtained by the method of steps: Given an initial function $\phi(t)$ integrable on $[\tau(t_0), t_0]$, define $x(t)$ on $[t_0, \tau^{-1}t_0]$ by integrating (1.1). Then repeat this process on $[\tau^{-1}t_0, \tau^{-2}t_0]$, and so on. A solution to (1.1) is said to be oscillatory if it has arbitrarily large zeros. If all solutions to an equation are oscillatory, the equation is said to be oscillatory.

Our goal is to obtain conditions on the $\limsup_{t\to\infty} \int_\tau^t p$ that imply the oscillation of all solutions to (1.1), independent of the $\liminf_{t\to\infty} \int_\tau^t p$. Our main tool is obtaining an estimate for $x(\tau(t))$ by integrating (1.1); then use this estimate in (1.1) and integrate again. Using these iterated of inequalities, we obtain a new oscillation criterion that can be applied even if $p(t)$ takes negative values in infinitely many intervals.

Now we present a summary of the known results. If $p(t) \geq 0$ and

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{e} \quad \text{or} \quad \beta := \limsup_{t\to\infty} \int_{\tau(t)}^t p(s)ds > 1, \tag{1.2}$$

then (1.1) is oscillatory [7]. On the other hand, when $\beta \leq 1/e$, equation (1.1) has a non-oscillatory solution [6]. Therefore, the interesting case happens when $\alpha \leq 1/e < \beta \leq 1$.

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In 1988, Erbe and Zhang [2] used an upper bound of the ratio $x(\tau(t))/x(t)$ to establish sufficient conditions for the oscillation of all solutions to (1.1):

$$p(t) \geq 0, \quad 0 < \alpha \leq \frac{1}{e}, \quad \beta > 1 - \frac{\alpha^2}{4}.$$

Since then several authors have improved these results by using upper bounds for $x(\tau(t))/x(t)$ when $\alpha \leq 1/e$; see for example [8, 11, 12, 14]. Let $\lambda_* \leq \lambda^*$ be the two roots of the equation $\lambda = e^{\alpha\lambda}$. The following results have been obtained for $p(t) \geq 0$ and $\alpha \leq 1/e$:

$$
\begin{aligned}
&\text{in [4]} \quad \beta > 1 - \frac{\alpha^2}{2(1-\alpha)} \geq 0.892,\\
&\text{in [13]} \quad \beta > 1 - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \geq 0.863,\\
&\text{in [1]} \quad \beta > 1 - \Big(1 - \frac{1}{\sqrt{\lambda_*}}\Big)^2 \quad \geq 0.845,\\
&\text{in [7]} \quad \beta > \frac{1+\ln\lambda_*}{\lambda_*} \geq 0.735,\\
&\text{in [9]} \quad \beta > 1 - \frac{\alpha^2}{2(1-\alpha)} - \frac{\alpha^2}{2}\lambda_* \geq 0.709,\\
&\text{in [3]} \quad \beta > \frac{1+\ln\lambda_*}{\lambda_*} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \geq 0.599,\\
&\text{in [5]} \quad \beta > 2\alpha + \frac{2}{\lambda_*} - 1 \geq 0.471,\\
&\text{in [10]} \quad \beta > \frac{\ln\lambda_* - 1 + \sqrt{5 - 2\lambda_* + 2\alpha\lambda_*}}{\lambda_*} \geq 0.459\,.
\end{aligned}
$$

Note that when $\alpha = 1/e$, the lowest bound for β is 0.459, so far. In 2004, Zhao et al [15] established the oscillation criterion

$$\limsup_{t\to\infty} \Big\{ \min_{\tau(t)\leq s\leq t} \int_{\tau(s)}^{s} p(\xi)d\xi \Big\} > \frac{1+\ln\lambda_*}{\lambda_*} - \frac{1}{\lambda^*}.$$

When $\alpha = 1/e$, we have $\lambda_* = \lambda^* = e$, and condition above reduces to

$$\limsup_{t\to\infty} \Big\{ \min_{\tau(t)\leq s\leq t} \int_{\tau(s)}^{s} p(\xi)d\xi \Big\} > \frac{1}{e}.$$

Note that as $\alpha \to 0$, almost all conditions reduce to the condition $\beta > 1$. However a condition in [10] leads to $\beta > \sqrt{3} - 1 \approx 0.732$, which is an essential improvement. A natural question arises: can we get a lower bound for β when $\alpha \leq 1/e$?

Motivated by [10, 11, 12, 15] and other publications, our goal is to study (1.1) when $p(t)$ takes negative values on infinitely many intervals.

2. Main results

Note that if $x(t)$ is a solution of (1.1), then $-x(t)$ is also a solution. Therefore, when $x(t)$ does not have zeros, we assume that $x(t)$ is positive. To use iterates of the delay, we define $\tau^0(t) = t$, $\tau^1(t) = \tau(t)$, $\tau^2(t) = \tau(\tau(t))$, etc. Also to use iterates of τ inverse, we define $\tau^{-2}(t) = \tau^{-1}(\tau^{-1}(t))$, $\tau^{-3}(t) = \tau^{-2}(\tau^{-1}(t))$, etc.

Following an idea in [12], we define the functions

$$\rho_i(t) = \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) \cdots \int_{\tau(s_{i-1})}^{\tau^{i-1}(t)} p(s_i) ds_i \cdots ds_1 \,. \tag{2.1}$$

However, the functions $\rho_i(t)$ in [12] are bounded below by non-negative constants; we do not use such assumption here. Note that computing $\rho_i(t)$ requires the evaluation of $p(s)$ for $s \in [\tau^i(t), t]$. Since $p(s)$ is defined only for $s \ge t_0$, the number of ρ_i's depend on the condition $\tau^i(t) \ge t_0$. For example when $\tau(t) = t - \text{const.}$ there are only finitely many ρ_i's, and when $\tau(t) = \sqrt{t}$ with $0 \le t_0 \le 1$ there are infinitely many ρ_i's. If $p(s) \ge 0$ on $[\tau^{-i}(t), t]$, then $\rho_i(t) \ge 0$. If $p(s) = 0$ on some interval $[\tau^n(t), \tau^{n-1}(t)]$, then $\rho_i(t) = 0$ for $i \ge n$. When $p(s) = p$ and $\tau(t) = t - \sigma$ with p and σ constants,

$$\rho_1(t) = \frac{p\sigma}{1!}, \quad \rho_2(t) = \frac{(p\sigma)^2}{2!}, \quad \dots .$$

We shall use the convention that for $j > n$, $\sum_{i=j}^{n} p = 0$ and $\prod_{i=j}^{n} p = 1$.

Lemma 2.1. *Let $x(t)$ be a solution of (1.1). Assume that there exit $n \ge 1$ and $t_1^* \ge t_1 > t_0$ such that: $\tau^{n+2}(t_1) \ge t_0$, $p(s) \ge 0$ for $s \in [\tau^{n+1}(t_1), t_1^*]$, and $x(t) > 0$ for $t \in [\tau^{n+2}(t_1), t_1^*]$. Let $\rho_i(t)$ be defined by (2.1), and $f_1(t) = 1/(1 - \rho_1(t))$. Then*

$$\frac{x(\tau(t))}{x(t)} \ge \frac{1}{1 - \sum_{i=1}^{n} \rho_i(t) \prod_{j=1}^{i-1} f_{n-j}(\tau^j(t))} =: f_n(t) > 0 \quad \forall t \in [t_1, t_1^*] \,, \tag{2.2}$$

where $f_2, f_3, \dots$ are defined recursively.

Proof. Integrating (1.1), we have

$$\begin{gathered} x(\tau(t)) = x(t) + \int_{\tau(t)}^{t} p(s_1) x(\tau(s_1))\, ds_1, \\ x(\tau(s_1)) = x(\tau(t)) + \int_{\tau(s_1)}^{\tau(t)} p(s_2) x(\tau(s_2))\, ds_2 \,, \\ x(\tau(s_2)) = x(\tau^2(t)) + \int_{\tau(s_2)}^{\tau^2(t)} p(s_3) x(\tau(s_3))\, ds_3 \,, \\ \dots . \end{gathered}$$

Using repeated substitutions we have

$$\begin{aligned} x(\tau(t)) = x(t) &+ \int_{\tau(t)}^{t} p(s_1) x(\tau(t))\, ds_1 + \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) x(\tau^2(t))\, ds_2\, ds_1 \\ &+ \int_{\tau(t)}^{t} p(s_1) \int_{\tau(s_1)}^{\tau(t)} p(s_2) \int_{\tau(s_2)}^{\tau^2(t)} p(s_3) x(\tau(s_3))\, ds_3\, ds_2\, ds_1 + \dots . \end{aligned}$$

Since $p(\cdot)$ and $x(\tau(\cdot))$ are non-negative on the interval $[\tau^{n+1}(t_1), t_1^*]$, from (1.1), we have that $x'(t) \le 0$ and hence $x(t)$ is non-increasing on the same interval. Using this property and (2.1) we obtain

$$x(\tau(t)) \ge x(t) + \rho_1(t) x(\tau(t)) + \rho_2(t) x(\tau^2(t)) + \rho_3(t) x(\tau^3(t)) + \dots . \tag{2.3}$$

Considering the first two terms in the right-hand side, we have

$$(1 - \rho_1(t)) x(\tau(t)) \ge x(t) > 0 \,;$$

thus

$$1-\rho_1(t)>0 \quad \text{and} \quad \frac{x(\tau(t))}{x(t)} \geq \frac{1}{1-\rho_1(t)} =: f_1(t)>0 \quad \text{for } t \in [\tau^{n-1}(t_1), t_1^*]\,.$$

With $\tau(t)$ instead of t in the above expression, we have

$$\frac{x(\tau^2(t))}{x(\tau(t))} \geq f_1(\tau(t)) \quad \text{for } t \in [\tau^{n-2}(t_1), t_1^*]\,. \tag{2.4}$$

Considering the first three terms in the right-hand side in (2.3), and using (2.4), we have

$$\Big(1-\rho_1(t)-\rho_2(t)f_1(\tau(t))\Big)x(\tau(t)) \geq x(t)>0 \quad \text{for } t \in [\tau^{n-2}(t_1), t_1^*]\,.$$

Thus $1-\rho_1(t)-\rho_2(t)f_1(\tau(t))>0$ and

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1-\rho_1(t)-\rho_2(t)f_1(\tau(t))} =: f_2(t)>0 \quad \text{for } t \in [\tau^{n-2}(t_1), t_1^*]\,.$$

With $\tau(t)$ instead of t in the above expression, we have

$$\frac{x(\tau^2(t))}{x(\tau(t))} \geq f_2(\tau(t)) \quad \text{for } t \in [\tau^{n-3}(t_1), t_1^*]\,. \tag{2.5}$$

With $\tau(t)$ instead of t in (2.4), we have

$$\frac{x(\tau^3(t))}{x(\tau^2(t))} \geq f_1(\tau^2(t)) \quad \text{for } t \in [\tau^{n-3}(t_1), t_1^*]\,. \tag{2.6}$$

Considering the first four terms in the right-hand side in (2.3), and using (2.5) and (2.6), we obtain

$$\Big(1-\rho_1(t)-\rho_2(t)f_2(\tau(t))-\rho_3(t)f_2(\tau(t))f_1(\tau^2(t))\Big)x(\tau(t)) \geq x(t)>0$$

for $t \in [\tau^{n-3}(t_1), t_1^*]$. Thus $1-\rho_1(t)-\rho_2(t)f_2(\tau(t))-\rho_3(t)f_2(\tau(t))f_1(\tau^2(t))>0$, and

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1-\rho_1(t)-\rho_2(t)f_2(\tau(t))-\rho_3(t)f_2(\tau(t))f_1(\tau^2(t))} =: f_3(t)>0$$

for $t \in [\tau^{n-3}(t_1), t_1^*]$. Doing the above process for n terms, we obtain (2.2), which completes the proof. □

Remark 2.2. Under the conditions of Lemma 2.1, we can obtain a lower bound that does not use recursion to define f_n:

$$\frac{x(\tau(t))}{x(t)} \geq \frac{1}{1-\sum_{i=1}^n \rho_i(t)\prod_{j=1}^{i-1} f_1(\tau^j(t))} =: f_n(t) \quad \forall t \in [t_1, t_1^*]\,. \tag{2.7}$$

This bound is easier to compute, but is not as sharp as the one in the lemma.

Let f_1 and (2.4) be as defined as in Lemma 2.1. From (2.4), we obtain

$$\begin{aligned} &\frac{x(\tau^3(t))}{x(\tau^2(t))} \geq f_1(\tau^2(t)) \quad \text{for } t \in [\tau^{n-3}(t_1), t_1^*]\,, \\ &\frac{x(\tau^4(t))}{x(\tau^3(t))} \geq f_1(\tau^3(t)) \quad \text{for } t \in [\tau^{n-4}(t_1), t_1^*]\,. \end{aligned} \tag{2.8}$$

Then considering the first four terms in the right-hand side in (2.3), and using (2.4) and (2.8) we have

$$\Big(1-\rho_1(t)-\rho_2(t)f_1(\tau(t))-\rho_3(t)f_1(\tau(t))f_1(\tau^2(t))\Big)x(\tau(t))\geq x(t)>0$$

for $t\in[\tau^{n-3}(t_1),t_1^*]$. Repeating this process for n terms, we obtain (2.7), which will be used in computations later.

The functions f_n defined in Lemma 2.1 have the following properties: Evaluating $f_n(t)$ requires evaluating $\rho_n(t)$ which in turn requires evaluating $p(s)$ for $s\in[\tau^n(t),t]$. In general, the functions f_n do not have monotonicity unless constant delay and constant coefficient. In addition, for a large index, $f_n(t)$ may not be defined, because of a division by zero in (2.2), or $f_n(t)$ may be negative.

Lemma 2.3. *Let $x(t)$ be a solution of* (1.1). *Assume that there exit $n\geq 1$, t_1, and t_1^* such that $\tau(t_1^*)\geq t_1>t_0$, $\tau^{n+2}(t_1)\geq t_0$, $p(s)\geq 0$ for $s\in[\tau^{n+1}(t_1),t_1^*]$, and $x(t)>0$ for $t\in[\tau^{n+2}(t_1),t_1^*]$. Let $\rho_i(t)$ be defined by* (2.1), *and $f_1(t)=1/(1-\rho_1(t))$. Then*

$$\frac{1-\rho_1(t)}{\sum_{i=2}^n\rho_i(t)\prod_{j=2}^{i-1}f_{n-j}(\tau^j(t))}\geq\frac{x(\tau^2(t))}{x(\tau(t))}\quad\forall t\in[\tau^{-1}(t_1),t_1^*]\,,\tag{2.9}$$

where $f_2,f_3,\dots$ are defined recursively by Lemma 2.1.

Proof. Note that the conditions for Lemma 2.1 are also satisfied in this lemma. Using (2.3) and ignoring the term $x(t)$ which is positive, we have

$$x(\tau(t))>\rho_1(t)x(\tau(t))+\rho_2(t)x(\tau^2(t))+\rho_3(t)x(\tau^3(t))+\rho_4(t)x(\tau^4(t))+\dots.$$

Using (2.5) and (2.6), we have

$$(1-\rho_1(t))x(\tau(t))>\Big(\rho_2(t)+\rho_3(t)f_2(\tau^2(t))+\rho_4(t)f_2(\tau^2(t))f_1(\tau^3(t))+\dots\Big)x(\tau^2(t))\,.$$

Then (2.9) follows by applying the same process n times. The proof is complete. □

Remark 2.4. Under the assumptions of Lemma 2.3, we can obtain a bound that does not use recursion to define f_n:

$$\frac{1-\rho_1(t)}{\sum_{i=2}^n\rho_i(t)\prod_{j=2}^{i-1}f_1(\tau^j(t))}\geq\frac{x(\tau^2(t))}{x(\tau(t))}\quad\forall t\in[\tau^{-1}(t_1),t_1^*]\,.\tag{2.10}$$

Of course this bound is easier to compute, but is not as sharp as the one in the lemma.

In the next theorems, we assume that there is time t_k^* and an integer n_k^* for which $f_n(t)$ may be negative or lower bound from (2.2) exceeds the upper bound from (2.9).

Theorem 2.5. *Assume that there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ approaching ∞, and a bounded sequence of positive integers $\{n_k\}_{k=1}^\infty$, such that for each k there exist t_k and t_k^* such that $t_k^*\geq t_k>t_0$, $\tau^{n_k+2}(t_k)\geq t_0$, and $p(s)\geq 0$ for $s\in[\tau^{n_k+1}(t_k),t_k^*]$. If there exists a sequence of integers $\{n_k^*\}_k$ with $1\leq n_k^*\leq n_k$ for which*

$$f_1(t)>0,\ f_2(t)>0,\ \dots,\ f_{n_k^*-1}(t)>0,\ f_{n_k^*}(t)<0,\tag{2.11}$$

where $t\in[t_k,t_k^*]$, $f_1(t)=1/(1-\rho_1(t))$ and $f_2(t),f_3(t),\dots$ are defined recursively by Lemma 2.1, then every solution of (1.1) *is oscillatory.*

Proof. Let $x(t)$ be a solution to (1.1). Without loss of generality, we may assume that $x(t) > 0$ for $t \in [\tau^{n_k^*+2}(t_k), t_k^*]$, where n_k^* satisfies (2.11). By Lemma 2.1, we get $f_{n_k^*} > 0$ for $t \in [t_k, t_k^*]$, which is a contradiction and completes the proof of Theorem 2.5. $\square$

Corollary 2.6. *Assume that $p(t) \geq p \geq 0$ on $[t_0, \infty)$ and $\tau(t) = t - \sigma$ for $\sigma > 0$. If there exists $n \geq 1$ such that*

$$\beta := \limsup_{t\to\infty} \int_{\tau(t)}^t p(s)ds > \frac{1}{1 + \sum_{i=2}^n \frac{\alpha^{i-1}}{i!} \prod_{j=1}^{i-1} f_{n-j}},$$

where $\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^t p(s)ds = p\sigma$ and $f_1, f_2, \dots f_{n-1}$ is defined by

$$f_1 = \frac{1}{1-\alpha}, \quad f_2 = \frac{1}{1 - \alpha - \frac{\alpha^2}{2!} f_1}, \quad \dots, \quad f_{n-1} = \frac{1}{1 - \sum_{i=1}^{n-1} \frac{\alpha^i}{i!} \prod_{j=1}^{i-1} f_{n-j-1}},$$

then every solution of (1.1) oscillates.

Remark 2.7. Under the conditions of Corollary 2.6, we can obtain a lower bound that use recursion to define f_n when $\alpha = 1/e$. A numerical trend can be found that the lowest bound for β is $\beta \to \frac{1}{e-1} \approx 0.5819$ since $f_n \to e$.

Theorem 2.8. *Assume that there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ approaching ∞, and a bounded sequence of positive integers $\{n_k\}_{k=1}^\infty$, such that for each k there exist t_k and t_k^* such that $\tau(t_k^*) \geq t_k > t_0$, $\tau^{n_k+2}(t_k) \geq t_0$, and $p(s) \geq 0$ for $s \in [\tau^{n_k+1}(t_k), t_k^*]$. If there exists a sequence of integers $\{n_k^*\}_k$ with $1 \leq n_k^* \leq n_k$ for which*

$$\frac{1 - \rho_1(t)}{\sum_{i=2}^{n_k^*} \rho_i(t) \prod_{j=2}^{i-1} f_{n-j}(\tau^j(t))} < \frac{1}{1 - \sum_{i=1}^{n_k^*} \rho_i(\tau(t)) \prod_{j=1}^{i-1} f_{n-j}(\tau^{j+1}(t))}, \tag{2.12}$$

where $t \in [\tau^{-1}(t_k), t_k^*]$, then every solution of (1.1) is oscillatory.

Proof. To reach a contradiction assume that a solution $x(t)$ does not have zeros on the interval $[\tau^{n+1}(t_k), t_k^*]$, and without loss of generality assume that $x(t) > 0$. Then the hypotheses for both Lemmas 2.1 and 2.3 are satisfied. Applying (2.7) with $f_n(\tau(t))$ instead of $f_n(t)$, we obtain an inequality that combined with (2.10) contradicts the definition of n_k^*; Therefore, each solution of (1.1) has a zero in $[\tau^{n_k+1}(t_k), t_k^*]$. This completes the proof. $\square$

In the next corollary, we assume that there is time t_k^* and an integer n_k^* for which lower bound from (2.7) exceeds the upper bound from (2.10). Note that to compare the two bounds we need $f_1(\tau(t_k^*))$, not $f_1(t_k^*)$, in (2.7).

Corollary 2.9. *Assume that there exists an increasing sequence $\{t_k\}_{k=1}^\infty$ approaching ∞, and a bounded sequence of positive integers $\{n_k\}_{k=1}^\infty$, such that for each k there exist t_k and t_k^* such that $\tau(t_k^*) \geq t_k > t_0$, $\tau^{n_k+2}(t_k) \geq t_0$, and $p(s) \geq 0$ for $s \in [\tau^{n_k+1}(t_k), t_k^*]$. If there exists a sequence of integers $\{n_k^*\}_k$ with $1 \leq n_k^* \leq n_k$ for which*

$$\frac{1 - \rho_1(t)}{\sum_{i=2}^{n_k^*} \rho_i(t) \prod_{j=2}^{i-1} f_1(\tau^j(t))} < \frac{1}{1 - \sum_{i=1}^{n_k^*} \rho_i(\tau(t)) \prod_{j=1}^{i-1} f_1(\tau^{j+1}(t))}, \tag{2.13}$$

where $t \in [\tau^{-1}(t_k), t_k^*]$, then every solution of (1.1) is oscillatory.

3. Examples

The following examples show equations for which Theorem 2.5 or Theorem 2.8 imply oscillation of all solutions, when $\limsup \int_{\tau(t)}^t p < 1$ and $\liminf \int_{\tau(t)}^t p < 1/e$. Thus oscillation criteria from the references cannot be applied to these examples.

Example 3.1. Consider the first-order delay differential equation with constant delay and sign-changing coefficient

$$x'(t) + p(t)x(t-1)) = 0, \quad t \geq 0, \tag{3.1}$$

where

$$p(t) = \begin{cases} -\beta + 2\beta t & \text{for } 0 \leq t < 1, \\ \beta & \text{for } 1 \leq t < 6, \\ \beta - 2\beta(t-6) & \text{for } 6 \leq t < 7, \\ -\beta & \text{for } 7 \leq t < 8, \end{cases}$$

and p is periodic with $p(t+8) = p(t)$. Then $\limsup_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds = \beta$ and $\liminf_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds = -\beta$. So the interesting case happens when $0 < \beta < 1$. For $t = t_k = 8k + 6$ and $i = 1, 2, 3, 4$, the functions $\rho_i(t) = \beta^i / i!$ and

$$f_1(t) = \frac{1}{1-\beta}, \quad f_2(t) = \frac{1}{1 - \beta - \frac{\beta^2}{2!} f_1(t-1)},$$

$$f_3(t) = \frac{1}{1 - \beta - \frac{\beta^2}{2!} f_2(t-1) - \frac{\beta^3}{3!} f_2(t-1) f_1(t-2)},$$

$$f_4(t) = \frac{1}{1 - \beta - \frac{\beta^2}{2!} f_3(t-1) - \frac{\beta^3}{3!} f_3(t-1) f_2(t-2) - \frac{\beta^4}{4!} f_3(t-1) f_2(t-2) f_1(t-3)}.$$

For $\beta = 0.439$ and $n_k^* = 4$, the software package Mathematica computes the lower bound to be

$$f_1(t) \approx 1.7825, \quad f_2(t) \approx 2.5691, \quad f_3(t) \approx 4.0183, \quad f_4(t) \approx -0.0038.$$

Therefore by Theorem 2.5 all solutions of (3.1) are oscillatory.

Looking for the lowest possible value of β, we consider the limit as $n_k^* \to \infty$. It is interesting to point out that the value of the lower bound of β appears as close as possible to value $1/e \approx 0.367879$. A numerical trend can be found in Table 1, Then by Table 1 and Theorem 2.5 every solution of (3.1) has at least one zero on $[t_k - (2 + n_k^*),\ t_k]$ for $k \in \mathbb{N}$, where n_k^* is given by the corresponding β. So every solution of (3.1) is oscillatory.

Table 1. Numerical results for different lower bound of $\beta > 1/e$

$\beta \to 1/e$	$f_1 > 0$	$f_2 > 0$	...	$f_{n_k^*-1} > 0$	$f_{n_k^*} \leq 0$
0.37200	1.5924	1.9312	...	$f_{19} \approx 6.3466$	$f_{20} \approx -7.2066$
0.36900	1.5848	1.9117	...	$f_{38} \approx 5.5387$	$f_{39} \approx -56.786$
0.36810	1.5825	1.9059	...	$f_{89} \approx 9.3745$	$f_{90} \approx -1.8723$
0.36795	1.5822	1.9049	...	$f_{159} \approx 37.17$	$f_{160} \approx -0.232$
0.36789	1.5820	1.9046	...	$f_{413} \approx 11.25$	$f_{414} \approx -1.270$

Example 3.2. Consider the first-order delay differential equation with constant delay and variable coefficient

$$x'(t) + b(1 + \sin(2t))x\big(t - \frac{\pi}{2}\big) = 0, \quad t \ge 0\,. \tag{3.2}$$

Then $\limsup_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds = \frac{b}{2}(\pi + 2)$ and $\liminf_{t\to\infty} \int_{\tau(t)}^t p(s)\,ds = \frac{b}{2}(\pi - 2)$. So the interesting case happens when $0 < b < \min\{\frac{2}{\pi+2}, \frac{2}{e(\pi-2)}\} \approx 0.388$.

Let $t_k = (k + \frac{1}{2})\pi$ for $k = 1, 2, 3, \dots$. Then using the software package Mathematica, we have

$$\rho_1(t_k) = \int_{t_k-\pi/2}^{t_k} b\big(1 + \sin(2s_1)\big)\,ds_1 = \frac{b}{2}(\pi + 2),$$

$$\rho_2(t_k) = \frac{b^2}{8}(\pi^2 - 4), \quad \rho_3(t_k) = \frac{b^3}{48}(\pi + 4)\big(\pi(\pi + 2) - 14\big),$$

$$\rho_4(t_k) = \frac{b^4}{384}\Big(\pi\big(\pi(\pi(5\pi - 32) - 24) - 272\big) - 48\Big)\,.$$

while $t_k^* = \tau^{-1}(t_k) = t_k + \frac{\pi}{2}$, and

$$\rho_1(t_k^*) = \int_{t_k}^{t_k+\pi/2} b\big(1 + \sin(2s_1)\big)\,ds_1 = \frac{b}{2}(\pi - 2),$$

$$\rho_2(t_k^*) = \frac{b^2}{8}(\pi^2 - 4), \quad \rho_3(t_k^*) = \frac{b^3}{48}(\pi - 4)\big(\pi(\pi - 2) - 14\big)\,,$$

$$\rho_4(t_k^*) = \frac{b^4}{384}\Big(\pi\big(\pi(\pi(5\pi - 32) - 24) - 272\big) - 48\Big)\,.$$

Then

$$f_1(t_k) = f_1(\tau^2(t_k)) = f_1(\tau^4(t_k)) = \dots = \frac{1}{1 - \frac{b}{2}(\pi + 2)},$$

$$f_1(\tau(t_k)) = f_1(\tau^3(t_k)) = f_1(\tau^5(t_k)) = \dots = \frac{1}{1 - \frac{b}{2}(\pi - 2)},$$

and

$$f_1(t_k^*) = f_1(\tau^2(t_k^*)) = f_1(\tau^4(t_k^*)) = \dots = \frac{1}{1 - \frac{b}{2}(\pi - 2)},$$

$$f_1(\tau(t_k^*)) = f_1(\tau^3(t_k^*)) = f_1(\tau^5(t_k^*)) = \dots = \frac{1}{1 - \frac{b}{2}(\pi + 2)}\,.$$

When $b = 0.3$, $n_k^* = 3$ and $t = t_k^*$, we have $\tau(t_k^*) = t_k$. Then computations show that lower bound given by (2.7) is 10.5472, while the upper bound given by (2.10) is 10.4542. Therefore by Corollary 2.9 all solutions of (3.2) are oscillatory.

Concluding remarks. The main contribution of Theorem 2.8 is that the result does not depend on the $\liminf \int_\tau^t p$. Example 3.1 shows that Theorem 2.5 can be applied for equations with $\limsup \int_\tau^t p = 0.439$, and $\liminf \int_\tau^t p < 0$. However, it does not imply that the theorem can be applied to every equation with this lim sup. Note that if $p(t)$ is greater and or equal to a coefficient $\tilde{p}(t)$ for which Theorems 2.5 and 2.8 apply, then the solutions to both equations are oscillatory. In the case of Example 3.1, if $p(t) \ge 0.439$ on four consecutive intervals of the form $[\tau(t), t]$ then all solutions are oscillatory. Therefore, lowering the bound for β when $0 \le \alpha \le 1/e$, remains an open question as mentioned in the introduction.

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References

[1] A. Elbert, I. P. Stavroulakis; *Oscillations of first-order differential equations with deviating arguments*, Recent trends in differential equations, World Sci. Ser. Appl. Anal., **1** (1992), 163–178.

[2] L. H. Erbe, B. G. Zhang; *Oscillation for first-order linear differential equations with deviating arguments*, Diff. Integral Equations, **1** (1988), 305–314.

[3] J. Jaroš, I. P. Stavroulakis; *Oscillation tests for delay equations*, Rocky Mountain J. Math., **29** (1999), 197–207.

[4] C. Jian; *On the oscillation of linear differential equations with deviating arguments*, Mathematics in Practice and Theory, **21** (1991), 32–41 (in chinese).

[5] M. Kon, Y. G. Sficas, I. P. Stavroulakis; *Oscillation criteria for delay equations*, Proc. Amer. Math. Soc., **128** (2000), 2989–2997.

[6] R. G. Koplatadze, T. A. Chanturija; *On oscillatory and monotonic solutions of first-order differential equations with deviating arguments*, Differ. Uravn., **18** (1982), 1463–1465 (in Russian).

[7] M. K. Kwong; *Oscillation of first order delay equations*, J. Math. Anal. Appl., **156** (1991), 274–286.

[8] A. D. Myshkis; *Linear homogeneous differential equations of first order with deviating arguments*, Uspekhi Mat. Nauk, **5** (1950), 160–162 (in Russian).

[9] C. G. Philos, Y. G. Sficas; *An oscillation criterion for first-order linear delay differential equations*, Canad. Math. Bull., **41** (1998), 207–213.

[10] Y. G. Sficas, I. P. Stavroulakis; *Oscillation criteria for first-order delay equations* Bull. London. Math. Soc., **35** (2003), 239-246.

[11] X. H. Tang, J. S. Yu; *The maximum existence interval of positive solutions of first order delay differential inequalities with applications*, Mathematics in Practice and Theory, **30** (2000), 447–452 (in chinese).

[12] H. W. Wu, L. Erbe, A. Peterson; *Upper bounds for the distances between consecutive zeros of solutions of first order delay differential equations*, J. Math. Anal. Appl., **429** (2015), 562–575.

[13] J. S. Yu, Z. C. Wang; *Some further results on oscillation of neutral differential equations*, Bull. Austral. Math. Soc., **46** (1992), 149–157.

[14] B. G. Zhang, Y. Zhou; *The distribution of zeros of solutions of differential equations with a variable delay*, J. Math. Anal. Appl., **256** (2001), 216–228.

[15] A. M. Zhao, X. H. Tang, J. R. Yan; *Oscillation of first-order delay differential equations*, ANZIAM J., **45** (2004), 593–599.

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