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EIGENVALUES OF STURM-LIOUVILLE PROBLEMS WITH DISCONTINUOUS BOUNDARY CONDITIONS

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Abstract. For classical regular two-point self-adjoint Sturm-Liouville problems (SLP) the dependence of the eigenvalues on the boundary conditions is well understood because of some surprisingly recent results. Recently there has been a lot of interest in problems with discontinuous boundary conditions. Such conditions are known by various names including transmission conditions, interface conditions, point interactions (in the physics literature), etc. Here we extend the known classical results to such problems.

1. Introduction

Regular Sturm-Liouville problems (SLP) with boundary conditions requiring a jump discontinuity at an interior point of the underlying interval are a very active current research area. Such conditions are known by various names including: transmission conditions [22, 23], discontinuous conditions [25, 16], interface conditions [19, 26, 32], multi-point conditions or multi-interval problems [13, 27, 18, 28], conditions on trees, point interactions, etc.

Consider the equation

$$My = -(py')' + qy = \lambda w\, yquad\text{on } J = [a,b],\ \lambda \in \mathbb{C},\ -\infty < a < b < \infty \tag{1.1}$$

with coefficients satisfying

$$\frac{1}{p}, q, w \in L(J,\mathbb{R}), \quad p > 0, \quad w > 0, \quad \text{a.e. on } J, \tag{1.2}$$

where $L(J,\mathbb{R})$ denotes the real-valued functions which are Lebesgue integrable on J.

Condition (1.2) implies that all solutions y and their quasi-derivatives $y^{[1]} = (py')$ of equation (1.1) are continuous on the whole interval J [33] and thus rules out any boundary condition requiring a discontinuity.

We call the study of equation (1.1) and its operators, under condition (1.2), the 1-interval theory. Of particular interest are the self-adjoint operator realizations S of equation (1.1) and their spectrum. These are operators S from $L^2(J,w)$ to $L^2(J,w)$ which satisfy

$$S_{\min} \subset S = S^* \subset S_{\max}, \tag{1.3}$$

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where $S_{\min}$ and $S_{\max}$ are the minimal and maximal operators of equation (1.1) under condition (1.2) in the space $L^2(J,w)$. For this and other definitions and basic properties of equation (1.1) see the book [33].

In this article we study equation (1.1) with boundary conditions

$$AY(a) + BY(b) = 0, \tag{1.4}$$

$$Y(c^+) = C\,Y(c^-), \quad a < c < b, \tag{1.5}$$

where $Y = \begin{pmatrix} y \\ y^{[1]} \end{pmatrix}$, $y^{[1]} = (py')$, and the matrices A, B, C satisfy $A, B \in M_2(\mathbb{C})$, $C \in M_2(\mathbb{R})$, $\det(C) = 1$,

$$AEA^* = BEB^*, \quad \operatorname{rank}(A : B) = 2, \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{1.6}$$

Here $\mathbb{C}$ and $\mathbb{R}$ denote the complex and real numbers, respectively, $(A : B)$ denotes the 2×4 matrix whose first two columns are those of A and the last two are the columns of B, and $M_2(\mathbb{S})$ denotes the 2×2 matrices with entries from $\mathbb{S}$.

It is well known [33] that the boundary value problem consisting of equation (1.1) with coefficients satisfying (1.2) and the boundary condition (1.4) and (1.6) generates a self-adjoint operator S satisfying (1.3) and that every operator S satisfying (1.3) is generated by a two point boundary condition (1.4) and (1.6). Thus every eigenfunction of every operator S satisfying (1.3) is continuous on J. Thus if C in (1.5) is not the identity matrix, how can we find eigenvalues whose eigenfunctions satisfy boundary conditions (1.4) and (1.5)? The next remark discusses this question.

Remark 1.1. In [29] it is shown that the boundary value problem (1.1), (1.2), (1.4), (1.5), (1.6) determines an operator S satisfying (1.3) i.e. is self-adjoint in the Hilbert space $H = L^2(J,w)$ and its spectrum is discrete consisting of an infinite number of eigenvalues. Thus if C is not the identity matrix I, then the eigenfunctions are not continuous at c by (1.5). This result is a special case of a much more general theorem from the 2-interval theory developed by Everitt and Zettl in [13]. See [29] for details. In this theory it is convenient to identify the Hilbert space H with the direct sum space $H = L^2(J_1,w_1) \dotplus L^2(J_2,w_2)$ where $J_1 = (a,c)$, $J_2 = (c,b)$ and w_1, w_2 are the restrictions of w to J_1, J_2, respectively. Strictly speaking, the 2-interval theory applied to J_1, J_2 extends (1.3) from the Hilbert space $L^2(J,w)$ to the direct sum space $L^2(J_1,w_1) \dotplus L^2(J_2,w_2)$. These two spaces consist of the same functions but the direct sum space emphasizes that these functions need not be continuous at c. We believe this clarifies the meaning of a statement commonly made in the literature when authors simply say we study the equation (1.1) on "$(a,c) \cup (c,b)$". See the next remark.

Remark 1.2. We comment on the nature of the solutions of equation (1.1) which satisfy condition (1.5) (and not necessarily (1.4) and (1.6)). Any initial condition at a determines a unique solution y and its quasi-derivative $y^{[1]}$ which are continuous on $[a,c^-]$. Condition (1.5) then determines $Y(c^+)$ and using $Y(c^+)$ as an initial condition y and $y^{[1]}$ are uniquely determined and continuous on $[c^+,b]$. Here, c^- denotes the limit from the left and c^+ the limit from the right. Therefore every initial condition at a determines a unique solution y on the interval $[a,b]$ which satisfies condition (1.5) and is continuous along with its quasi-derivative (py') on the intervals $[a, c^-]$ and $[c^+, b]$. We call this solution y the 'extended' solution, or C-extended solution, on $[a, b]$ and continue to denote it by y. Thus for any fixed matrix C with $\det C = 1$ there is a 2-dimensional space of extended solutions of equation (1.1) on the interval $[a, b]$.

In this article we develop a method for studying Sturm-Liouville problems (1.1), (1.2), (1.4), (1.5), (1.6) by constructing operators $C_{\min}$ and $C_{\max}$ which depend on the jump condition (1.5) and then prove that, for any fixed condition (1.5), all self-adjoint operators S in $L^2(J, w)$ generated by the boundary conditions (1.4) (1.6) are characterized by

$$C_{\min} \subset S = S^* \subset C_{\max}. \tag{1.7}$$

This essentially reduces problems with boundary conditions (1.4) (1.6) and (1.5) to the study of problems with condition (1.4) (1.6)) only and allows us to generalize known results for boundary conditions (1.4) (1.6) to problems (1.4) (1.6) and (1.5). For fixed C in (1.5) the well known inequalities among eigenvalues for different boundary conditions (1.4) (1.6) established by Eastham, Kong, Wu, Zettl [7], the characterization of the eigenvalues as zeros of an entire function, the continuous and discontinuous dependence of the eigenvalues on the boundary conditions (1.4) (1.6) are extended to (1.4) (1.6) (1.5). We make no attempt to state all of these extensions here. When C is not the identity matrix then the eigenfunctions are extended solutions as described in Remark 1.2. When C is the identity then the extended results reduce to the known results for (1.4) (1.6).

A key difference between the operators $S_{\min}$ and $S_{\max}$ in (1.3) and $C_{\min}$ and $C_{\max}$ in (1.7) is that the former do not depend on the boundary conditions and the latter do depend on condition (1.5). Because of this dependence the proof of (1.7) is rather technical. But it can readily be extended to any finite number of interior jump conditions (1.5) but we do not pursue this extension here. It can also be extended to an infinite number of conditions (1.5) but this requires some additional technical considerations.

The organization of the paper is as follows: In Section 2 we construct $C_{\min}$ and $C_{\max}$ and establish (1.7), in Section 3 prove the transcendental characterization of the eigenvalues. Section 4 contains a brief review of the canonical forms of the boundary conditions (1.4) (1.6), existence of eigenvalues is discussed in Section 5. The other sections contain 'applications' of (1.7): Inequalities in Section 6, Continuity in Section 7, differentiability in Section 8, monotonicity in 9, and multiplicity in 10.

2. Minimal and maximal operators for discontinuous boundary conditions

In this section we construct the operators $C_{\min}$ and $C_{\max}$ and characterize the boundary conditions which generate the operators S in the Hilbert space $H = L^2(J, w)$ satisfying (1.7). Our construction is based on the 2-interval theory applied to the intervals

$$J_1 = (a, c), \quad J_2 = (c, b).$$

For a detailed discussion of this theory and its application to intervals which have a common endpoint see the recent paper [29]. In this application the Hilbert space H is identified with the direct sum space $L^2(J_1, w_1) \dotplus L^2(J_2, w_2)$ where w_1, w_2 are the restrictions of w to the intervals J_1, J_2, respectively. We briefly summarize this two interval theory next. The 2-interval definitions of the minimal and maximal operators and their basic properties used below.

Definition 2.1.

$$D(S_{\min}(J)) = D(S_{\min}(J_1)) \dotplus D(S_{\min}(J_2)),$$
$$D(S_{\max}(J)) = D(S_{\max}(J_1)) \dotplus D(S_{\max}(J_2)),$$

and the corresponding operators $S_{\min}(J)$ and $S_{\max}(J)$ have these domains.

As in the 1-interval case the Lagrange sesquilinear form is fundamental in the study of boundary value problems. It is defined by

$$[f, g] = [f_1, g_1](c^-) - [f_1, g_1](a) + [f_2, g_2](b) - [f_2, g_2](c^+)$$

where

$$[f_r, g_r] = f_r(p_r \overline{g_r'}) - \overline{g_r}(p_r f_r').$$

Here f_r, g_r, p_r denote the the restrictions of f, g, p to J_r, $r = 1, 2$.

From the 2-interval theory [29], [33] we have the following two lemmas. To simplify the notation we let $S_{\min} = S_{\min}(J)$ and $S_{\max} = S_{\max}(J)$.

Lemma 2.2. (1) *The minimal operator $S_{\min}$ is a closed, densely defined, symmetric operator in the Hilbert space H.*
(2)

$$S^*_{\min} = S^*_{1,\min} \dotplus S^*_{2,\min} = S_{1,\max} \dotplus S_{2,\max} = S_{\max};$$
$$S^*_{\max} = S^*_{1,\max} \dotplus S^*_{2,\max} = S_{1,\min} \dotplus S_{2,\min} = S_{\min}.$$

Lemma 2.3. *The operators $S_{\min}$ and $S_{\max}$ have the properties:*

(1) *The generalized Green's formula holds*

$$(S_{\max} f, g) - (f, S_{\max} g) = [f, g] \quad (f, g \in D(S_{\max})); \tag{2.1}$$

(2) *$D(S_{\min})$ can be characterized as*

$$D(S_{\min}) = \{f \in D(S_{\max}) : [f, g] = 0 \ \text{ for all } g \in D(S_{\max})\}.$$

For a proof of the above lemma, see [33]. Next we define the operators $C_{\max}$ and $C_{\min}$ which depend on the interior discontinuous condition (1.5) in the space H.

Definition 2.4. Let (1.1), (1.2), (1.4) and (1.5) hold. Define the operator $C_{\max}$ in the Hilbert space H by

$$D(C_{\max}) = \{y = \{y_1, y_2\} \in D(S_{\max}) : Y(c^+) = CY(c^-)\}$$

and $C_{\max}$ is the restriction of the 2-interval maximal operator $S_{\max}$ to the domain $D(C_{\max})$.

Definition 2.5. Let (1.1), (1.2), (1.4) and (1.5) hold. Define the operator $C_{\min}$ in the Hilbert space H by

$$D(C_{\min}) = \{y = \{y_1, y_2\} \in D(C_{\max}) : Y(a) = 0 = Y(b)\}$$

and $C_{\min}$ is the restriction of the 2-interval maximal operator $S_{\max}$ to the domain $D(C_{\min})$.

Lemma 2.6 (Naimark Patching Lemma). Given any $c_k \in \mathbb{C}, k = 1, 2, \dots, 8$ there exists a maximal domain function $g = \{g_1, g_2\} \in D(S_{\max})$ such that

$$
\begin{aligned}
&g_1(a) = c_1, \quad (p_1 g_1')(a) = c_2, \quad g_1(c^-) = c_3, \quad (p_1 g_1')(c^-) = c_4, \\
&g_2(c^+) = c_5, \quad (p_2 g_2')(c^+) = c_6, \quad g_2(b) = c_7, \quad (p_2 g_2')(b) = c_8.
\end{aligned}
$$

For a proof of the above lemma see the two-interval S-L theory [28, 33] . From Lemma 2.6, one can obtain the following conclusion.

Lemma 2.7. *Given any complex numbers α_i, $i = 1, 2, 3, 4$ there exists a function $g = \{g_1, g_2\} \in D(C_{\max})$ such that*

$$g_1(a) = \alpha_1, \quad (p_1 g_1')(a) = \alpha_2, \quad g_2(b) = \alpha_3, \quad (p_2 g_2')(b) = \alpha_4.$$

Proof. This lemma is a special case of Lemma 2.6, where the function g satisfies the interior discontinuous condition i.e.

$$\begin{pmatrix} g_2(c^+) \\ (p_2 g_2')(c^+) \end{pmatrix} = C \begin{pmatrix} g_1(c^-) \\ (p_1 g_1')(c^-) \end{pmatrix}.$$

□

The well known GKN theorem and its extensions are powerful tools for characterizing all self-adjoint realizations S of equation (1.1) i.e. all operators S satisfying (1.3), in terms of two point boundary conditions. The next theorems in this section, especially Theorem 2.14, establish a correspondingly powerful tool which can be used to characterize all self-adjoint realizations S satisfying (1.7) in terms of two point boundary conditions for any fixed C. This new tool is used in Sections 6 to 10 to extend the known classical results to the boundary value problem (1.1), (1.2), (1.4), (1.5), (1.6).

Theorem 2.8. *Let the operators $C_{\min}$ and $C_{\max}$ be defined as above. Then we have*

1. *$D(S_{\min}) \subset D(C_{\min}) \subset D(C_{\max}) \subset D(S_{\max})$ and $S_{\min} \subset C_{\min} \subset C_{\max} \subset S_{\max}$;*
2. *$D(C_{\min})$ and $D(C_{\max})$ are dense in H;*
3. *For any $f, g \in D(C_{\max})$,*
$$(C_{\max} f, g) - (f, C_{\max} g) = [f, g] = [f_2, g_2](b) - [f_1, g_1](a); \tag{2.2}$$
4. *For any $f, g \in D(C_{\min})$, $[f, g] = 0$;*
5. *The operator $C_{\min}$ is a closed symmetric extension of the two-interval minimal operator $S_{\min}$;*
6. *$C_{\min}^* = C_{\max}$ and $C_{\max}^* = C_{\min}$;*
7. *$C_{\max}$ is closed in H.*

Proof. Properties (1) and (2) follow from the definition of $C_{\min}$ and $C_{\max}$ and the fact that $D(S_{\min})$ is dense in H.

For any $f, g \in D(C_{\max})$, functions f and g satisfy the interior discontinuous condition, i.e.

$$\begin{pmatrix} f_2(c^+) \\ f_2^{[1]}(c^+) \end{pmatrix} = C \begin{pmatrix} f_1(c^-) \\ f_1^{[1]}(c^-) \end{pmatrix}, \quad \begin{pmatrix} g_2(c^+) \\ g_2^{[1]}(c^+) \end{pmatrix} = C \begin{pmatrix} g_1(c^-) \\ g_1^{[1]}(c^-) \end{pmatrix},$$

$$\begin{aligned}[f_2,g_2](c^+) &= \big(f_2(p_2\overline{g_2'}) - \overline{g_2}(p_2f_2')\big)(c^+)\\ &= \det(C)\big(f_1(p_1\overline{g_1'}) - \overline{g_1}(p_1f_1')\big)(c^-)\\ &= [f_1,g_1](c^-).\end{aligned}$$

It follows from the generalized Green's formula (2.1) that, for any $f,g\in D(C_{\max})\subset D(S_{\max})$,

$$(C_{\max}f,g)-(f,C_{\max}g)=[f,g]=[f_2,g_2](b)-[f_1,g_1](a).$$

Therefore, for all $f,g\in D(C_{\min})$,

$$(C_{\min}f,g)-(f,C_{\min}g)=[f,g]=0,$$

which shows that the densely defined operator $C_{\min}$ is symmetric.

It is obvious that

$$(C_{\min}f,g)-(f,C_{\max}g)=[f_2,g_2](b)-[f_1,g_1](a)=0,\quad \forall f\in D(C_{\min}),\ g\in D(C_{\max})$$

Hence $C_{\max}\subset C_{\min}^*$. Next we prove $C_{\min}^*\subset C_{\max}$.

Since $S_{\min}\subset C_{\min}\subset C_{\max}\subset S_{\max}$, we have

$$S_{\min}=S_{\max}^*\subset C_{\max}^*\subset C_{\min}^*\subset S_{\min}^*=S_{\max}. \tag{2.3}$$

Let $g\in D(C_{\min}^*)$, then for any $f\in D(C_{\min})$, it follows from (2.1) that

$$\begin{aligned}0&=(C_{\min}f,g)-(f,C_{\min}^*g)\\ &=[f,g]\\ &=[f_1,g_1](c^-)-[f_1,g_1](a)+[f_2,g_2](b)-[f_2,g_2](c^+)\\ &=[f_1,g_1](c^-)-[f_2,g_2](c^+)\\ &=\big(f_1(p_1\overline{g_1'})-\overline{g_1}(p_1f_1')\big)(c^-)-\big(f_2(p_2\overline{g_2'})-\overline{g_2}(p_2f_2')\big)(c^+).\end{aligned} \tag{2.4}$$

Since $f\in D(C_{\min})$, the function f satisfies

$$\begin{pmatrix} f_2(c^+)\\ (p_2f_2')(c^+)\end{pmatrix}=C\begin{pmatrix} f_1(c^-)\\ (p_1f_1')(c^-)\end{pmatrix},$$

and by substituting it into equation (2.4), it follows that

$$\begin{aligned}&\Big((p_1\overline{g_1'})(c^-)-c_{11}(p_2\overline{g_2'})(c^+)+c_{21}\overline{g_2}(c^+)\Big)f_1(c^-)\\ &+\Big(-\overline{g_1}(c^-)-c_{12}(p_2\overline{g_2'})(c^+)+c_{22}\overline{g_2}(c^+)\Big)(p_1f_1')(c^-)=0.\end{aligned} \tag{2.5}$$

From the arbitrariness of function $f\in D(C_{\min})$ and the Naimark Patching Lemma 2.7, it follows that

$$\begin{gathered}(p_1\overline{g_1'})(c^-)-c_{11}(p_2\overline{g_2'})(c^+)+c_{21}\overline{g_2}(c^+)=0,\\ \overline{g_1}(c^-)+c_{12}(p_2\overline{g_2'})(c^+)-c_{22}\overline{g_2}(c^+)=0.\end{gathered}$$

Then

$$\begin{pmatrix} g_2(c^+)\\ (p_2g_2')(c^+)\end{pmatrix}=C\begin{pmatrix} g_1(c^-)\\ (p_1g_1')(c^-)\end{pmatrix},$$

i.e. $g\in D(C_{\max})$ and $C_{\min}^*g=C_{\max}g$. Thus $C_{\min}^*\subset C_{\max}$. Hence $C_{\min}^*=C_{\max}$. From the facts that the adjoint of any densely defined operator is automatically closed and $C_{\min}^*=C_{\max}$, it follows that $C_{\max}$ is a closed operator in H.

Since $D(C_{\min})$ and $D(C_{\max})(=D(C_{\min}^*))$ are dense in H, we have $C_{\min}\subset C_{\min}^{**}=C_{\max}^*$. In the following we prove that $C_{\max}^*\subset C_{\min}$.

Let $g = \{g_1, g_2\} \in D(C^*_{\max})$. Then for all $f \in D(C_{\max})$,

$$(C_{\max} f, g) = (f, C^*_{\max} g).$$

From (2.3), one obtains that $C^*_{\max} \subset C^*_{\min} = C_{\max}$, $g \in D(C_{\max})$ and then

$$(C_{\max} f, g) = (f, C_{\max} g).$$

From (2.2), one has

$$[f_2, g_2](b) - [f_1, g_1](a) = 0, \quad \text{for all } f \in D(C_{\max}),$$

i.e.

$$\begin{aligned} &f_2(b)(p_2\overline{g_2'})(b) - \overline{g_2}(b)(p_2 f_2')(b) \\ &= f_1(a)(p_1\overline{g_1'})(a) - \overline{g_1}(a)(p_1 f_1')(a), \quad \text{for all } f = \{f_1, f_2\} \in D(C_{\max}). \end{aligned} \tag{2.6}$$

In particular, using Patching Lemma 2.7, one can select $f \in D(C_{\max})$ which satisfies $f_1(a) = (p_1 f_1')(a) = 0$, $f_2(b) = 1$, $(p_2 f_2')(b) = 0$. Then from (2.6), it follows that $(p_2 g_2')(b) = 0$. In the same way, one has $g_2(b) = g_1(a) = (p_1 g_1')(a) = 0$. Therefore $g \in D(C_{\min})$ and $C^*_{\max} g = C_{\max} g = C_{\min} g$. Hence $C^*_{\max} \subset C_{\min}$, then $C^*_{\max} = C_{\min}$ and $C_{\min}$ is closed in H. □

Corollary 2.9. *$D(C_{\min})$ can be characterized as*

$$D(C_{\min}) = \{g \in D(C_{\max}) : [f, g] = 0 \textit{ for all } f \in D(C_{\max})\}.$$

Proof. If $g \in D(C_{\min})$, then from (2.2) it is clear that for all $f \in D(C_{\max})$,

$$[f, g] = [f_2, g_2](b) - [f_1, g_1](a) = 0.$$

On the other hand, if $g \in D(C_{\max})$ and, for all $f \in D(C_{\max})$, $[f, g] = 0$, i.e.

$$[f_2, g_2](b) - [f_1, g_1](a) = 0,$$

then by the last part proof of Theorem 2.8, it follows that $g \in D(C_{\min})$. □

Remark 2.10. The operators $C_{\min}$ and $C_{\max}$ defined above are our 'new' minimal and maximal operators, they play the roles of $S_{\min}$ and $S_{\max}$ in the 'standard' GKN theory as developed in the classic book of Naimark [24]. Our characterization of self-adjoint realizations of Sturm-Liouville problems with interior conditions is based on the operators $C_{\min}$ and $C_{\max}$ rather than $S_{\min}$ and $S_{\max}$. The key difference between ($S_{\min}$, $S_{\max}$) and ($C_{\min}, C_{\max}$) is that $S_{\min}$ and $S_{\max}$ depend only on the coefficients $1/p, q, w$ whereas $C_{\min}$ and $C_{\max}$ depend on these coefficients and on the interior discontinuous boundary conditions. Thus the study of the multi-point boundary conditions is reduced to the study of two point boundary conditions, the two points being the two 'outer' endpoints of the underlying interval.

Next we make some further observations. If S is a symmetric extension of $C_{\min}$, then we have

$$S_{\min} \subset C_{\min} \subset S \subset S^* \subset C_{\max} \subset S_{\max}.$$

Thus S is a self-adjoint extensions of the minimal operator $C_{\min}$ and of the 'standard' 2-interval minimal operator $S_{\min}$.

Each such operator S satisfies

$$S_{\min} \subset C_{\min} \subset S = S^* \subset C_{\max} \subset S_{\max}.$$

and is an extension of the 'new' minimal operator $C_{\min}$ or, equivalently, a restriction of the 'new' maximal operator $C_{\max}$. The next theorem characterizes all such operators S.

Theorem 2.11. *A linear manifold D of H is the domain of a self-adjoint extension of $C_{\min}$ if and only if*

1. $D(C_{\min}) \subset D \subset D(C_{\max})$*;*
2. *For any* $f, g \in D$*,* $[f, g] = 0$*;*
3. *If* $f \in D(C_{\max})$ *and* $[f, g] = 0$ *for any* $g \in D$*, then* $f \in D$*.*

Proof. Necessity. Let S be a self-adjoint extension of $C_{\min}$. Let $D(S) = D$ be the domain of S. Obviously $C_{\min} \subset S = S^* \subset C^*_{\min} = C_{\max}$, i.e.

$$D(C_{\min}) \subset D(S) \subset D(C_{\max}).$$

For any $f, g \in D(S)$, since S is a restriction of the 'new' maximal operator $C_{\max}$ and S is self-adjoint and hence symmetric, combing (2.1), it follows that

$$[f, g] = (Sf, g) - (f, Sg) = 0.$$

Let $f \in D(C_{\max})$. If $g \in D(S) \subset D(C_{\max})$, from (2.1), one can obtain

$$[f, g] = (C_{\max} f, g) - (f, C_{\max} g) = (C_{\max} f, g) - (f, Sg).$$

Since for any $g \in D(S)$, $[f, g] = 0$, i.e.

$$(C_{\max} f, g) - (f, Sg) = 0, \quad \text{for all } g \in D(S),$$

Therefore $f \in D(S^*) = D(S)$.

Sufficiency. Let the linear manifold D satisfy conditions (1), (2) and (3) of Theorem 2.11. Since $D(C_{\min})$ is dense in H then D is also dense in H. We define the operator S: $D(S) = D \to H$ and $Sf = C_{\max} f$ ($f \in D(S)$).

For any $f, g \in D(S)$,

$$0 = [f, g] = (Sf, g) - (f, Sg).$$

Therefore $S \subset S^*$.

Assume that $f \in D(C_{\max})$ and for any $g \in D(S)$, $[f, g] = 0$, i.e.

$$[f, g] = (C_{\max} f, g) - (f, Sg) = 0,$$

which shows that $f \in D(S^*)$. From (3), we know $f \in D(S)$. Thus $S^* \subset S$ and then $S = S^*$, i.e. S is a self-adjoint operator in H. □

Next we characterize all self-adjoint extensions of $C_{\min}$ in H or, equivalently, all self-adjoint restrictions of $C_{\max}$ in H. These extensions (or restrictions) differ only by their domains. These domains are characterized by boundary conditions. How many ? And what are they? These two questions are answered below. The number of independent boundary conditions depends on the deficiency index which we study next.

The deficiency subspaces $\{N_\lambda : \lambda \in \mathbb{C}\}$ of the closed symmetric operator $C_{\min}$ are defined by

$$N_\lambda = \{f \in D(C_{\max}) : C_{\max} f = \lambda f\},$$

where $\lambda \in \mathbb{C}$, $Im\lambda \neq 0$, and recall that $C^*_{\min} = C_{\max}$. Similar to [24], for any $\lambda \in \mathbb{C}$ with $Im\lambda \neq 0$, the deficiency indices of $C_{\min}$ are defined by

$$d^+ = \dim N_\lambda, \quad d^- = \dim N_{\overline{\lambda}},$$

and d^+, d^- are independent of λ. Since the differential expression is real, it follows that $d^+ = d^- = d$.

It follows from the classical Von Neumann formula that, for any fixed $\lambda \in \mathbb{C}$ with $Im\lambda \neq 0$,

$$D(C_{\max}) = D(C_{\min}) + N_\lambda + N_{\overline{\lambda}},$$

where the linear manifolds $D(C_{\min})$, N_λ and $N_{\overline{\lambda}}$ are linearly independent.

From the general theory [24], we obtain that an operator S is a self-adjoint extension of $C_{\min}$ if and only if its domain

$$D(S) = \{y \in D(C_{\max}) : y = y_0 + \phi + V\phi \text{ for all } y_0 \in D(C_{\min}) \text{ and for all } \phi \in N_\lambda\}, \tag{2.7}$$

where V is any unitary map with the property that

$$V:\ N_\lambda \to N_{\overline{\lambda}}, \quad V^* = V^{-1} : N_{\overline{\lambda}} \to N_\lambda,$$

and $Sf = C_{\max}f$, $f \in D(S)$.

Let $\{\phi_1, \dots, \phi_d\}$ be an orthonormal basis for N_λ in H, and then $\{V\phi_1, \dots, V\phi_d\}$ is an orthonormal basis for $N_{\overline{\lambda}}$ in H (see [30, 9]).

From what has been stated above, we present the following results.

Theorem 2.12. *Let the operator S be a self-adjoint extension of $C_{\min}$. Then the domain of S can be described as follows:*

$$D(S) = \{y \in D(C_{\max}) : y = y_0 + \sum_{r=1}^{d} \alpha_r \psi_r\}, \tag{2.8}$$

where $y_0 \in D(C_{\min})$, $\alpha_r \in \mathbb{C}$ and $\psi_r = \phi_r + V\phi_r$ $(r = 1, \dots, d)$.

Proof. We just need to prove that the two domains (2.7) and (2.8) are identical. Let $\phi \in N_\lambda$ and $\{\phi_1, \dots, \phi_d\}$ be an orthonormal basis for N_λ, then there exist $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ such that $\phi = \alpha_1\phi_1 + \dots + \alpha_d\phi_d$. Therefore $V\phi = \alpha_1 V\phi_1 + \dots + \alpha_d V\phi_d$ and

$$\phi + V\phi = \alpha_1(\phi_1 + V\phi_1) + \dots + \alpha_d(\phi_d + V\phi_d) = \sum_{r=1}^{d} \alpha_r\psi_r.$$

Conversely, it follows from $\sum_{r=1}^{d} \alpha_r\psi_r = \sum_{r=1}^{d} \alpha_r(\phi_r + V\phi_r)$ that $\sum_{r=1}^{d} \alpha_r\phi_r = \phi \in N_\lambda$ and $\sum_{r=1}^{d} \alpha_r V\phi_r = V\phi \in N_{\overline{\lambda}}$. Therefore $\sum_{r=1}^{d} \alpha_r\psi_r = \phi + V\phi$. $\square$

Theorem 2.13. *Let S be a self-adjoint extension of $C_{\min}$ with domain*

$$D(S) = \{y \in D(C_{\max}) : y = y_0 + \sum_{r=1}^{d} \alpha_r\psi_r, \quad \alpha_r \in \mathbb{C}\}.$$

Then $D(S)$ is given by

$$\{y \in D(C_{\max}) : [y, \psi_r] = 0,\ r = 1, \dots, d\}.$$

Proof. Let $D = \{y \in D(C_{\max}) : [y, \psi_r] = 0, \quad r = 1, \dots, d\}$. It is easy to see that $\psi_1, \dots, \psi_d \in D(S)$. For $y \in D(S)$, it follows from (2.1) that

$$[y, \psi_r] = (Sy, \psi_r) - (y, S\psi_r) = 0, \quad r = 1, 2, \dots, d.$$

Therefore $y \in D$, and then $D(S) \subset D$.

On the other hand, let $y \in D \subset D(C_{\max})$ and $g \in D(S)$ then there exist $g_0 \in D(C_{\min})$, $\alpha_1, \dots, \alpha_d \in \mathbb{C}$ such that $g = g_0 + \alpha_1\psi_1 + \dots + \alpha_d\psi_d$. Combining with Corollary 2.9, we deduce that

$$[y, g] = [y, g_0] + [y, \alpha_1\psi_1 + \dots + \alpha_d\psi_d] = 0.$$

Hence for $y \in D$ and any $g \in D(S)$, it follows that

$$0 = [y, g] = (C_{\max} y, g) - (y, C_{\max} g) = (C_{\max} y, g) - (y, Sg).$$

Therefore $y \in D(S^*) = D(S)$. So $D \subset D(S)$ and then $D(S) = D$. $\square$

Theorem 2.14 (New GKN-TYPE Theorem). *Let d denote the deficiency index of $C_{\min}$. A linear submanifold $D(S)$ of $D(C_{\max})$ is the domain of a self-adjoint extension S of $C_{\min}$ if and only if there exist functions $v_1 = \{v_{11}, v_{12}\}, \dots, v_d = \{v_{d1}, v_{d2}\} \in D(C_{\max})$ satisfying the following conditions:*

(1) *$v_1, \dots, v_d$ are linearly independent modulo $D(C_{\min})$;*
(2) *$[v_i, v_j] = 0$, $i, j = 1, \dots, d$;*
(3) *$D(S) = \{y \in D(C_{\max}) : [y, v_i] = 0, i = 1, \dots, d\}$.*

Proof. Necessity. Using Theorems 2.12 and 2.13, we set $v_1 = \psi_1, \dots, v_d = \psi_d$, then $v_1, \dots, v_d$ satisfy the conditions (1) and (2), and the self-adjoint domain can be denoted by (3).

Sufficiency. Assume there exist functions $v_1, \dots, v_d \in D(C_{\max})$ satisfying the conditions (1), (2) and (3). Now we prove that $D(S)$ is a self-adjoint domain.

Conditions $[y, v_i] = 0$ $(i = 1, \dots, d)$ are linearly independent. If not, there exist constants $c_1, \dots, c_d$, not all zero, such that for all $y \in D(C_{\max})$,

$$c_1[y, v_1] + \dots + c_d[y, v_d] = 0,$$

i.e. $[y, \bar{c}_1 v_1 + \dots + \bar{c}_d v_d] = 0$. It follows from Corollary 2.9 that $\bar{c}_1 v_1 + \dots + \bar{c}_d v_d \in D(C_{\min})$. This contradicts the linear independence of $v_1, \dots, v_d$ modulo $D(C_{\min})$.

Let

$$\widehat{D} = \{y : \ y = y_0 + c_1 v_1 + \dots + c_d v_d\},$$

where $y_0 \in D(C_{\min})$ and $c_1, \dots, c_d$ are any complex constants. From condition (2) and Corollary 2.9, it follows that $\widehat{D} \subset D(S)$. Since $D(S)$ is obtained from $D(C_{\max})$ by imposing d linearly independent conditions, one can deduce that $\dim\big(D(S)/D(C_{\min})\big) = 2d - d = d$. Moreover, $\dim\big(\widehat{D}/D(C_{\min})\big) = d$. Thus $\widehat{D} = D(S)$.

Note that $D(C_{\min}) \subset \widehat{D} \subset D(C_{\max})$. Since $v_1, \dots, v_d$ satisfy condition (2), we obtain

$$[f, g] = 0, \ \text{ for any } f, g \in \widehat{D}.$$

If $f \in D(C_{\max})$ and for any $g \in \widehat{D}$, $[f, g] = 0$, then for $g = v_i (i = 1, \dots, d)$, we have $[f, v_i] = 0, i = 1, \dots, d$. Hence $f \in D(S) = \widehat{D}$. It follows from Theorem 2.11 that $\widehat{D}(= D(S))$ is a self-adjoint domain. $\square$

Note that

$$\dim\big(D(S_{\max})/D(S_{\min})\big) = 2d_0 = 8,$$

where d_0 is the deficiency index of the two-interval minimal operator $S_{\min}$,

$$\dim\big(D(S_{\max})/D(C_{\max})\big) = 2, \quad \dim\big(D(C_{\min})/D(S_{\min})\big) = 2.$$

Therefore,

$$\dim\big(D(C_{\max})/D(C_{\min})\big) = 2d_0 - 4 = d^+ + d^- = 2d$$

and then $d = 2$.

Theorem 2.15. *An operator S in H satisfies* (1.7) *if and only if its domain $D = D(S)$ is given as*
$$D(S) = \{y = \{y_1, y_2\} \in D(C_{\max}) : AY(a) + BY(b) = 0\}, \tag{2.9}$$
where matrices A, B satisfy (1.6) *i.e. $A, B \in M_2(\mathbb{C})$, $\operatorname{rank}(A : B) = 2$ and $AEA^* = BEB^*$.*

Proof. The deficiency index of $C_{\min}$ is $d = 2$.

Necessity. Let $D(S)$ be the domain of a self-adjoint extension S of $C_{\min}$. By Theorem 2.14, there exist functions $w_1 = \{w_{11}, w_{12}\}$, $w_2 = \{w_{21}, w_{22}\} \in D(C_{\max})$ satisfying conditions (1),(2) and (3) of Theorem 2.14. For any $y = \{y_1, y_2\} \in D(C_{\max})$ satisfying condition (3), we have
$$0 = \begin{pmatrix} [y, w_1] \\ [y, w_2] \end{pmatrix} = \begin{pmatrix} [y_2, w_{12}](b) - [y_1, w_{11}](a) \\ [y_2, w_{22}](b) - [y_1, w_{21}](a) \end{pmatrix},$$
i.e.
$$\begin{pmatrix} [y_1, w_{11}](a) \\ [y_1, w_{21}](a) \end{pmatrix} = \begin{pmatrix} [y_2, w_{12}](b) \\ [y_2, w_{22}](b) \end{pmatrix}.$$
Therefore
$$\begin{pmatrix} \overline{w}_{11}(a) & \overline{w}_{11}^{[1]}(a) \\ \overline{w}_{21}(a) & \overline{w}_{21}^{[1]}(a) \end{pmatrix} EY(a) - \begin{pmatrix} \overline{w}_{12}(b) & \overline{w}_{12}^{[1]}(b) \\ \overline{w}_{22}(b) & \overline{w}_{22}^{[1]}(b) \end{pmatrix} EY(b) = 0.$$
Set
$$A = \begin{pmatrix} \overline{w}_{11}(a) & \overline{w}_{11}^{[1]}(a) \\ \overline{w}_{21}(a) & \overline{w}_{21}^{[1]}(a) \end{pmatrix} E, \quad B = -\begin{pmatrix} \overline{w}_{12}(b) & \overline{w}_{12}^{[1]}(b) \\ \overline{w}_{22}(b) & \overline{w}_{22}^{[1]}(b) \end{pmatrix} E.$$
Hence boundary conditions (3) of Theorem 2.14 is equivalent to $AY(a)+BY(b) = 0$. Compute
$$\begin{aligned} AEA^* &= \begin{pmatrix} \overline{w}_{11}(a) & \overline{w}_{11}^{[1]}(a) \\ \overline{w}_{21}(a) & \overline{w}_{21}^{[1]}(a) \end{pmatrix} E \begin{pmatrix} w_{11}(a) & w_{21}(a) \\ w_{11}^{[1]}(a) & w_{21}^{[1]}(a) \end{pmatrix}, \\ BEB^* &= \begin{pmatrix} \overline{w}_{12}(b) & \overline{w}_{12}^{[1]}(b) \\ \overline{w}_{22}(b) & \overline{w}_{22}^{[1]}(b) \end{pmatrix} E \begin{pmatrix} w_{12}(b) & w_{22}(b) \\ w_{12}^{[1]}(b) & w_{22}^{[1]}(b) \end{pmatrix}. \end{aligned}$$
From
$$\begin{aligned} 0 &= \begin{pmatrix} [w_1, w_1] & [w_2, w_1] \\ [w_1, w_2] & [w_2, w_2] \end{pmatrix} \\ &= \begin{pmatrix} [w_{12}, w_{12}](b) - [w_{11}, w_{11}](a) & [w_{22}, w_{12}](b) - [w_{21}, w_{11}](a) \\ [w_{12}, w_{22}](b) - [w_{11}, w_{21}](a) & [w_{22}, w_{22}](b) - [w_{21}, w_{21}](a) \end{pmatrix} \\ &= BEB^* - AEA^*, \end{aligned}$$
it follows that $AEA^* = BEB^*$.

It is obvious that $\operatorname{rank}(A : B) \le 2$. If $\operatorname{rank}(A : B) < 2$, then there exist constants c and d, not all zero, such that $\begin{pmatrix} c & d \end{pmatrix} (A : B) = 0$. Therefore
$$\begin{pmatrix} c & d \end{pmatrix} A = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} \overline{w}_{11}(a) & \overline{w}_{11}^{[1]}(a) \\ \overline{w}_{21}(a) & \overline{w}_{21}^{[1]}(a) \end{pmatrix} E = 0,$$
i.e.
$$c\overline{w}_{11}^{[1]}(a) + d\overline{w}_{21}^{[1]}(a) = 0, \quad c\overline{w}_{11}(a) + d\overline{w}_{21}(a) = 0. \tag{2.10}$$

Similarly,

$$\begin{pmatrix} c & d \end{pmatrix} B = \begin{pmatrix} c & d \end{pmatrix} \begin{pmatrix} \overline{w}_{12}(b) & \overline{w}_{12}^{[1]}(b) \\ \overline{w}_{22}(b) & \overline{w}_{22}^{[1]}(b) \end{pmatrix} (-E) = 0,$$

i.e.

$$c\overline{w}_{12}^{[1]}(b) + d\overline{w}_{22}^{[1]}(b) = 0, \quad c\overline{w}_{12}(b) + d\overline{w}_{22}(b) = 0. \tag{2.11}$$

Let $g = \{g_1, g_2\} = \bar{c}w_1 + \bar{d}w_2 \in D(C_{\max})$. Therefore for any $f = \{f_1, f_2\} \in D(C_{\max})$, from (2.10) and (2.11), one can obtain that

$$\begin{aligned} [f, g] &= [f_2, g_2](b) - [f_1, g_1](a) \\ &= [f_2, \bar{c}w_{12} + \bar{d}w_{22}](b) - [f_1, \bar{c}w_{11} + \bar{d}w_{21}](a) = 0. \end{aligned}$$

It follows from Corollary 2.9 that $g \in D(C_{\min})$. This contradicts the fact that w_1, w_2 are linearly independent modulo $D(C_{\min})$. Thus $\operatorname{rank}(A : B) = 2$.

Sufficiency. If there exist complex 2×2 matrices A and B satisfy $\operatorname{rank}(A : B) = 2$, $AEA^* = BEB^*$ and (2.9). We just need to prove that $D(S)$ defined by (2.9) is a self-adjoint domain.

Let $A = (a_{ij})_{2\times 2}$ and $B = (b_{ij})_{2\times 2}$. From Lemma 2.7, there exist functions $w_1 = \{w_{11}, w_{12}\}$, $w_2 = \{w_{21}, w_{22}\} \in D(C_{\max})$ such that

$$\begin{gathered} w_{11}(a) = -\bar{a}_{12}, \quad w_{11}^{[1]}(a) = \bar{a}_{11}, \quad w_{12}(b) = \bar{b}_{12}, \quad w_{12}^{[1]}(b) = -\bar{b}_{11}, \\ w_{21}(a) = -\bar{a}_{22}, \quad w_{21}^{[1]}(a) = \bar{a}_{21}, \quad w_{22}(b) = \bar{b}_{22}, \quad w_{22}^{[1]}(b) = -\bar{b}_{21}. \end{gathered}$$

For $y = \{y_1, y_2\} \in D(C_{\max})$, we have

$$\begin{aligned} \begin{pmatrix} [y, w_1] \\ [y, w_2] \end{pmatrix} &= \begin{pmatrix} [y_2, w_{12}](b) \\ [y_2, w_{22}](b) \end{pmatrix} - \begin{pmatrix} [y_1, w_{11}](a) \\ [y_1, w_{21}](a) \end{pmatrix} \\ &= \begin{pmatrix} \overline{w}_{12}(b) & \overline{w}_{12}^{[1]}(b) \\ \overline{w}_{22}(b) & \overline{w}_{22}^{[1]}(b) \end{pmatrix} E \begin{pmatrix} y_2(b) \\ y_2^{[1]}(b) \end{pmatrix} \\ &\quad - \begin{pmatrix} \overline{w}_{11}(a) & \overline{w}_{11}^{[1]}(a) \\ \overline{w}_{21}(a) & \overline{w}_{21}^{[1]}(a) \end{pmatrix} E \begin{pmatrix} y_1(a) \\ y_1^{[1]}(a) \end{pmatrix} \\ &= -BY(b) - AY(a) \end{aligned}$$

Hence the boundary conditions $AY(a) + BY(b) = 0$ are equivalent to $[y, w_i] = 0$, $i = 1, 2$.

Now we prove $[w_i, w_j] = 0$, $i, j = 1, 2$. Compute

$$\begin{aligned} &\begin{pmatrix} [w_1, w_1] & [w_2, w_1] \\ [w_1, w_2] & [w_2, w_2] \end{pmatrix} \\ &= \begin{pmatrix} [w_{12}, w_{12}](b) & [w_{22}, w_{12}](b) \\ [w_{12}, w_{22}](b) & [w_{22}, w_{22}](b) \end{pmatrix} - \begin{pmatrix} [w_{11}, w_{11}](a) & [w_{21}, w_{11}](a) \\ [w_{11}, w_{21}](a) & [w_{21}, w_{21}](a) \end{pmatrix} \\ &= \begin{pmatrix} -\bar{b}_{12}b_{11} + \bar{b}_{11}b_{12} & -\bar{b}_{22}b_{11} + \bar{b}_{21}b_{12} \\ -\bar{b}_{12}b_{21} + \bar{b}_{11}b_{22} & -\bar{b}_{22}b_{21} + \bar{b}_{21}b_{22} \end{pmatrix} \\ &\quad - \begin{pmatrix} -\bar{a}_{12}a_{11} + \bar{a}_{11}a_{12} & -\bar{a}_{22}a_{11} + \bar{a}_{21}a_{12} \\ -\bar{a}_{12}a_{21} + \bar{a}_{11}a_{22} & -\bar{a}_{22}a_{21} + \bar{a}_{21}a_{22} \end{pmatrix} \\ &= BEB^* - AEA^*. \end{aligned}$$

Hence, it follows from $AEA^* = BEB^*$ that $[w_i, w_j] = 0$, $i, j = 1, 2$.

Next we prove that w_1, w_2 are linearly independent modulo $D(C_{\min})$. If not, there exist constants c and d, not all zero, such that $cw_1 + dw_2 \in D(C_{\min})$.

By the Patching Lemma 2.7, we may construct $f = \{f_1, f_2\}$, $g = \{g_1, g_2\} \in D(C_{\max})$ such that

$$f_1(a) = 0, \quad f_1^{[1]}(a) = -1, \quad f_2(b) = 0, \quad f_2^{[1]}(b) = 1,$$
$$g_1(a) = 1, \quad g_1^{[1]}(a) = 0, \quad g_2(b) = -1, \quad g_2^{[1]}(b) = 0.$$

Therefore

$$[cw_1 + dw_2, f] = 0, \quad [cw_1 + dw_2, g] = 0,$$

i.e.

$$[cw_{11} + dw_{21}, f_1](a) = 0, \quad [cw_{11} + dw_{21}, g_1](a) = 0,$$
$$[cw_{12} + dw_{22}, f_2](b) = 0, \quad [cw_{12} + dw_{22}, g_2](b) = 0.$$

It is seen from simple computation that

$$\begin{pmatrix} \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} a_{12} & -a_{11} & b_{12} & -b_{11} \\ a_{22} & -a_{21} & b_{22} & -b_{21} \end{pmatrix} = 0.$$

Namely

$$\begin{pmatrix} \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} AE : BE \end{pmatrix} = \begin{pmatrix} \overline{c} & \overline{d} \end{pmatrix} \begin{pmatrix} A : B \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix} = 0.$$

Since c and d are not both zero and E is nonsingular, we have $\operatorname{rank}(A : B) < 2$. This contradicts the fact that $\operatorname{rank}(A : B) = 2$. Therefore w_1, w_2 are linearly independent modulo $D(C_{\min})$. From the New GKN-TYPE Theorem 2.14, it follows that $D(S)$ defined by (2.9) is the domain of a self-adjoint extension of $C_{\min}$. $\square$

3. Transcendental characterization of the eigenvalues for self-adjoint discontinuous boundary conditions

In this section we extend the well known characterization of the eigenvalues of boundary value problems consisting of equation (1.1) with boundary condition (1.4) to problems with boundary conditions (1.4) and (1.5). This characterization will be used below to extend the very general Eastham, Kong, Wu, Zettl [7] inequalities for boundary conditions (1.4) to boundary conditions (1.4) and (1.5) for fixed C.

Consider the equation

$$My = -(py')' + qy = \lambda wy \quad \text{on } J = [a, b],\ \lambda \in \mathbb{C},\ -\infty < a < b < \infty \tag{3.1}$$

with coefficients satisfying

$$p^{-1}, q, w \in L(J, \mathbb{R}), \quad p > 0, \quad w > 0 \quad \text{a.e. on } J, \tag{3.2}$$

and boundary conditions

$$AY(a) + BY(b) = 0, \tag{3.3}$$
$$Y(c^+) = CY(c^-), \quad a < c < b, \tag{3.4}$$

and the matrices A, B, C satisfy

$$AEA^* = BEB^*, \quad \operatorname{rank}(A : B) = 2,\ \det(C) = 1, \tag{3.5}$$

where

$$A, B \in M_2(\mathbb{C}), \quad C \in M_2(\mathbb{R}), \quad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{3.6}$$

Although the next result follows from the standard linear ODE theory we state it as a theorem here since it plays a major role below.

Theorem 3.1. *Let* (3.1) *to* (3.4) *hold and let* $\lambda \in \mathbb{C}$. *Every initial condition at* a *determines a unique solution on* $[a, b]$ *which satisfies the jump condition* (3.4) *and there are exactly two such linearly independent solutions of equation* (3.1) *for every* $\lambda \in \mathbb{C}$.

Proof. See Remark 1.2. The proof that there are exactly two such linearly independent solutions is similar to the proof in the general linear ode theory for the case when $C = I$ and hence omitted. □

Definition 3.2. *A solution on* $[a, b]$ *satisfying* (3.4) *is called a* C *jump solution or just a jump solution when* C *remains fixed. A complex number* λ *is an eigenvalue of problem* (3.1) *to* (3.6) *if there exists a nontrivial* C *jump solution* y *on* $[a, b]$ *which satisfies both boundary conditions* (3.3) *and* (3.4).

As mentioned in Section 1, condition (3.2) implies that all solutions are continuous on $[a, b]$. So if $C \neq I$, the identity matrix, how can we get an eigenfunction satisfying both conditions (3.3) and (3.4))? The next theorem answers this question.

Notation. Below, for a fixed boundary condition (3.4), we extend solutions y from $[a, c]$ to $[c, b]$ as in Remark 1.2, and continue to use the same notation y for the extended solution. Thus if y is an eigenfunction satisfying (3.3) then it is such an extended solution.

Let

$$P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}. \tag{3.7}$$

Then the scalar equation (3.1) is equivalent to the first order system

$$Y' = (P - \lambda W)Y = \begin{pmatrix} 0 & 1/p \\ q - \lambda w & 0 \end{pmatrix} Y, \quad Y = \begin{pmatrix} y \\ (py') \end{pmatrix}. \tag{3.8}$$

For fixed boundary condition (3.4) let u, v be the extended solutions of (3.1) on $[a, b]$ determined by the initial conditions:

$$u(a) = 1 = v^{[1]}(a), \ v(a) = 0 = u^{[1]}(a)$$

Let

$$\Phi = \begin{pmatrix} u & v \\ u^{[1]} & v^{[1]} \end{pmatrix}.$$

Then

$$\Phi' = (P - \lambda W)\,\Phi \quad \text{on } J, \quad \Phi(a, \lambda) = I, \ \lambda \in \mathbb{C}.$$

Define the characteristic function δ by

$$\delta(\lambda) = \det[A + B\,\Phi(b, a, \lambda)], \quad \lambda \in \mathbb{C}. \tag{3.9}$$

This function δ is a transcendental function whose zeros characterize the eigenvalues as we will see below.

Lemma 3.3. *The characteristic function* δ *is well defined and is an entire function of* λ *for fixed* (a, b, A, B, C, P, W).

The proof of the above lemma is similar to the case when $C = I$, see [33, Chapter 2].

Lemma 3.4. *For fixed boundary condition* (3.4) *and* $\delta(\lambda)$ *defined as above in* (3.9) *we have:*

1. *A complex number* λ *is an eigenvalue of the boundary value problem* (3.1) *to* (3.6) *if and only if* $\delta(\lambda) = 0$.
2. *The geometric multiplicity of an eigenvalue* λ *is equal to the number of linearly independent vector solutions* $C = Y(a)$ *of the linear algebra system*

$$[A + B\Phi(b, a, \lambda)]C = 0. \tag{3.10}$$

Proof. Suppose $\delta(\lambda) = 0$. Then (3.10) has a nontrivial vector solution for C. Solve the IVP

$$Y' = (P - \lambda W)Y \quad \text{on } J,\ Y(a) = C.$$

Then

$$Y(b) = \Phi(b, a, \lambda)\, Y(a) \quad \text{and} \quad [A + B\Phi(b, a, \lambda)]Y(a) = 0.$$

From this it follows that the top component of Y, say, y is an eigenfunction of (3.1) to (3.6) and λ is an eigenvalue of this BVP. (Recall that the eigenfunctions are extended solutions on $[a, b]$.

Conversely, if λ is an eigenvalue and y an eigenvector of λ, then $Y = \begin{pmatrix} y \\ py' \end{pmatrix}$ satisfies $Y(b) = \Phi(b, a, \lambda)Y(a)$ and consequently $[A + B\Phi(b, a, \lambda)]Y(a) = 0$. Since $Y(a) = 0$ would imply that y is the trivial solution in contradiction to it being an eigenfunction, we have that $\det[A + B\Phi(b, a, \lambda)] = 0$. If (3.10) has two linearly independent solutions for C, say C_1, C_2, then solve the IVP with the initial conditions $Y(a) = C_1$, $Y(a) = C_2$ to obtain solutions Y_1, Y_2. Then Y_1, Y_2 are linearly independent vector solutions of (3.8) and their top components y_1, y_2 are linearly independent solutions of (3.1). Conversely, if y_1, y_2 are linearly dependent solutions of (3.1) we can reverse the steps above to obtain two linearly independent vector solutions of the algebraic system (3.10). □

It is convenient to classify the boundary conditions (BC) (3.3), (3.5) into two mutually exclusive classes: separated and coupled. Note that, since the BC are homogeneous, multiplication on the left by a nonzero constant or a nonsingular matrix leads to equivalent boundary conditions.

Lemma 3.5 (Separated boundary conditions)**.** *Assume*

$$A = \begin{pmatrix} A_1 & A_2 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ B_1 & B_2 \end{pmatrix}.$$

Then for $\lambda \in \mathbb{C}$,

$$\delta(\lambda) = -A_2\, B_1\phi_{11}(b, a, \lambda) - A_2B_2\phi_{21}(b, a, \lambda) + A_1B_1\phi_{12}(b, a, \lambda) + A_1B_2\phi_{22}(b, a, \lambda).$$

The proof of the above lemma follows from the definition of δ and a direct computation. The characterization of the eigenvalues as zeros of an entire function given by Lemma 3.4 reduces to a simpler and more informative form when the boundary conditions are self-adjoint and coupled. This reduction is given by the next lemma.

Theorem 3.6. *Let* (3.1) *to* (3.8) *hold and fix* (3.4) *and* P, W, J. *Define* $\Phi = (\phi_{ij})$ *as above and suppose that*

$$B = -I, \quad A = e^{i\gamma}\, K, \quad 0 \le \gamma \le \pi, \quad K \in M_2(\mathbb{R}), \quad \det K = 1. \tag{3.11}$$

Let $K = (k_{ij})$ and define

$$D(\lambda, K) = k_{11}\,\phi_{22}(b,a,\lambda) - k_{12}\,\phi_{21}(b,a,\lambda) - k_{21}\,\phi_{12}(b,a,\lambda) + k_{22}\,\phi_{11}(b,a,\lambda), \tag{3.12}$$

for $\lambda \in \mathbb{C}$. Then

(1) The complex number λ is an eigenvalue of BVP (3.1) to (3.6) if and only if
$$D(\lambda, K) = 2\cos\gamma, \quad 0 \le \gamma \le \pi. \tag{3.13}$$
(2) If λ is an eigenvalue for $A = e^{i\gamma}K$, $B = -I$, $0 < \gamma < \pi$, with eigenfunction u, then λ is also an eigenvalue for $A = e^{-i\gamma}K$, $B = -I$, but with eigenfunction $\overline{u}$.

Proof of Theorem 3.6. From the basic theory of linear ordinary differential equations, see [33], we have $\det \Phi(b,a,\lambda) = 1$. We abbreviate $(\phi_{ij}(b,a,\lambda))$ to ϕ_{ij} and $D(\lambda, K)$ to D for simplicity of exposition. By (3.9) and (3.11) and recalling that $\det K = 1$ we get

$$\begin{aligned} \delta(\lambda) &= \det(e^{i\gamma}\,K - \Phi) = \begin{pmatrix} e^{i\gamma}k_{11} - \phi_{11} & e^{i\gamma}k_{12} - \phi_{12} \\ e^{i\gamma}k_{21} - \phi_{21} & e^{i\gamma}k_{22} - \phi_{22} \end{pmatrix} \\ &= (e^{i\gamma}k_{11} - \phi_{11})(e^{i\gamma}k_{22} - \phi_{22}) - (e^{i\gamma}k_{12} - \phi_{12})(e^{i\gamma}k_{21} - \phi_{21}) \\ &= e^{2i\gamma}(k_{11}k_{22} - k_{12}k_{21}) - e^{i\gamma}D + \det\Phi. \end{aligned} \tag{3.14}$$

By Lemma 3.4, λ is an eigenvalue if and only if $\delta(\lambda) = 0$. Therefore λ is an eigenvalue if and only if

$$\begin{aligned} D(\lambda) &= (1 + e^{2i\gamma})/e^{i\gamma} = e^{-i\gamma} + e^{i\gamma} \\ &= \cos(-\gamma) + i\sin(-\lambda) + \cos(\gamma) + i\sin(\gamma) \\ &= 2\cos(\gamma). \end{aligned}$$

This proves part (1). Part (2) follows from (3.14) and by taking conjugates of equation (3.1). $\square$

Corollary 3.7. *Let the hypotheses and notation of Theorem 3.6 hold. If λ is any eigenvalue and $D(\lambda, K)$ is given by* (3.12) *then*

$$-2 \le D(\lambda, K) \le 2. \tag{3.15}$$

The above corollary follows directly from (3.13).

Corollary 3.8. *Let the hypotheses and notation of Theorem 3.6 and let I denote the identity matrix. Then*

(1) *A complex number λ is an eigenvalue of the periodic boundary condition*
$$Y(b) = Y(a)$$
if and only if $D(\lambda, I) = 2$.
(2) *A complex number λ is an eigenvalue of the semi-periodic boundary condition*
$$Y(b) = -Y(a)$$
if and only if $D(\lambda, -I) = -2$.
(3) *A complex number λ is an eigenvalue of the complex self-adjoint boundary condition*
$$Y(b) = e^{i\gamma}\,Y(a), \ 0 < \gamma < \pi$$
if and only if $D(\lambda, I) = 2\cos(\gamma)$.

The above corollary follows directly from (3.13). Next we comment on the remarkable characterization (3.13).

Remark 3.9. Note that in (3.13) $D(\lambda, K)$ on the left is defined for any on $K \in SL_2(\mathbb{R})$ and the right side depends only on $\gamma \in [0, \pi]$. Recall the canonical form of the coupled boundary conditions with A, B given by (3.11). When $\gamma = 0$, $D(\lambda, K) = 2$ characterizes the eigenvalues when $A = K$; when $\gamma = \pi$, $D(\lambda, K) = -2$ characterizes the eigenvalues when $A = -K$; when $\gamma \in (0, \pi)$ we have the complex coupled boundary condition: $A = e^{i\gamma} K$. Thus the characterization $D(\lambda, K) = 2 \cos \gamma$ suggests a close relationship between the eigenvalues of the complex coupled condition with $A = e^{i\gamma} K$ and the eigenvalues of the two real coupled conditions with $A = K$ and $A = -K$. Below we explore this relationship in some detail for the special case when $K = I$, the identity matrix. Another project we plan to pursue is to study this relationship for other $K \in SL_2(\mathbb{R})$ using the special features of this well known special linear group of order 2 over the reals, i.e. $SL_2(\mathbb{R})$.

4. Canonical forms of self-adjoint boundary conditions

The boundary condition (1.4), (1.6) is homogeneous and thus clearly invariant under multiplication by a nonsingular matrix or nonzero constant. This is a serious obstacle to studying the dependence of the eigenvalues on this condition. The conditions (1.4), (1.6) can be divided into three mutually exclusive classes: separated, real coupled and complex coupled. We refer to all nonseparated conditions as coupled. These three classes are:

Separated self-adjoint BC. These are

$$\begin{aligned} A_1 y(a) + A_2 (py')(a) = 0, \quad & A_1, A_2 \in \mathbb{R},\ (A_1, A_2) \neq (0,0), \\ B_1 y(b) + B_2 (py')(b) = 0, \quad & B_1, B_2 \in \mathbb{R},\ (B_1, B_2) \neq (0,0). \end{aligned}$$

These separated conditions can be parameterized as follows:

$$\cos \alpha y(a) - \sin \alpha (py')(a) = 0, \quad 0 \leq \alpha < \pi, \tag{4.1}$$

$$\cos \beta y(b) - \sin \beta (py')(b) = 0, \quad 0 < \beta \leq \pi, \tag{4.2}$$

choose $\alpha \in [0, \pi)$ such that

$$\tan \alpha = \frac{-A_2}{A_1} \text{ if } A_1 \neq 0, \quad \text{and} \quad \alpha = \pi/2 \text{ if } A_1 = 0,$$

similarly, choose $\beta \in (0, \pi]$ such that

$$\tan \beta = \frac{-B_2}{B_1} \text{ if } B_1 \neq 0, \quad \text{and} \quad \beta = \pi/2 \text{ if } B_1 = 0.$$

Note the different normalization in (4.2) for β than that used for α in (4.1). This is for convenience in using the Prüfer transformation which is widely used for the theoretical studies of eigenvalues and their eigenfunction and for the numerical computation of these. For example the FORTRAN code SLEIGN2 [1, 5, 2, 3, 4] uses this normalization.

All real coupled self-adjoint BC. These can be formulated as follows:

$$Y(b) = K\,Y(a), \quad Y = \begin{pmatrix} y \\ (py') \end{pmatrix},$$

where $K \in SL_2(\mathbb{R})$, i.e. K satisfies

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}, \quad k_{ij} \in \mathbb{R}, \quad \det K = 1. \tag{4.3}$$

All complex coupled self-adjoint BC. These are:

$$Y(b) = e^{i\gamma} K Y(a),$$

where K satisfies (4.3) and $-\pi < \gamma < 0$, or $0 < \gamma < \pi$.

Lemma 4.1. *Given a boundary condition* (1.4), (1.6) *it is equivalent to exactly one of the separated, real coupled, or complex coupled boundary conditions defined above and each of these conditions can be written in the form* (1.4), (1.6).

For a proof of the above lemma, see [33].

Notation 4.2. For fixed coefficients p, q, w, fixed endpoints a, b and a fixed jump condition (1.5) we use the following notation for the eigenvalues of the boundary conditions (1.4), (1.6):

$$\lambda_n(\alpha,\beta), \quad \lambda_n(K), \quad \lambda_n(\gamma, K), \quad n \in \mathbb{N}_0. \tag{4.4}$$

Here and below $\mathbb{N}_0 = \{0, 1, 2, 3, \cdots\}$. Note that λ_n is uniquely defined although its eigenfunction may not be unique and this notation covers all self-adjoint boundary conditions (1.4), (1.6). Since each of these has a unique representation as a separated, real coupled, or complex coupled condition we can study how the eigenvalues change when this boundary condition changes. The existence of eigenvalues is discussed in the next section.

5. Existence of eigenvalues

Theorem 5.1. *Let* (1.1) *to* (1.6) *hold and let S satisfy* (1.7)*. Then the spectrum of S is real, discrete, bounded below and not bounded above. We have*

(1) *There are an infinite but countable number of eigenvalues with no finite accumulation point.*

(2) *The eigenvalues can be ordered to satisfy*

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \dots; \quad \lambda_n \to +\infty, \quad \text{as } n \to \infty. \tag{5.1}$$

Each eigenvalue may be simple or double but there cannot be two consecutive equalities in (5.1) *since, as pointed out in Theorem 3.1, for any value of λ, the equation* (1.1) *has exactly two linearly independent extended solutions. Note that λ_n is well defined for each $n \in \mathbb{N}_0$ but there is some arbitrariness in the indexing of the eigenfunctions corresponding to a double eigenvalue since every nontrivial extended solution of the equation for such an eigenvalue is an eigenfunction. Let $\sigma(S) = \{\lambda_n : n \in \mathbb{N}_0\}$ where the eigenvlaues are ordered to satisfy* (5.1).

(3) *If the boundary condition is separated then strict inequality holds everywhere in* (5.1)*. Furthermore, if u_n is an eigenfunction of λ_n, then u_n is unique up to constant multiples and has exactly n zeros in the open interval (a, b) for each $n \in \mathbb{N}_0$.*

(4) *Let S be determined by a real coupled boundary condition matrix K and u_n be a real-valued eigenfunction of $\lambda_n(K)$. Then the number of zeros of u_n in the open interval (a,b) is 0 or 1, if $n=0$, and $n-1$ or n or $n+1$ if $n \geq 1$.*

(5) *Let S be determined by a complex coupled boundary condition (K,γ) and let $\sigma(S)=\{\lambda_n : n \in \mathbb{N}_0\}$. Then all eigenvalues are simple and strict inequality holds everywhere in (5.1). Moreover, if u_n is an eigenfunction of λ_n then the number of zeros of $\operatorname{Re} u_n$ on $[a,b)$ is 0 or 1 if $n=0$, and $n-1$ or n or $n+1$ if $n \geq 1$. The same conclusion holds for $\operatorname{Im} u_n$. Moreover, u_n has no zero in $[a,b]$, $n \in \mathbb{N}_0$.*

See [33] for a proof or a reference to a proof and note that these proofs can be generalized to the boundary conditions used here.

Remark 5.2. Note that Theorem 5.1 justifies the notation 4.2. Thus for each S satisfy (1.7) we have that the spectrum $\sigma(S)$ of S is given by

(1) $\sigma(S)=\{\lambda_n(\alpha,\beta),\, n \in \mathbb{N}_0\}$ if the boundary condition of S is separated and determined by the parameters α,β;

(2) $\sigma(S)=\{\lambda_n(K),\, n \in \mathbb{N}_0\}$ if the boundary condition of S is real coupled with coupling constant K;

(3) $\sigma(S)=\{\,\lambda_n(\gamma,K),\, n \in \mathbb{N}_0\}$ if the boundary condition of S is complex coupled with coupling constants K,γ.

Remark 5.3. It is the canonical forms of the boundary conditions which make it possible to introduce the notation of Remark 5.2. This notation identifies λ_n uniquely and makes it possible to study the dependence of the eigenvalues on the boundary conditions and on the equations. No comparable canonical representation of all self-adjoint boundary conditions is known for higher order ordinary differential equations. There are some recent results [14, 15] but these are much more complicated and thus more difficult to use for the study of the dependence of the eigenvalues on the problem. But note that the jump condition (1.5) determined by C at the point c remains fixed as A and B vary.

6. Eigenvalue inequalities

In this section we give a complete description of how, for a fixed equation and fixed matrix C, the eigenvalues change as the boundary conditions (1.4) determined the matrices A,B vary. Since the Dirichlet and Neumann boundary conditions play a special role we introduce the notation

$$\lambda_n^D=\lambda_n(0,\pi), \quad \lambda_n^N=\lambda_n(\pi/2,\pi/2), \quad n \in \mathbb{N}_0. \tag{6.1}$$

Theorem 6.1. *Let (1.1) to (1.6) hold, let S satisfy (1.7) and let λ_n^D be defined by (6.1). Then for all (A,B) satisfying (1.4) we have*

(1)
$$\lambda_n(A,B) \leq \lambda_n^D, \quad n \in \mathbb{N}_0. \tag{6.2}$$
Equality can hold in (6.2) for non Dirichlet eigenvalues.

(2) *For all (A,B) satisfying (1.4) we have*
$$\lambda_n^D \leq \lambda_{n+2}(A,B), \quad n \in \mathbb{N}_0.$$

(3) *The range of $\lambda_0(A,B)$ is $(-\infty,\lambda_0^D]$.*

(4) *The range of $\lambda_1(A,B)$ is $(-\infty,\lambda_0^D]$.*

(5) *The range of* $\lambda_n(A,B)$ *is* $(\lambda_{n-2}^D, \lambda_n^D]$ *for* $n \geq 2$.
Moreover, (3), (4), (5) still hold when A, B *are restricted to be real.*

For a proof of the above theorem, see [33]. Next we investigate how the eigenvalues change when the boundary conditions change more closely.

According to a well-known classical result (see [8] and [6] for the case of smooth coefficients and [31] for the general case), we have the following inequalities for $K = I$, the identity matrix:

$$\begin{aligned} \lambda_0^N &\leq \lambda_0(I) < \lambda_0(e^{i\gamma}I) < \lambda_0(-I) \leq \{\lambda_0^D, \lambda_1^N\} \\ &\leq \lambda_1(-I) < \lambda_1(e^{i\gamma}I) < \lambda_1(I) \leq \{\lambda_1^D, \lambda_2^N\} \\ &\leq \lambda_2(I) < \lambda_2(e^{i\gamma}I) < \lambda_2(-I) \leq \{\lambda_2^D, \lambda_3^N\} \\ &\leq \lambda_3(-I) < \lambda_3(e^{i\gamma}I) < \lambda_3(I) \leq \{\lambda_3^D, \lambda_4^N\} \leq \dots, \end{aligned} \tag{6.3}$$

where $\gamma \in (-\pi, \pi)$ and $\gamma \neq 0$. In (6.3) notation $\{\lambda_n^D, \lambda_{n+1}^N\}$ means either of λ_n^D and λ_{n+1}^N and there is no comparison made between these two. These inequalities are well known in Flochet theory.

Eastham, Kong, Wu and Zettl [7] extended these inequalities to general $K \in SL_2(\mathbb{R})$. A key feature of this extension is the identification of separated boundary conditions which play the role of the Dirichlet and Neumann conditions in (6.3). These are given next.

For $K \in SL_2(\mathbb{R})$, $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix}$, denote by $\mu_n = \mu_n(K)$ and $\nu_n = \nu_n(K)$, $n \in \mathbb{N}_0$, the eigenvalues for the separated boundary conditions

$$y(a) = 0, \quad k_{22}y(b) - k_{12}y^{[1]}(b) = 0; \tag{6.4}$$

$$y^{[1]}(a) = 0, \quad k_{21}y(b) - k_{11}y^{[1]}(b) = 0; \tag{6.5}$$

respectively. Note that $(k_{22}, k_{12}) \neq (0,0) \neq (k_{21}, k_{11})$ since $\det K = 1$. Therefore each of these is a self-adjoint separated boundary condition with a countably infinite number of only real eigenvalues.

Theorem 6.2. *Let* (1.1) *to* (1.7) *hold. Let* μ_n *and* ν_n, $n \in \mathbb{N}_0$ *be the eigenvalues for* (6.4), *and* (6.5), *respectively. Then we have*

- *Suppose that* $k_{12} < 0$ *and* $k_{11} \leq 0$. *Then*

(1) $\lambda_0(K)$ *is simple;*
(2) $\lambda_0(K) < \lambda_0(-K)$;
(3) *The following inequalities hold for* $-\pi < \gamma < 0$ *and* $0 < \gamma < \pi$:

$$\begin{aligned} -\infty &< \lambda_0(K) < \lambda_0(\gamma, K) < \lambda_0(-K) \leq \{\mu_0, \nu_0\} \\ &\leq \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \leq \{\mu_1, \nu_1\} \\ &\leq \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \leq \{\mu_2, \nu_2\} \\ &\leq \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(-K) \leq \{\mu_3, \nu_3\} \leq \dots \end{aligned}$$

- *Suppose that* $k_{12} \leq 0$ *and* $k_{11} > 0$. *Then*

(1) $\lambda_0(K)$ *is simple;*
(2) $\lambda_0(K) < \lambda_0(-K)$
(3) *The following inequalities hold for* $-\pi < \gamma < 0$ *and* $0 < \gamma < \pi$:

$$\nu_0 \leq \lambda_0(K) < \lambda_0(\gamma, K) < \lambda_0(-K) \leq \{\mu_0, \nu_1\}$$

$$\begin{aligned}
&< \lambda_1(-K) < \lambda_1(\gamma, K) < \lambda_1(K) \le \{\mu_1, \nu_2\} \\
&\le \lambda_2(K) < \lambda_2(\gamma, K) < \lambda_2(-K) \le \{\mu_2, \nu_3\} \\
&\le \lambda_3(-K) < \lambda_3(\gamma, K) < \lambda_3(K) \le \{\mu_3, \nu_4\} \le \dots
\end{aligned}$$

- *Furthermore, for $0 < \alpha < \beta < \pi$ we have*
$$\begin{aligned}
\lambda_0(\beta, K) &< \lambda_0(\alpha, K) < \lambda_1(\alpha, K) < \lambda_1(\beta, K) < \lambda_2(\beta, K) < \lambda_2(\alpha, K) \\
&< \lambda_3(\alpha, K) < \lambda_3(\beta, K) < \dots
\end{aligned}$$
- *If neither of the above cases holds for K then one of them must hold for $-K$. The notation $\{\mu_n, \nu_m\}$ is used to indicate either υ_n or ν_m but no comparison is made between μ_n and ν_m.*

Proof. For K a diagonal matrix these inequalities were established in Weidmann [31]. The general result is proven in Eastham, Kong, Wu and Zettl [7]. □

Next we mention some interesting consequences of Theorem 6.2.

Remark 6.3. For separated boundary conditions the Prüfer transformation is a powerful tool for proving the existence of eigenvalues, studying their properties and computing them numerically. There is no comparable tool for coupled conditions. For coupled conditions the standard existence proof for the eigenvalues is based on operator theory in Hilbert space; the Green's function is constructed and used as a kernel in the definition of an integral operator whose eigenvalues are those of the problem or their reciprocals, see Coddington and Levinson [6] or Weidmann [31].

A proof based on Theorem 6.2 was given in [7] and goes as follows: Starting with the eigenvalues μ_n and ν_n, $n \in \mathbb{N}_0$, of the separated BC (6.4), (6.5) the proof of [7, Theorem 4.8.1] (although this is not explicitly pointed out there) actually shows that there is one and only one eigenvalue of the coupled condition determined by K in the interval $(-\infty, \mu_0]$ and it is $\lambda_0(\gamma, K)$; there is exactly one eigenvalue in the interval $[\mu_0, \mu_1]$ and it is $\lambda_1(\gamma, K)$; there is exactly one eigenvalue in the interval $[\mu_n, \mu_{n+1}]$ and it is $\lambda_{n+1}(\gamma, K)$, for $n \in \mathbb{N}_0$. This not only proves the existence of the eigenvalues of K but can be used to construct an algorithm to compute them. Such an algorithm is used by SLEIGN2, see [1], see also [2, 3]. This is the first existence proof for coupled eigenvalues which does not use the self-adjoint operator in Hilbert space and thus can be described as the first 'elementary' existence proof.

Remark 6.4. By Theorem 6.2 for *any* $K \in SL_2(\mathbb{R})$ either $\lambda_0(K)$ or $\lambda_0(-K)$ is simple. This extends the classical result that the lowest periodic eigenvalue is simple, to the general case of arbitrary coupled self-adjoint BC's. Here simple refers to both the algebraic and geometric multiplicities, since these are equal.

Theorem 6.5. *Let* (1.1) *to* (1.7) *hold. Let μ_n and ν_n, $n \in \mathbb{N}_0$ be the eigenvalues for* (6.4)*, and* (6.5)*, respectively. Then*

(1) *An eigenvalue $\lambda_n(K)$ is double if and only if there exist $k, m \in \mathbb{N}_0$ such that*
$$\lambda_n(K) = \mu_k = \nu_m;$$
(2) *Given eigenvalues $\lambda_n(K)$ and $\lambda_{n+1}(K)$ of K, distinct or not, there exist eigenvalues υ_k, ν_m of the separated boundary conditions* (6.4)*,* (6.5) *such that*
$$\lambda_n(K) \le \{\mu_k, \nu_m\} \le \lambda_{n+1}(K).$$

For a proof of the above theorem, see [17, Theorem 4.3 and Corollary 4.2].

7. Continuity of eigenvalues

In this section we study the continuity of the eigenvalues as functions of each parameter of the problem. We extend the notation (4.4) for the eigenvalues to include the coefficients and the endpoints

$$\lambda_n(a,b,\alpha,\beta,p,q,w), \quad \lambda_n(a,b,K,p,q,w), \quad \lambda_n(a,b,\gamma,K,p,q,w), \quad n \in \mathbb{N}_0. \tag{7.1}$$

When we study the dependence on one parameter x with the others fixed we abbreviate the notation to $\lambda_n(x)$; thus $\lambda_n(q)$ indicates that we are studying λ_n as a function of $q \in L(J,\mathbb{R})$ with all other parameters of the problem fixed, $\lambda_n(a)$ indicates that we are studying λ_n as a function of the left endpoint with all other parameters fixed, etc. Since C is fixed in our results below we do not include it in the notation (7.1).

The eigenvalues are continuous functions of each of $\frac{1}{p}, q, w, a, b$; they are not continuous functions of the boundary conditions, in general. The continuity on the coefficients $\frac{1}{p}, q, w$ is with respect to the $L(J,\mathbb{R})$ norm; the continuity on K is with respect to any matrix norm and the continuity with respect to a,b,α,β,γ is in the reals $\mathbb{R}$. We will see below that even though, in general, λ_n is not a continuous function of the boundary conditions for fixed n, it can always be embedded in a "continuous branch" of eigenvalues by varying the index n. For separated boundary conditions there is a jump discontinuity when either $y(a)=0$ or $y(b)=0$. We also characterize the coupled boundary conditions at which the eigenvalues are not continuous and show that all discontinuities are finite or infinite jumps. The set of boundary conditions at which the eigenvalues have discontinuities we call "the jump set" since all discontinuities are of the jump type.

We start with the continuous dependence on the coefficients and the endpoints. For all results in this section C is fixed.

Theorem 7.1. *Let* (1.1) *to* (1.7) *hold and let* $n \in \mathbb{N}_0$. *Then*

1. $\lambda_n(1/p)$ *is a continuous function of* $1/p \in L(J,\mathbb{R})$*;*
2. $\lambda_n(q)$ *is a continuous function of* $q \in L(J,\mathbb{R})$*;*
3. $\lambda_n(w)$ *is a continuous function of* $w \in L(J,\mathbb{R})$*;*
4. $\lambda_n(a)$ *is a continuous function of* a.
5. $\lambda_n(b)$ *is a continuous function of* b.

For a proof of the above theorem, see Kong, Wu and Zettl [17, Section 2]. Next we characterize the boundary conditions at which λ_n is not continuous, we call this set the "jump" set since all discontinuities are of jump type.

Definition 7.2 (Jump set of boundary conditions)**.** The jump set of boundary conditions $\mathbb{J}$ is the union of

1. the (real and complex) coupled conditions
$$Y(b) = e^{i\gamma} K\, Y(a), \quad Y = \begin{pmatrix} y \\ (py') \end{pmatrix}, \quad -\pi < \gamma \leq \pi,$$
where the 2×2 matrix $K = (k_{ij}) \in SL_2(\mathbb{R})$ satisfies $k_{12}=0$, and
2. the separated boundary conditions
$$\begin{aligned} A_1 y(a) + A_2(py')(a) = 0, &\quad A_1, A_2 \in \mathbb{R},\ (A_1,A_2) \neq (0,0) \\ B_1 y(b) + B_2(py')(b) = 0, &\quad B_1, B_2 \in \mathbb{R},\ (B_1,B_2) \neq (0,0) \end{aligned} \tag{7.2}$$

satisfying $A_2B_2 = 0$. Note that these are precisely the conditions where either $\alpha = 0$ or $\beta = \pi$ or both $\alpha = 0$ and $\beta = \pi$.

Theorem 7.3. *Let* (1.1) *to* (1.7) *hold and let $n \in \mathbb{N}_0$. Let $\mathbb{J}$ be given by Definition7.2. Then*

(1) *If the boundary condition is not on the jump set $\mathbb{J}$, then λ_n is a continuous function of the boundary condition.*
(2) *If $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$, $k_{12} = 0$ and $\lambda_n = \lambda_{n-1}$, then λ_n is continuous at K.*
(3) *The lowest eigenvalue λ_0 has an infinite jump discontinuity at each separated or (real or complex) coupled boundary condition in $\mathbb{J}$.*
(4) *Let $n \in \mathbb{N}$. If the boundary condition is in $\mathbb{J}$ and λ_n is simple, then λ_n has a finite jump discontinuity at this boundary condition.*

For a proof of the above theorem, see [17, Section 3]. For the important special case of separated boundary conditions in canonical form (4.1), (4.2) there is a stronger result.

Lemma 7.4. *For any $n \in \mathbb{N}_0$, $\lambda_n(\alpha, \beta)$ is jointly continuous on $[0, \alpha) \times (0, \pi]$ and strictly decreasing in α for each fixed β and strictly increasing in β for each fixed α.*

The proof of the above lemma can be found in [17]. The next theorem gives more detailed information about separated boundary conditions (7.2) not in canonical form, in particular for the separated jump boundary conditions.

Theorem 7.5 (Everitt-Möller-Zettl)**.** *Fix a, b, p, q, w and consider the conditions* (7.2)*.*

- *Fix B_1, B_2 and let $A_1 = 1$. Consider $\lambda_n = \lambda_n(A_2)$ as a function of $A_2 \in \mathbb{R}$. Then for each $n \in \mathbb{N}_0$, $\lambda_n(A_2)$ is continuous at A_2 for $A_2 > 0$ and $A_2 < 0$ but has a jump discontinuity at $A_2 = 0$. More precisely we have*
  (1) $\lambda_n(A_2) \to \lambda_n(0)$ *as* $A_2 \to 0^-$, $n \in \mathbb{N}_0$.
  (2) $\lambda_0(A_2) \to -\infty$ *as* $A_2 \to 0^+$.
  (3) $\lambda_{n+1}(A_2) \to \lambda_n(0)$ *as* $A_2 \to 0^+$.
- *Fix A_1, A_2 and let $B_1 = 1$. Consider $\lambda_n = \lambda_n(B_2)$ as a function of $B_2 \in \mathbb{R}$. Then for each $n \in \mathbb{N}_0$, $\lambda_n(B_2)$ is continuous at B_2 for $B_2 > 0$ and $B_2 < 0$ but has a jump discontinuity at $B_2 = 0$. More precisely we have:*
  (1) $\lambda_n(B_2) \to \lambda_n(0)$ *as* $B_2 \to 0^+$, $n \in \mathbb{N}_0$.
  (2) $\lambda_0(B_2) \to -\infty$ *as* $B_2 \to 0^-$.
  (3) $\lambda_{n+1}(B_2) \to \lambda_n(0)$ *as* $B_2 \to 0^-$.

For a proof of the above theorem, see Everitt, Möller and Zettl [10, 11, 12].

Remark 7.6. Note that $\lambda_0(A_2)$ has an infinite jump discontinuity at $A_2 = 0$, but for all $n \geq 1$, $\lambda_n(A_2)$ has a finite jump discontinuity at $A_2 = 0$, $\lambda_n(A_2)$ is left but not right continuous at 0. Similarly, $\lambda_0(B_2)$ has an infinite jump discontinuity at $B_2 = 0$, but for all $n \geq 1$, $\lambda_n(B_2)$ has a finite jump discontinuity at $B_2 = 0$ $\lambda_n(B_2)$ is right but not left continuous at 0. In all cases $\lambda_n(0)$ is embedded in a continuous branch of eigenvalues as A_2 or B_2 passes through zero but this branch is not given by a fixed index n; in order to preserve continuity the index "jumps" from n to $n + 1$ as A_2 or B_2 pass through zero from the appropriate direction.

Remark 7.7. This forced "index jumping" in order to stay on a continuous branch of eigenvalues plays an important role in some of the algorithms and their numerical implementations used in the code SLEIGN2 [1] for the numerical approximation of the spectrum of regular and singular SLP.

Remark 7.8. This "index jumping" phenomenon in order to stay on a "continuous eigenvalue branch" is quite general: It applies to all simple eigenvalues for all boundary conditions on the jump set $\mathbb{J}$, separated, real coupled, or complex coupled. For details the reader is referred to [17, Theorems 3.39, 3.73, 3.76, Propositions 3.71, 3.72].

Remark 7.9. Kong and Zettl [20] have shown that each continuous eigenvalue branch is in fact differentiable everywhere including the point $A_2 = 0$ (or $B_2 = 0$) where the index jumps. This also follows from Möller and Zettl [21].

Remark 7.10. Remarkably, if the boundary condition is in $\mathbb{J}$ and λ_n is simple then it can be embedded in a continuous eigenvalue branch and this branch is differentiable. Möller-Zettl [21] extended this result to abstract operators in Banach space.

8. Differentiability of eigenvalues

Now that the continuities of λ_n have been characterized it is natural to investigate the differentiability of λ_n as a function of the parameters of the problem. This we embark upon next. Here for each $n \in \mathbb{N}_0$, u_n denotes a normalized eigenfunction of λ_n. For all cases except when $\gamma \neq 0$ we choose u_n to be real valued. Again C is fixed in this section.

Theorem 8.1. *Let* (1.1) *to* (1.7) *hold. Let* $n \in \mathbb{N}_0$.

(1) *Assume that* p, q, w *are continuous at* a *and* $p(a) \neq 0$*, then* $\lambda_n(a)$ *is differentiable at* a *and*
$$\lambda_n'(a) = \frac{1}{p(a)}|pu_n'|^2(a) - |u_n|^2(a)[q(a) - \lambda_n(a)w(a)].$$
(2) *Assume that* p, q, w *are continuous at* b *and* $p(b) \neq 0$*, then* $\lambda_n(b)$ *is differentiable at* b *and*
$$\lambda_n'(b) = -\frac{1}{p(b)}|pu_n'|^2(b) + |u_n|^2(b)\,[q(b) - \lambda_n(b)w(b)].$$
(3) *Let* $-\pi < \gamma < 0$ *or* $0 < \gamma < \pi$*. Then* $\lambda_n(\gamma)$ *is differentiable at* γ *and*
$$\lambda_n'(\gamma) = -2\ \mathrm{Im}[u_n(b)\ (pu_n')(b)],$$
where $\mathrm{Im}[z]$ *denotes the imaginary part of* z.

(4) *Let* $\alpha \in (0, \pi)$*. Then* $\lambda_n(\alpha)$ *is differentiable and its derivative is given by*
$$\lambda_n'(\alpha) = -u^2(a) - (pu')^2(a).$$
(5) *Let* $\beta \in (0, \pi)$*. Then* $\lambda_n(\beta)$ *is differentiable and its derivative is given by*
$$\lambda_n'(\beta) = u^2(b) + (pu')^2(b).$$

For a proof of the above theorem, see [20]. Next we study the differentiability of the eigenvalues with respect to the remaining parameters: $\frac{1}{p}, q, w$ and K.

Theorem 8.2. *Let* (1.1) *to* (1.7) *hold. Let* $n \in \mathbb{N}_0$.

(1) *Assume that $\lambda_n(q)$ is a simple eigenvalue with real valued normalized eigenfunction $u_n(\cdot,q)$. Then $\lambda_n(\cdot,q)$ is differentiable in $L(J,\mathbb{R})$ and its Frechet derivative is given by*

$$\lambda_n'(q)h=\int_a^b |u_n(\cdot,q)|^2h,\quad h\in L(J,\mathbb{R}). \tag{8.1}$$

(2) *Assume that $\lambda_n(1/p)$ is a simple eigenvalue with real valued normalized eigenfunction $u_n(\cdot,\frac{1}{p})$. Then $\lambda_n(\cdot,1/p)$ is differentiable in $L(J,\mathbb{R})$ and its Frechet derivative is given by*

$$\lambda_n'(1/p)\,h=-\int_a^b |u_n^{[1]}(\cdot,1/p)|^2h,\quad h\in L(J,\mathbb{R}).$$

(3) *Assume that $\lambda_n(w)$ is a simple eigenvalue with real valued normalized eigenfunction $u_n(\cdot,w)$. Then $\lambda_n(\cdot,w)$ is differentiable in $L(J,\mathbb{R})$ and its Frechet derivative is given by*

$$\lambda_n'(w)h=-\lambda_n(w)\int_a^b|u_n(\cdot,w)|^2\,h,\quad h\in L(J,\mathbb{R}).$$

(4) *Assume that $\lambda_n(K)$ is a simple eigenvalue with real valued normalized eigenfunction $u_n(\cdot,K)$. Then $\lambda_n(\cdot,K)$ is differentiable and its Frechet derivative is given by the bounded linear transformation defined by*

$$\lambda_n'(K)\,H=[p\overline{u_n}'(b),-\overline{u}_n(b)]HK^{-1}\begin{pmatrix}u_n(b)\\(pu_n')(b)\end{pmatrix},\quad H\in M_{2,2}(\mathbb{C}).$$

For the proof of (1), (2), (3), see [20], and for (4) see [21].

9. Monotonicity of eigenvalues

In this section we fix a boundary condition and study how the eigenvalues change when coefficient changes monotonically.

Theorem 9.1. *Let* (1.1) *to* (1.7) *hold, let $n\in\mathbb{N}_0$.*

(1) *Fix p,w. Suppose $Q\in L([a,b],\mathbb{R}\ )$ and assume that $Q\geq q$ a.e. on $[a,b]$. Then $\lambda_n(Q)\geq\lambda_n(q)$. If $Q>q$ on a subset of $[a,b]$ having positive Lebesgue measure, then $\lambda_n(Q)>\lambda_n(q)$.*
(2) *Fix q,w. Suppose $1/P\in L([a,b],\mathbb{R})$ and $0<P\leq p$ a.e. on $[a,b]$. Then $\lambda_n(1/P)\geq\lambda_n(1/p)$; if $1/P<1/p$ on a subset of $[a,b]$ having positive Lebesgue measure, then $\lambda_n(1/P)<\lambda_n(1/p)$.*
(3) *Fix p,q. Suppose $W\in L([a,b],\mathbb{R})$ and $W\geq w>0$ a.e. on $[a,b]$. Then $\lambda_n(W)\geq\lambda_n(w)$ if $\lambda_n(W)<0$ and $\lambda_n(w)<0$; but $\lambda_n(W)\leq\lambda_n(w)$ if $\lambda_n(W)>0$ and $\lambda_n(w)>0$. Furthermore, if strict inequality holds in the hypothesis on a set of positive Lebesgue measure, then strict inequality holds in the conclusion.*

Proof. We give the proof for (1), the proofs of (2) and (3) are similar. Define a function $f:\mathbb{R}\to\mathbb{R}$ by

$$f(t)=\lambda_n(s(t)),\ s(t)=q+t(Q-q),\quad t\in[0,1].$$

Then $s(t) \in L((a,b),\mathbb{R})$ for each $t \in [0,1]$. From the chain rule in Banach space and formula (8.1) for $\lambda_n'(q)$ we have

$$f'(t) = \lambda_n'((s(t))\, s'(t) = \int_a^b |u^2(r,s(t))|\,(Q(r)-q(r))\,dr \geq 0,\ t \in [0,1].$$

Hence f is nondecreasing on $[0,1]$ and $f(1) = \lambda_n(Q) \geq \lambda_n(q) = f(0)$. The strict inequality part of the theorem also follows from this argument. $\square$

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