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# SPECTRAL FUNCTION FOR A NONSYMMETRIC DIFFERENTIAL OPERATOR ON THE HALF LINE

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Abstract. In this article we study the spectral function for a nonsymmetric differential operator on the half line. Two cases of the coefficient matrix are considered, and for each case we prove by Marchenko's method that, to the boundary value problem, there corresponds a spectral function related to which a Marchenko-Parseval equality and an expansion formula are established. Our results extend the classical spectral theory for self-adjoint Sturm-Liouville operators and Dirac operators.

## 1. Introduction

As a very essential mathematical problem, the Weyl-Stone eigenfunction expansion [29, 32] in which the key role is the spectral function for singular self-adjoint second order linear differential operators, has been studied deeply by many renowned mathematicians: Kodaira [14], Levinson [17], Levitan [18], Titchmarsh [30], Yosida [34] and others. As for spectral function, one of the well-known classical results is about the Sturm-Liouville problem on the half line:

$$-y'' + q(x)y = \lambda y, \quad x > 0; \quad y(0) = 1, \quad y'(0) = h.$$

Let $y = y(x, \lambda)$ be the solution to the Sturm-Liouville problem. Then there exists a spectral function $\rho(\lambda)$ (see e.g. [5]) such that, for all real $f \in L^2(0, \infty)$, it holds that

$$\int_0^\infty f^2(x)\mathrm{d}x = \int_{-\infty}^\infty \Big[\lim_{n\to\infty}\int_0^n f(x)y(x,\lambda)\mathrm{d}x\Big]^2 \mathrm{d}\rho(\lambda).$$

The above equality can be derived as a limiting case of the classical Sturm-Liouville expansion theorem for the regular operators (see e.g. [5, 19]), where the Parseval equality for the regular operators plays a very important role for the proofs. Similar ideas can also be applied to singular self-adjoint first order systems, for example, the Dirac operators [16, 19]. For general theory of eigenfunction expansion for self-adjoint and regular non-self-adjoint operators in Hilbert space, we refer to [2, 15, 22]. For multidimensional cases, see, e.g., [11]. Moreover, a two-fold spectral expansion in terms of principal functions of a Schrödinger operator has been derived in [1].

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Recently, Kirsten and Loya [13] obtained some interesting results on the spectral zeta function for a Schrödinger operator on the half line.

However, to the author's knowledge, for singular nonsymmetric differential operators there are only a few results on eigenfunction expansion. The limiting approach for self-adjoint case can not be applied even for very simple case of nonsymmetric differential operators, since in general the corresponding regular spectrum has irregular behavior on the complex plane. To extend expansion theory to general case, Marchenko [20, 21] established an excellent method in dealing with the singular Sturm-Liouville operator with complex-valued potential. In this paper, inspired by the idea of Marchenko, we are going to establish expansion theorem in two cases for a singular nonsymmetric differential operator, where the key is to prove the existence of the corresponding spectral function. Our results can be extended to $2n \times 2n$ systems, and for simplicity we here will only consider the case of $n = 1$. For the regular case of this nonsymmetric differential operator, recently we have obtained some results on inverse spectral problems with applications to inverse problems for one-dimensional hyperbolic systems, see [24]–[27]. It is well known that for many differential operators there are intrinsic relations between their spectral functions and the corresponding Weyl functions (often called $m$-functions), and for the recent interesting results on Weyl functions see, e.g., [3, 6, 7, 12, 28, 35, 36]. For the asymptotic behavior of spectral functions for elliptic operators we refer to [8, 10, 23].

In this article we consider boundary value problems generated by a nonsymmetric differential operator on the half line $0 \le x < \infty$:

$$(A_P\varphi)(x) := B\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) + P(x)\varphi(x) = \lambda\varphi(x),$$

where $B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ in $(C^1[0,\infty))^4$. Both the matrix-valued function $P$ and parameter $\lambda$ are complex-valued. In this article, we consider only the $C^1$-class case for $P$, because in this case it is easier to prove the transformation formula (see Lemma 2.1) while in general case it will be very complicated. It is directly checked that the adjoint operator of $A_P$ in some suitable Hilbert space is $-B\frac{\mathrm{d}}{\mathrm{d}x} + \overline{P^T(x)}$ and consequently $A_P$ is nonsymmetric. Here and henceforth, $\overline{c}$ denotes the complex conjugate of $c$ and $\cdot^T$ denotes the transpose of a vector or matrix under consideration. Here we point out that the spectrum problem for $A_P$ with compact matrix-valued function $P$ has been studied in [31].

To describe our results properly, we first give some information on distributions and we refer to [21] for more details. Let $\mathbb{K}^2(0,\infty)$ denote the set of all square integrable functions in $(0,\infty)$ with compact support. For $\sigma > 0$, we set $\mathbb{K}^2_\sigma(0,\infty) = \{f \in \mathbb{K}^2(0,\infty) : f(x) = 0 \text{ for } x > \sigma\}$. The entire function $e(\rho)$ is called the *function of exponential type* if $|e(\rho)| \le C\exp(\sigma|\mathrm{Im}\rho|)$ where the positive constants $C$ and $\sigma$ depend on $e(\rho)$. Moreover, the index

$$\sigma_e = \limsup_{r\to\infty} r^{-1}\ln\big(\max_{|\rho|=r}|e(\rho)|\big)$$

is called the type of entire function $e(\rho)$. Let linear topological space $Z$ be the set of all entire exponential type functions integrable on the real line. The sequence $e_n$ converges to $e$ in $Z$ if $\lim_{n\to\infty}\int_{-\infty}^{\infty}|e_n(\rho) - e(\rho)|\mathrm{d}\rho = 0$ and the types $\sigma_n$ of the functions $e_n(\rho)$ are bounded: $\sup\sigma_n < \infty$. The set of all linear continuous functionals defined on the test space $Z$ will be denoted by $Z'$ whose components are called *distributions* (generalized functions). The sequence $D_n$ converges to $D$ in $Z'$ if $\lim_{n\to\infty} < D_n, e(\rho) >=< D, e(\rho) >$ for all test functions $e \in Z$.

In this article we consider two cases of the coefficient matrix $P$. The first case is special and will be described as follows. Let $P$ be a continuously differentiable matrix-valued function satisfying $BP = PB$ and $\mu$ be a complex constant. Here it is easy to see that $P$ is of the form $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$. We consider the boundary value problem

$$\begin{gathered} B\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x) + P(x)\varphi(x) = \lambda\varphi(x),\ 0 < x < \infty, \\ \varphi(0) = \begin{pmatrix} \cosh\mu & \sinh\mu \\ \sinh\mu & \cosh\mu \end{pmatrix}. \end{gathered} \tag{1.1}$$

Let $\varphi = \varphi(x,\lambda)$ be the solution to (1.1) and

$$\varphi_{[1]} = \begin{pmatrix} \varphi_{[1]}^{(1)} \\ \varphi_{[1]}^{(2)} \end{pmatrix} \quad \text{and} \quad \varphi_{[2]} = \begin{pmatrix} \varphi_{[2]}^{(1)} \\ \varphi_{[2]}^{(2)} \end{pmatrix}$$

be the first and the second column vector of the matrix $\varphi$, i.e., $\varphi = (\varphi_{[1]}\ \varphi_{[2]})$. Similarly we denote the matrix inverse of $\varphi$ by $\psi = \varphi^{-1} = (\psi_{[1]}\ \psi_{[2]})$. Now for

$$f = \begin{pmatrix} f^{(1)} \\ f^{(2)} \end{pmatrix} \in \left(L^2(0,\infty)\right)^2, \quad g = \begin{pmatrix} g^{(1)} \\ g^{(2)} \end{pmatrix} \in \left(L^2(0,\infty)\right)^2$$

where $\left(L^2(0,\infty)\right)^2$ denotes the product space of $L^2(0,\infty)$, we set

$$\omega_f^k(\rho) = \int_0^\infty f^T(x)\psi_{[k]}(x,i\rho)\mathrm{d}x, \quad \eta_g^k(\rho) = \int_0^\infty \varphi_{[k]}^T(x,i\rho)g(x)\mathrm{d}x \quad (k=1,2),$$

where $i = \sqrt{-1}$, $\rho \in \mathbb{R}$. Then we have the first main result of this article.

**Theorem 1.1.** *It holds for the boundary value problem* (1.1) *that*

$$\int_0^\infty f^T(x)\overline{g(x)}\mathrm{d}x = \frac{1}{2\pi}\sum_{k=1}^2 \int_{-\infty}^\infty \omega_f^k(\rho)\eta_g^k(\rho)\mathrm{d}\rho. \tag{1.2}$$

*Moreover, for $f \in \left(\mathbb{K}^2(0,\infty)\right)^2$ with $\omega_f^k(\rho), \eta_f^k(\rho) \in Z$ $(k=1,2)$, the following expansion formula holds:*

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\sum_{k=1}^2 \int_{-\infty}^\infty \omega_f^k(\rho)\varphi_{[k]}(x,i\rho)\mathrm{d}\rho \\ &= \frac{1}{2\pi}\sum_{k=1}^2 \int_{-\infty}^\infty \eta_f^k(\rho)\psi_{[k]}(x,i\rho)\mathrm{d}\rho. \end{aligned} \tag{1.3}$$

We often call (1.2) (or (1.7)) the *Marchenko-Parseval equality* which means that a spectral function exists in corresponding boundary value problem. Historically, the concept of spectral function came from the classical theory of Weyl. Theorem 1.1 implies that $\frac{1}{2\pi}E$ is a spectral function corresponding to problem (1.1) with $P$ satisfying $BP = PB$, which is the same as the case of $P = 0$. Here and henceforth $E$ denotes the $2\times 2$ unit matrix.

For general matrix function $P \in \left(C^1[0,\infty)\right)^4$ without the constraint $BP = PB$, we also can show the existence of the corresponding spectral function. More precisely, let $Q$ be a $2\times 2$ matrix satisfying $QB + BQ = B$ and $Q^2 = Q$. It is seen by simple computation that there exists matrix $Q$ satisfying the above conditions, and the simplest one is $Q=$diag$(1,0)$. It follows easily from $\det B=-1$ that $\det Q=0$. We consider the boundary value problems

$$\begin{gathered} B\frac{\mathrm{d}\varphi}{\mathrm{d}x}(x)+P(x)\varphi(x)=\lambda\varphi(x), \quad 0<x<\infty, \\ \varphi(0)=Q, \end{gathered} \tag{1.4}$$

and

$$\begin{gathered} -\frac{\mathrm{d}\widetilde{\varphi}}{\mathrm{d}x}(x)B+\widetilde{\varphi}(x)P(x)=\lambda\widetilde{\varphi}(x), \quad 0<x<\infty, \\ \widetilde{\varphi}(0)=Q. \end{gathered} \tag{1.5}$$

We denote the solutions to problems (1.4) and (1.5) by $\varphi(x,\lambda)$ and $\widetilde{\varphi}(x,\lambda)$, respectively. For all $2\times 2$ matrices $f,g\in\left(L^2(0,\infty)\right)^4$, we set

$$\Phi_f(\rho)=\int_0^\infty f(x)\varphi(x,i\rho)\mathrm{d}x, \quad \widetilde{\Phi}_g(\rho)=\int_0^\infty \widetilde{\varphi}(x,i\rho)g(x)\mathrm{d}x, \tag{1.6}$$

where $i=\sqrt{-1}$, $\rho\in\mathbb{R}$. Then we have another main result of this paper.

**Theorem 1.2.** *To the problems* (1.4) *and* (1.5) *there corresponds a distribution-valued spectral function* $D=(D_{kl})_{1\le k,l\le 2}$ *such that* $D=QDQ$, $D_{kl}\in Z'$ *and*

$$\int_0^\infty f(x)g(x)\mathrm{d}x=\int_{-\infty}^\infty \Phi_f(\rho)D(\rho)\widetilde{\Phi}_g(\rho)\mathrm{d}\rho. \tag{1.7}$$

*Moreover, for* $f\in\left(\mathbb{K}^2(0,\infty)\right)^4$ *with* $\Phi_f(\rho),\widetilde{\Phi}_f(\rho)\in Z^4$, *the following expansion formula holds:*

$$f(x)=\int_{-\infty}^\infty \Phi_f(\rho)D(\rho)\widetilde{\varphi}(x,i\rho)\mathrm{d}\rho=\int_{-\infty}^\infty \varphi(x,i\rho)D(\rho)\widetilde{\Phi}_f(\rho)\mathrm{d}\rho. \tag{1.8}$$

Although Theorems 1.1 and 1.2 have shown the existence of spectral function for the singular nonsymmetric differential operator in two cases, we point out that the uniqueness of spectral function for the operator does not hold generally, which is the same as that for Sturm-Liouville operators (see, e.g., [19]). Moreover, since the spectral function is distribution-valued, it is not a measure in general, which is different from the case of self-adjoint Sturm-Liouville operators. Besides, given singular nonsymmetric differential operators with general $P$, it is still an open problem to prove the existence of spectral functions under general boundary conditions. On the other hand, it is interesting to investigate the corresponding inverse problems, namely, given spectral functions or Weyl functions, find the differential operators. See [5] for the classical inverse problem to determine the potential of the Sturm-Liouville operator from its spectral function and [4] for determination of singular differential pencils from the Weyl function. Theorem 1.1 has implied that the uniqueness does not hold generally for the inverse problems, and we need impose other assumptions for uniqueness. In a forthcoming paper we will study the inverse problems for the singular nonsymmetric differential operator.

This article is composed of four sections. In Section 2 we establish transformation formulae for our boundary value problems. Sections 3 and 4 are devoted to prove Theorems 1.1 and 1.2 by transformation formulae, respectively.

## 2. Transformation formulae

Set

$$\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < y < x\}. \tag{2.1}$$

For $P_j = (P_{j,kl})_{1\le k,l\le 2} \in \big(C^1[0,\infty)\big)^4$ $(j=1,2)$, we define

$$\theta_1(x) = \frac{1}{2}\int_0^x (P_{2,12} + P_{2,21} - P_{1,12} - P_{1,21})(s)\mathrm{d}s, \tag{2.2}$$

$$\theta_2(x) = \frac{1}{2}\int_0^x (P_{2,11} + P_{2,22} - P_{1,11} - P_{1,22})(s)\mathrm{d}s. \tag{2.3}$$

Moreover let us put

$$R(P_1,P_2)(x) = \exp(-\theta_1(x))\begin{pmatrix} \cosh\theta_2(x) & -\sinh\theta_2(x) \\ -\sinh\theta_2(x) & \cosh\theta_2(x)\end{pmatrix}. \tag{2.4}$$

Here we remark that $R(P_1,P_2)(0) = E$, $R(P_1,P_2)(x) = R^{-1}(P_2,P_1)(x)$ and that $R\big(-\overline{P_1^T}, -\overline{P_2^T}\big)(x) = \overline{R\big(P_2,P_1\big)(x)}$. Let $M_2(\mathbb{C})$ be the set of all $2\times 2$ complex-valued matrices. We first prove the following lemma.

**Lemma 2.1.** *For any $\lambda \in \mathbb{C}$, $Q \in M_2(\mathbb{C})$ with $\det Q = 0$ and $P_j \in \big(C^1[0,\infty)\big)^4$ $(j=1,2)$, let $\varphi_j = \varphi_j(x,\lambda)$ satisfy*

$$\begin{gathered} B\frac{\mathrm{d}\varphi_j(x)}{\mathrm{d}x} + P_j(x)\varphi_j(x) = \lambda\varphi_j(x), \quad 0 < x < \infty, \\ \varphi_j(0) = Q. \end{gathered} \tag{2.5}$$

*Then there exists a unique $K(P_1,P_2;Q) = (K_{kl}(P_1,P_2;Q))_{1\le k,l\le 2} \in \big(C^1(\overline{\Omega})\big)^4$ independent of $\lambda$ such that for $0 \le x < \infty$ and all $\lambda \in \mathbb{C}$*

$$\varphi_2(x,\lambda) = R(P_1,P_2)(x)\varphi_1(x,\lambda) + \int_0^x K(P_1,P_2;Q)(x,y)\varphi_1(y,\lambda)\mathrm{d}y. \tag{2.6}$$

*(transformation formula). Here $R(P_1,P_2)(x)$ is defined by (2.4).*

*Moreover, the kernel $K(P_1,P_2;Q)$ is the unique solution to the following problem of first order system (2.7)–(2.9):*

$$\begin{aligned} &B\frac{\partial K(P_1,P_2;Q)}{\partial x}(x,y) + \frac{\partial K(P_1,P_2;Q)}{\partial y}(x,y)B \\ &+P_2(x)K(P_1,P_2;Q)(x,y) - K(P_1,P_2;Q)(x,y)P_1(y) = 0, \quad (x,y)\in\Omega. \end{aligned} \tag{2.7}$$

$$K(P_1,P_2;Q)(x,0)BQ = 0 \quad (0 \le x < \infty), \tag{2.8}$$

$$\begin{aligned} &K(P_1,P_2;Q)(x,x)B - BK(P_1,P_2;Q)(x,x) \\ &= B\frac{\mathrm{d}R(P_1,P_2)}{\mathrm{d}x}(x) + P_2(x)R(P_1,P_2)(x) - R(P_1,P_2)(x)P_1(x) \\ &(0 \le x < \infty). \end{aligned} \tag{2.9}$$

*Proof.* We prove this lemma using ideas from [33]. Since $P_j \in \big(C^1[0,\infty)\big)^4$ $(j = 1,2)$, it can be verified directly that, if $K(P_1,P_2;Q) \in \big(C^1(\overline{\Omega})\big)^4$ is the unique solution to problem (2.7)–(2.9), then (2.6) holds. Therefore, it is sufficient to prove the existence and the uniqueness of the solution to problem (2.7)–(2.9) for each $P_1,P_2 \in \big(C^1[0,\infty)\big)^4$.

For clarity, we reduce the proof to a special case. By the condition $\det Q = 0$, we may assume that a complex constant $c$ exists such that $q_2 = cq_1$ where $q_1,q_2$ are the first column vector and the second one of $Q$, respectively. Then it is sufficient to prove the existence and the uniqueness of the solution to problem (2.7)–(2.9) in the case $\varphi_j(0,\lambda) = q_1$, since problem (2.5) is linear. Moreover, since a complex constant $c^*$ exists such that $q_1 = c^* \begin{pmatrix} \cosh\mu \\ \sinh\mu \end{pmatrix}$ where $\mu \in \mathbb{C}$, it can be reduced to the case $\varphi_j(0,\lambda) = \begin{pmatrix} \cosh\mu \\ \sinh\mu \end{pmatrix}$. In this case, we denote the the solution to problem (2.7)–(2.9) by $K(P_1, P_2, \mu)(x, y)$, and (2.8) has the form

$$\begin{aligned} K_{12}(P_1,P_2,\mu)(x,0) &= -\tanh\mu\ K_{11}(P_1,P_2,\mu)(x,0), \\ K_{22}(P_1,P_2,\mu)(x,0) &= -\tanh\mu\ K_{21}(P_1,P_2,\mu)(x,0). \end{aligned} \tag{2.10}$$

If we set

$$\begin{aligned} L_1(x,y) &= K_{12}(P_1,P_2,\mu)(x,y) - K_{21}(P_1,P_2,\mu)(x,y), \\ L_2(x,y) &= K_{11}(P_1,P_2,\mu)(x,y) - K_{22}(P_1,P_2,\mu)(x,y), \\ L_3(x,y) &= K_{11}(P_1,P_2,\mu)(x,y) + K_{22}(P_1,P_2,\mu)(x,y), \\ L_4(x,y) &= K_{12}(P_1,P_2,\mu)(x,y) + K_{21}(P_1,P_2,\mu)(x,y) \end{aligned} \tag{2.11}$$

and $L = L(x,y) = (L_1(x,y), L_2(x,y), L_3(x,y), L_4(x,y))$, then we can rewrite (2.7)–(2.9) as follows:

$$\frac{\partial L_k(x,y)}{\partial x} - \frac{\partial L_k(x,y)}{\partial y} = f_k(x,y,L) \quad ((x,y)\in\Omega,\ k=1,2), \tag{2.12}$$

$$\frac{\partial L_k(x,y)}{\partial x} + \frac{\partial L_k(x,y)}{\partial y} = f_k(x,y,L) \quad ((x,y)\in\Omega,\ k=3,4), \tag{2.13}$$

$$L_k(x,x) = r_k(x) \quad (0 \le x < \infty,\ k=1,2), \tag{2.14}$$

$$\begin{aligned} L_3(x,0) &= \sinh(2\mu)L_1(x,0) + \cosh(2\mu)L_2(x,0) \quad (0\le x<\infty), \\ L_4(x,0) &= -\cosh(2\mu)L_1(x,0) - \sinh(2\mu)L_2(x,0) \quad (0\le x<\infty), \end{aligned} \tag{2.15}$$

where $f_k(x,y,L) = \frac{1}{2}\sum_{m=1}^4 (a_{km}(y) + b_{km}(x))\, L_m(x,y)$ $(1 \le k \le 4)$, here $a_{km}(y)$, $b_{km}(x)$ $(1 \le k,m \le 4)$ are linear combinations of two elements of the matrix functions $P_1(y)$ and $P_2(x)$ respectively, and $r_k \in C^1[0,\infty)$ $(k=1,2)$ are dependent only on $P_1$ and $P_2$.

Integrating (2.12), (2.13) with (2.14) and (2.15) along the characteristics $x+y=$const. and $x-y=$const. respectively, we obtain the following integral equations:

$$\begin{gathered} L_k(x,y) = \int_y^{\frac{x+y}{2}} f_k(-s+x+y,s,L)\mathrm{d}s + r_k(\frac{x+y}{2}) \\ \big((x,y)\in\overline{\Omega}, k=1,2\big), \end{gathered} \tag{2.16}$$

and

$$\begin{aligned} L_k(x,y) = \int_0^y f_k(s+x-y,s,L)\mathrm{d}s + \int_0^{\frac{x-y}{2}} \Big\{&\alpha_k f_1(-s+x-y,s,L) \\ &+ \beta_k f_2(-s+x-y,s,L)\Big\}\mathrm{d}s + \alpha_k r_1(\frac{x-y}{2}) + \beta_k r_2(\frac{x-y}{2}) \\ &\big((x,y)\in\overline{\Omega},\ k=3,4\big), \end{aligned} \tag{2.17}$$

where $\alpha_3 = \sinh(2\mu)$, $\beta_3 = \cosh(2\mu)$ and $\alpha_4 = -\cosh(2\mu)$, $\beta_4 = -\sinh(2\mu)$.

The unique solution $L \in \left(C^1(\overline{\Omega})\right)^4$ to (2.16) and (2.17) can be obtained by the iteration method. In fact, setting

$$
\begin{gathered}
L_k^{(0)}(x,y) = 0 \quad \left((x,y) \in \overline{\Omega}, 1 \le k \le 4\right), \\
L_k^{(n)}(x,y) = \int_y^{\frac{x+y}{2}} f_k\left(-s+x+y, s, L^{(n-1)}\right) \mathrm{d}s + r_k(\frac{x+y}{2}) \\
\left((x,y) \in \overline{\Omega}, n \ge 1,\ k=1,2\right),
\end{gathered}
$$

and

$$
\begin{aligned}
&L_k^{(n)}(x,y) \\
&= \int_0^y f_k\left(s+x-y, s, L^{(n-1)}\right) \mathrm{d}s \\
&\quad + \int_0^{\frac{x-y}{2}} \left\{\alpha_k f_1\left(-s+x-y, s, L^{(n-1)}\right) + \beta_k f_2\left(-s+x-y, s, L^{(n-1)}\right)\right\} \mathrm{d}s \\
&\quad +\alpha_k r_1(\frac{x-y}{2}) + \beta_k r_2(\frac{x-y}{2}) \quad \left((x,y) \in \overline{\Omega}, n \ge 1, k=3,4\right),
\end{aligned}
$$

we can obtain by induction the estimates for each $n \ge 1$,

$$
\left|L_k^{(n)}(x,y) - L_k^{(n-1)}(x,y)\right| \le \omega(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad \left((x,y) \in \overline{\Omega},\ 1 \le k \le 4\right), \tag{2.18}
$$

where

$$
\begin{gathered}
\omega(x) = \left(|\sinh(2\mu)| + |\cosh(2\mu)| + 1\right) \max_{0 \le s \le x} \left(|r_1(s)| + |r_2(s)|\right), \\
\zeta(x) = \left(|\sinh(2\mu)| + |\cosh(2\mu)| + 1\right) x \max_{0 \le s \le x} \frac{1}{2} \sum_{k,l=1}^{2} \left(|P_{1,kl}(s)| + |P_{2,kl}(s)|\right).
\end{gathered}
$$

Thus $L_k(x,y) = \lim_{n\to\infty} L_k^{(n)}(x,y)$ $(1 \le k \le 4)$ exist uniformly for $(x,y) \in \overline{\Omega}$ and we see that $L_k(x,y)$ $(1 \le k \le 4)$ satisfy (2.16) and (2.17) with the bound $|L_k(x,y)| \le \omega(x)\exp(\sigma(x))$.

Moreover, differentiating (2.16) and (2.17) with respect to $x$ and $y$; similarly we can obtain by induction the following estimates

$$
\Big|\frac{\partial L_k^{(n)}(x,y)}{\partial x} - \frac{\partial L_k^{(n-1)}(x,y)}{\partial x}\Big| \le \xi(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad \left((x,y) \in \overline{\Omega}, 1 \le k \le 4\right), \tag{2.19}
$$

$$
\Big|\frac{\partial L_k^{(n)}(x,y)}{\partial y} - \frac{\partial L_k^{(n-1)}(x,y)}{\partial y}\Big| \le \xi(x) \frac{\zeta^{n-1}(x)}{(n-1)!} \quad \left((x,y) \in \overline{\Omega}, 1 \le k \le 4\right), \tag{2.20}
$$

where

$$
\begin{aligned}
\xi(x) = &\frac{1}{2}\left(|\sinh(2\mu)| + |\cosh(2\mu)| + 1\right) \\
&\times \Big\{ \max_{0 \le s \le x} \left(|r_1'(s)| + |r_2'(s)|\right) + \frac{1}{2}\omega(x)\exp(\zeta(x)) \\
&\times \max_{0 \le s \le x} \sum_{k,l=1}^{2} \left(|P_{1,kl}(s)| + |P_{2,kl}(s)| + \left(|P_{1,kl}'(s)| + |P_{2,kl}'(s)|\right) x\right) \Big\}.
\end{aligned}
$$

Therefore, from (2.19) and (2.20) it follows that $L \in (C^1(\overline{\Omega}))^4$. The uniqueness of the solution to (2.7)–(2.9) is shown by (2.18). $\square$

**Corollary 2.2.** *For $j = 1, 2$, let $\varphi_j$ be the solution to problem (1.1) with $P = P_j \in (C^1[0,\infty))^4$ satisfying $P_jB = BP_j$. Then the following transformation formula holds:*

$$\varphi_2(x,\lambda) = R(P_1,P_2)(x)\varphi_1(x,\lambda) \tag{2.21}$$

*where $R(P_1,P_2)(x)$ is defined by (2.4).*

The above corollary follows from the fact that $K(P_1,P_2,\mu) \equiv 0$, which can be derived easily by observing that the right hand side of (2.9) is 0 (in this case the condition $\det Q = 0$ is not necessary). Or one may directly verify (2.21). Here we omit the details.

**Corollary 2.3.** *Let $S$ and $\widetilde{S}$ be the solutions corresponding to $P = 0$ in (1.4) and (1.5), respectively. Then the following transformation formulae hold.*

*(1) For problem (1.4) we have*

$$S(x,i\rho) = R(P,0)(x)\varphi(x,i\rho) + \int_0^x K(P,0;Q)(x,y)\varphi_(y,i\rho)\mathrm{d}y \tag{2.22}$$

*where the kernel $K(P,0;Q) \in \left(C^1(\overline{\Omega})\right)^4$ satisfies*

$$BK_x(P,0;Q)(x,y) + K_y(P,0;Q)(x,y)B - K(P,0;Q)(x,y)P(y) = 0, \tag{2.23}$$

*for $(x,y) \in \Omega$, as well as the conditions*

$$K(P,0;Q)(x,0)Q = K(P,0;Q)(x,0), \tag{2.24}$$

$$K(P,0;Q)(x,x)B - BK(P,0;Q)(x,x) = BR'(P,0)(x) - R(P,0)(x)P(x) \tag{2.25}$$

*for $0 \le x < \infty$.*

*(2) For problem (1.5) we have*

$$\widetilde{S}(x,i\rho) = \widetilde{\varphi}(x,i\rho)R(0,P)(x) + \int_0^x \widetilde{\varphi}(y,i\rho)\overline{K^T(-\overline{P^T},0;\overline{Q^T})(x,y)}\mathrm{d}y \tag{2.26}$$

*where the kernel $\overline{K^T(-\overline{P^T},0;\overline{Q^T})(x,y)}$ satisfies*

$$Q\overline{K^T(-\overline{P^T},0;\overline{Q^T})(x,0)} = \overline{K^T(-\overline{P^T},0;\overline{Q^T})(x,0)}. \tag{2.27}$$

*Proof.* (1) is obvious, since $\det Q = 0$ and then Lemma 2.1 can be applied. Here (2.24) follows from (2.8), $BQ = B - QB$ and $B^2 = E$. Now we prove (2). Note that by (1.5) the function $\widetilde{\varphi}(x,i\rho)$ statisfies

$$B\frac{\mathrm{d}\overline{\widetilde{\varphi}^T}}{\mathrm{d}x}(x) - \overline{P^T(x)\widetilde{\varphi}^T(x)} = i\rho\overline{\widetilde{\varphi}^T(x)},\ 0 < x < \infty,$$
$$\overline{\widetilde{\varphi}^T(0)} = \overline{Q^T}.$$

Then one obtains (2.26) by (1) using that $R(0,P)(x) = \overline{R(-\overline{P^T},0)(x)}$. ☐

Since the solutions to the boundary value problems with $P = 0$ are entire in $\lambda$, by the transformation formulae we obtain easily the following result.

**Corollary 2.4.** *For each fixed $x$, all solutions to the boundary value problems under consideration are entire in $\lambda$.*

## 3. Proof of Theorem 1.1

We divide the proof into four steps.

**First step.** We construct a regular spectral function. Let $S$ denote the solution of (1.1) corresponding to $P=0$. Set

$$\rho = -i\lambda \quad \text{and} \quad \nu = -i\mu.$$

It is easy to see that

$$\begin{aligned} S = S(x,\lambda) &= \begin{pmatrix} \cosh(\lambda x+\mu) & \sinh(\lambda x+\mu) \\ \sinh(\lambda x+\mu) & \cosh(\lambda x+\mu) \end{pmatrix} \\ &= \begin{pmatrix} \cos(\rho x+\nu) & i\sin(\rho x+\nu) \\ i\sin(\rho x+\nu) & \cos(\rho x+\nu) \end{pmatrix} \end{aligned} \tag{3.1}$$

and

$$S^{-1} = S^{-1}(x,\lambda) = \begin{pmatrix} \cos(\rho x+\nu) & -i\sin(\rho x+\nu) \\ -i\sin(\rho x+\nu) & \cos(\rho x+\nu) \end{pmatrix}. \tag{3.2}$$

We choose two sufficiently smooth real-valued functions $\delta_n(x)$ and $\gamma_\sigma(x)$ subject to the following conditions:

$$\begin{gathered} \int_0^\infty \delta_n(x)\mathrm{d}x = 1, \\ \delta_n(x) = 0 \text{ for } x=0 \text{ and } x \ge \frac{1}{n}, \quad \delta_n(x) > 0 \text{ for } 0 < x < \frac{1}{n}, \\ \gamma_\sigma(x) = \begin{cases} 1 & \text{for } 0 \le x \le \sigma, \\ 0 & \text{for } x > \sigma+1, \end{cases} \end{gathered} \tag{3.3}$$

and it is obvious that $\delta_n(x)$ tends to the Dirac delta function $\delta(x)$ as $n\to\infty$. We set

$$\begin{aligned} D_n^\sigma(\rho) &= \big(D_{n,jm}^\sigma(\rho)\big)_{1\le j,m\le 2} \\ &= \frac{1}{2\pi}\int_0^\infty \begin{pmatrix} \cos(\rho x+\nu) & -i\sin(\rho x+\nu) \\ -i\sin(\rho x+\nu) & \cos(\rho x+\nu) \end{pmatrix} \\ &\quad \times R(P,0)(x)\delta_n(x)E\gamma_\sigma(x)\begin{pmatrix} \cos\nu & i\sin\nu \\ i\sin\nu & \cos\nu \end{pmatrix}\mathrm{d}x. \end{aligned} \tag{3.4}$$

Since the Fourier transform is a one-to-one mapping on the space of bounded continuous Lebesgue-integrable functions and $R(P,0)(x)\delta_n(x)E\gamma_\sigma(x)$ is a continuously differentiable matrix function with compact support, it is not hard to see that the matrix function $D_n^\sigma(\rho)$ is bounded and Lebesgue-integrable on the real line $-\infty<\rho<\infty$. Hence the integral

$$\int_{-\infty}^\infty S(x,i\rho)D_n^\sigma(\rho)\begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix}\mathrm{d}\rho$$

converges absolutely. By Corollary 2.2 we have $\varphi(x,i\rho) = R(0,P)(x)S(x,i\rho) = R^{-1}(P,0)(x)S(x,i\rho)$, which implies by the Fourier inverse transform that

$$\int_{-\infty}^\infty \varphi(x,i\rho)D_n^\sigma(\rho)\begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix}\mathrm{d}\rho = \delta_n(x)E \quad (0\le x\le\sigma). \tag{3.5}$$

Here and henceforth we repeatedly make use of the fact that two matrices $P_1$ and $P_2$ in the form of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are interchangeable: $P_1P_2 = P_2P_1$.

**Second step.** Now we investigate the asymptotic behavior of the matrix function as $n \to \infty$

$$\begin{aligned} U_n^\sigma(x,y) &= \big(U_{n,kl}^\sigma(x,y)\big)_{1\le k,l\le 2} \\ &:= \int_{-\infty}^{\infty} \varphi(x,i\rho)D_n^\sigma(\rho)\varphi^{-1}(y,i\rho)\mathrm{d}\rho \quad (0\le x,y\le\sigma). \end{aligned} \tag{3.6}$$

It is easy to find that

$$U_n^\sigma(x,0) = \int_{-\infty}^{\infty} \varphi(x,i\rho)D_n^\sigma(\rho)\begin{pmatrix} \cos\nu & -i\sin\nu \\ -i\sin\nu & \cos\nu \end{pmatrix}\mathrm{d}\rho = \delta_n(x)E$$

for $0\le x\le\sigma$, and

$$U_n^\sigma(0,y) = \int_{-\infty}^{\infty} \begin{pmatrix} \cos\nu & i\sin\nu \\ i\sin\nu & \cos\nu \end{pmatrix} D_n^\sigma(\rho)\varphi^{-1}(y,i\rho)\mathrm{d}\rho \quad (0\le y\le\sigma).$$

Now we show that $U_n^\sigma(0,y)=0$ for all $y\ge 0$. Indeed, first one can see from (3.4) that

$$\begin{aligned} D_n^\sigma(\rho) = \frac{1}{2\pi}\int_0^\infty &\begin{pmatrix} \cos(\rho x) & -i\sin(\rho x) \\ -i\sin(\rho x) & \cos(\rho x) \end{pmatrix} \\ &\times\{R_{11}(P,0)(x)E + R_{12}(P,0)(x)B\}\,\delta_n(x)\gamma_\sigma(x)\mathrm{d}x. \end{aligned}$$

Moreover, for any continuous scalar function $u(x)$ with compact support and $u(0)=0$, it follows easily from the theory of the Fourier cosine and sine transforms that

$$\begin{aligned} &\int_{-\infty}^{\infty}\int_0^\infty \begin{pmatrix} \cos(\rho x) & -i\sin(\rho x) \\ -i\sin(\rho x) & \cos(\rho x) \end{pmatrix} u(x) \begin{pmatrix} \cos(\rho y) & -i\sin(\rho y) \\ -i\sin(\rho y) & \cos(\rho y) \end{pmatrix}\mathrm{d}x\,\mathrm{d}\rho \\ &= 0. \end{aligned} \tag{3.7}$$

From (3.7) and $\varphi^{-1}(\cdot,i\rho) = S^{-1}(\cdot,i\rho)R(P,0)(\cdot)$ it follows that $U_n^\sigma(0,y)R(0,P)(y) = 0$ and hence $U_n^\sigma(0,y)=0$, since $R(0,P)(y)$ is invertible.

On the other hand, by (3.6) it is easy to see that, for fixed $n$ and $\sigma$, $U_n^\sigma(\sigma,\cdot)$ is a bounded differentiable function on $[0,\sigma]$ and denoted by $\Xi_n(\cdot)$ for simplicity. Therefore, since by (1.1) we easily show that

$$\frac{\mathrm{d}\varphi^{-1}(x)}{\mathrm{d}x}B - \varphi^{-1}(x)P(x) = -i\rho\varphi^{-1}(x),$$

the above argument implies that the functions

$$U_{nN}^\sigma(x,y) := \int_{-N}^{N} \varphi(x,i\rho)D_n^\sigma(\rho)\varphi^{-1}(y,i\rho)\mathrm{d}\rho \tag{3.8}$$

are continuously differentiable and satisfy the equation

$$B\frac{\partial U}{\partial x}(x,y) + \frac{\partial U}{\partial y}(x,y)B + P(x)U(x,y) - U(x,y)P(y) = 0 \text{ in } \Pi_\sigma \tag{3.9}$$

and the following conditions

$$U(x,0) = \delta_{nN}(x)E \quad (0\le x\le\sigma), \tag{3.10}$$

$$U(0,y) = \Gamma_{nN}(y), \quad U(\sigma,y) = \Xi_{nN}(y) \quad (0\le y\le\sigma), \tag{3.11}$$

where $\Pi_\sigma = \{(x,y)\in\mathbb{R}^2 : 0<x,y<\sigma\}$, the functions $\delta_{nN}, \Gamma_{nN}$ and $\Xi_{nN}$ satisfy the compatibility conditions and $\lim_{N\to\infty}\delta_{nN}(x) = \delta_n(x)$, $\lim_{N\to\infty}\Gamma_{nN}(y) = 0$ and

$\lim_{N\to\infty} \Xi_{nN}(y) = \Xi_n(y)$. We should note that problem (3.9), (3.10) and (3.11) can be rewritten as a symmetric hyperbolic system:

$$\begin{gathered}\frac{\partial V}{\partial y}(x,y) + \begin{pmatrix} 0 & E \\ E & 0 \end{pmatrix} \frac{\partial V}{\partial x}(x,y) + C(x,y)V(x,y) = 0 \quad \text{in } \Pi_\sigma, \\ V(x,0) = \delta_{nN}(x)\overrightarrow{H} \quad (0 \le x \le \sigma), \\ V(0,y) = \overrightarrow{\Gamma}_{nN}(y), \quad V(\sigma,y) = \overrightarrow{\Xi}_{nN}(y) \quad (0 \le y \le \sigma),\end{gathered} \tag{3.12}$$

where

$$V(x,y) = \begin{pmatrix} U_{11}(x,y) \\ U_{12}(x,y) \\ U_{21}(x,y) \\ U_{22}(x,y) \end{pmatrix}, \quad \overrightarrow{H} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\overrightarrow{\Gamma}_{nN}(y) = \begin{pmatrix} \Gamma_{nN,11}(y) \\ \Gamma_{nN,12}(y) \\ \Gamma_{nN,21}(y) \\ \Gamma_{nN,22}(y) \end{pmatrix}, \quad \overrightarrow{\Xi}_{nN}(y) = \begin{pmatrix} \Xi_{nN,11}(y) \\ \Xi_{nN,12}(y) \\ \Xi_{nN,21}(y) \\ \Xi_{nN,22}(y) \end{pmatrix}$$

and $C(x,y)$ is the $4 \times 4$ matrix-valued function

$$\begin{pmatrix} -P_{12}(y) & P_{11}(x) - P_{22}(y) & 0 & P_{12}(x) \\ P_{11}(x) - P_{11}(y) & -P_{21}(y) & P_{12}(x) & 0 \\ 0 & P_{21}(x) & -P_{12}(y) & P_{22}(x) - P_{22}(y) \\ P_{21}(x) & 0 & P_{22}(x) - P_{11}(y) & -P_{21}(y) \end{pmatrix}.$$

Since $BU^\sigma_{nN}(x,y) = U^\sigma_{nN}(x,y)B$, a direct calculation shows that the symmetric hyperbolic system (3.12) is actually equivalent to the following normal hyperbolic system

$$\begin{gathered}\frac{\partial v}{\partial y}(x,y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial v}{\partial x}(x,y) + c(x,y)v(x,y) \quad \text{in } \Pi_\sigma, \\ v(x,0) = \delta_{nN}(x)\overrightarrow{h} \quad (0 \le x \le \sigma), \\ v_2(0,y) = 2v_1(0,y) - \Gamma_{nN,11}(y) + 3\Gamma_{nN,12}(y), \\ v_2(\sigma,y) = 2v_1(\sigma,y) - \Xi_{nN,11}(y) + 3\Xi_{nN,12}(y) \quad (0 \le y \le \sigma),\end{gathered} \tag{3.13}$$

where

$$v(x,y) = \begin{pmatrix} v_1(x,y) \\ v_2(x,y) \end{pmatrix} = \begin{pmatrix} U_{11}(x,y) - U_{12}(x,y) \\ U_{11}(x,y) + U_{12}(x,y) \end{pmatrix}, \quad \overrightarrow{h} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

and

$$\begin{aligned} &c(x,y) \\ &= \begin{pmatrix} (P_{11} - P_{12})(x) + (P_{12} - P_{11})(y) & (P_{12} - P_{11})(x) + (P_{22} - P_{21})(y) \\ (P_{11} + P_{12})(y) - (P_{11} + P_{12})(x) & (P_{22} + P_{21})(y) - (P_{11} + P_{12})(x) \end{pmatrix}. \end{aligned}$$

If we take the variable $y$ as time, then it is not hard to verify that the classical Uniform Kreiss Condition holds, and hence from the well-known results of well-posedness on linear hyperbolic systems (cf. [9] and references therein) we see that (3.13) has a unique solution; that is, there exists a unique solution $U^\sigma_{nN}(x,y)$ to problem (3.9), (3.10) and (3.11) such that $U^\sigma_{nN}(x,y) \to U^\sigma_n(x,y)$ as $N \to \infty$.

On the other hand, if we set $W_{nN}^\sigma(x,y)=U_{nN}^\sigma(x,y)-\delta_{nN}(x-y)E$ for $0\le x,y\le \sigma$ where $\delta_{nN}(x-y)=0$ for $0\le x<y\le\sigma$, then $W_{nN}^\sigma(x,y)$ satisfies the equation

$$\begin{aligned} &B\frac{\partial W}{\partial x}(x,y)+\frac{\partial W}{\partial y}(x,y)B+P(x)W(x,y)-W(x,y)P(y)\\ &=\delta_{nN}(x-y)\left(P(y)-P(x)\right) \end{aligned} \tag{3.14}$$

and $W(x,0)=0$. From the compatibility conditions it follows that $W_{nN}^\sigma(0,y)\to 0$ as $N,n\to\infty$. Next we show that

$$W_{nN}^\sigma(\sigma,y)\to 0 \quad (N,n\to\infty). \tag{3.15}$$

From (3.1), (3.2), (3.5) and the transformation formulae $\varphi(\cdot,i\rho)= R(0,P)(\cdot)S(\cdot,i\rho)= R^{-1}(P,0)(\cdot)S(\cdot,i\rho)$, $\varphi^{-1}(\cdot,i\rho)=S^{-1}(\cdot,i\rho)R(P,0)(\cdot)$ it follows that

$$\begin{aligned} \Xi_n(y)&= R(0,P)(\sigma)R(P,0)(y)R(P,0)(\sigma-y)\\ &\quad\times\int_{-\infty}^{\infty}\varphi(\sigma-y,i\rho)D_n^\sigma(\rho)\begin{pmatrix}\cos\nu & -i\sin\nu\\ -i\sin\nu & \cos\nu\end{pmatrix}\mathrm{d}\rho\\ &= R(0,P)(\sigma)R(P,0)(y)R(P,0)(\sigma-y)\delta_n(\sigma-y)E \quad (0\le x\le\sigma). \end{aligned}$$

Therefore,

$$\begin{aligned} &\Xi_n(y)-\delta_n(\sigma-y)E\\ &=\delta_n(\sigma-y)R(0,P)(\sigma)[R(P,0)(y)R(P,0)(\sigma-y)-R(P,0)(\sigma)] \end{aligned} \tag{3.16}$$

whence (3.15) follows easily. Consequently, by the well-posedness of symmetric hyperbolic linear differential equations, we have

$$W_{nN}^\sigma(x,y)\to 0 \quad \text{as } N,n\to\infty$$

since $\delta_{nN}(x-y)\to\delta(x-y)$ as $N,n\to\infty$ and hence the right hand side of (3.14) tends to 0. Therefore, for $0\le x,y\le\sigma$,

$$U_n^\sigma(x,y)\to\delta(x-y)E \quad (n\to\infty). \tag{3.17}$$

We remark that there is another and simpler way to prove (3.17) in which it is not needed to consider (3.9). The key idea is based on considering (3.6), (3.7) and (3.16) with replacing $\sigma$ by $x$. We leave the details to the reader.

**Third step.** We prove the Marchenko-Parseval equality (1.2). Assuming that $f,g\in\left(\mathbb{K}_\sigma^2(0,\infty)\right)^2$ have compact support, by changing the order of integration we have

$$\begin{aligned} &\int_0^\infty\int_0^\infty\sum_{k,l=1}^2 U_{n,kl}^\sigma(x,y)\overline{g^{(k)}(x)}f^{(l)}(y)\mathrm{d}x\mathrm{d}y\\ &=\int_0^\infty\int_0^\infty\sum_{k,l=1}^2\Big(\int_{-\infty}^\infty\sum_{j,m=1}^2 D_{n,jm}^\sigma(\rho)\varphi_{[j]}^{(k)}(x,i\rho)\psi_{[m]}^{(l)}(y,i\rho)\mathrm{d}\rho\Big)\overline{g^{(k)}(x)}f^{(l)}(y)\mathrm{d}x\mathrm{d}y\\ &=\sum_{j,m=1}^2\int_{-\infty}^\infty\mathrm{d}\rho D_{n,jm}^\sigma(\rho)\Big(\int_0^\infty\int_0^\infty\sum_{k,l=1}^2 f^{(l)}(y)\psi_{[m]}^{(l)}(y,i\rho)\varphi_{[j]}^{(k)}(x,i\rho)\overline{g^{(k)}(x)}\mathrm{d}x\mathrm{d}y\Big)\\ &=\sum_{j,m=1}^2\int_{-\infty}^\infty D_{n,jm}^\sigma(\rho)\omega_f^m(\rho)\eta_{\overline{g}}^j(\rho)\mathrm{d}\rho. \end{aligned}$$

Therefore, in view of (3.17), by letting $n \to \infty$ we obtain that for any $f, g$ in $\big(\mathbb{K}^2_\sigma(0,\infty)\big)^2$,

$$\int_0^\infty f^T(x)\overline{g(x)}\mathrm{d}x = \lim_{n\to\infty}\sum_{j,m=1}^{2}\int_{-\infty}^{\infty} D^\sigma_{n,jm}(\rho)\omega^m_f(\rho)\eta^j_g(\rho)\mathrm{d}\rho. \tag{3.18}$$

By the definition of $D^\sigma_n(\rho)$ (see (3.4)), we easily see that

$$\lim_{n\to\infty} D^\sigma_n(\rho) = \frac{1}{2\pi}R(P,0)(0) = \frac{1}{2\pi}E. \tag{3.19}$$

On the other hand, since the Fourier transform is a continuous mapping of $L^2(\mathbb{R})$ into $L^2(\mathbb{R})$, it follows easily from Corollary 2.2 and the zero extensions of $f$ and $g$ on $\mathbb{R}$ that both $\omega^m_f(\rho)$ and $\eta^j_g(\rho)$ belong to $L^2(\mathbb{R})$. Therefore, combining (3.18) and (3.19) and letting $\sigma \to \infty$, we can assert (1.2) by the boundedness of $D^\sigma_n(\cdot)$, the dominated convergence theorem and the fact that $\big(\mathbb{K}^2(0,\infty)\big)^2$ is dense in $\big(L^2(0,\infty)\big)^2$.

**Forth step.** We prove the expansion (1.3). First we assume that $f \in (C_0[0,\infty))^2$, where $(C_0[0,\infty))^2$ denotes the product space of the set of all continuous functions with compact support. For any fixed real number $x \geq 0$ and $\delta > 0$, set

$$\varsigma(t) = \begin{cases} 1/\delta & \text{for } t \in (x, x+\delta), \\ 0 & \text{otherwise.} \end{cases} \tag{3.20}$$

In (1.2) first letting $g^{(1)}(t) = \varsigma(t), g^{(2)}(t) = 0$ and then letting $g^{(1)}(t) = 0, g^{(2)}(t) = \varsigma(t)$, we have

$$\frac{1}{\delta}\int_x^{x+\delta} f(t)\mathrm{d}t = \frac{1}{2\pi}\sum_{k=1}^{2}\int_{-\infty}^{\infty}\omega^k_f(\rho)\frac{1}{\delta}\int_x^{x+\delta}\varphi_{[k]}(t,i\rho)\mathrm{d}t\,\mathrm{d}\rho.$$

Since

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_x^{x+\delta} f(t)\mathrm{d}t = f(x)$$

and in $Z$,

$$\lim_{\delta\to 0}\omega^k_f(\rho)\frac{1}{\delta}\int_x^{x+\delta}\varphi_{[k]}(t,i\rho)\mathrm{d}t = \omega^k_f(\rho)\varphi_{[k]}(x,i\rho),$$

we prove the first part of (1.3) by the dominated convergence theorem if $f \in (C_0[0,\infty))^2$. For the case of $f \in \big(\mathbb{K}^2(0,\infty)\big)^2$ we can approximate $f$ by the functions in $(C_0[0,\infty))^2$. The second part of (1.3) can be proved similarly.

## 4. Proof of Theorem 1.2

First we prove the theorem for a special case. Recall that $S$ and $\widetilde{S}$ are the solutions corresponding to $P = 0$ in (1.4) and (1.5), respectively.

**Lemma 4.1.** *For $f, g \in \big(L^2(0,\infty)\big)^4$, it holds*

$$\int_0^\infty f(x)g(x)\mathrm{d}x = \frac{1}{\pi}\int_{-\infty}^{\infty}\Theta_f(\rho)\widetilde{\Theta}_g(\rho)\mathrm{d}\rho = \frac{1}{\pi}\int_{-\infty}^{\infty}\Theta_f(\rho)Q\widetilde{\Theta}_g(\rho)\mathrm{d}\rho$$

*and for $x > 0$,*

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty}\Theta_f(\rho)\widetilde{S}(x,i\rho)\mathrm{d}\rho = \frac{1}{\pi}\int_{-\infty}^{\infty}S(x,i\rho)\widetilde{\Theta}_f(\rho)\mathrm{d}\rho,$$

*where $\Theta_f(\rho)$ and $\widetilde{\Theta}_g(\rho)$ are defined by*

$$\Theta_f(\rho) = \int_0^\infty f(x)S(x,i\rho)\mathrm{d}x, \quad \widetilde{\Theta}_g(\rho) = \int_0^\infty \widetilde{S}(x,i\rho)g(x)\mathrm{d}x. \tag{4.1}$$

*Proof.* It is easy to find that

$$\begin{aligned} S(x,i\rho) &= Q\cosh(i\rho x) + BQ\sinh(i\rho x),\\ \widetilde{S}(x,i\rho) &= Q\cosh(i\rho x) - QB\sinh(i\rho x), \end{aligned} \tag{4.2}$$

then we have

$$\begin{aligned} \Theta_f(\rho) &= \int_0^\infty f(x)S(x,i\rho)\mathrm{d}x = \frac{1}{2}\widehat{f}(\rho)(Q-BQ) + \frac{1}{2}\widehat{f}(-\rho)(Q+BQ),\\ \widetilde{\Theta}_g(\rho) &= \int_0^\infty \widetilde{S}(x,i\rho)g(x)\mathrm{d}x = \frac{1}{2}(Q+QB)\widehat{g}(\rho) + \frac{1}{2}(Q-QB)\widehat{g}(-\rho), \end{aligned}$$

where $\widehat{f}(\rho) = \int_0^\infty f(x)\exp(-i\rho x)\mathrm{d}x$ denotes the Fourier transform of $f(x)$. Therefore, by the well-known Parseval equality

$$\int_0^\infty f(x)g(x)\mathrm{d}x = \frac{1}{2\pi}\int_{-\infty}^\infty \widehat{f}(\rho)\widehat{g}(-\rho)\mathrm{d}\rho$$

and the identity for $u,v \in L^2(0,\infty)$

$$\int_{-\infty}^\infty \widehat{u}(\rho)\widehat{v}(\rho)\mathrm{d}\rho = 0,$$

we obtain easily (note that $Q^2 = Q$)

$$\frac{1}{\pi}\int_{-\infty}^\infty \Theta_f(\rho)\widetilde{\Theta}_g(\rho)\mathrm{d}\rho = \frac{1}{\pi}\int_{-\infty}^\infty \Theta_f(\rho)Q\widetilde{\Theta}_g(\rho)\mathrm{d}\rho = \int_0^\infty f(x)g(x)\mathrm{d}x.$$

On the other hand, since for all $u \in L^2(0,\infty)$ and $x > 0$ it holds that

$$\int_{-\infty}^\infty \widehat{u}(\rho)\exp(-i\rho x)\mathrm{d}\rho = \int_{-\infty}^\infty \widehat{u}(-\rho)\exp(i\rho x)\mathrm{d}\rho = 0,$$

we have

$$\begin{aligned} &\frac{1}{\pi}\int_{-\infty}^\infty \Theta_f(\rho)\widetilde{S}(x,i\rho)\mathrm{d}\rho\\ &= \frac{1}{2}f(x)(Q-BQ)(Q-QB) + \frac{1}{2}f(x)(Q+BQ)(Q+QB) = f(x). \end{aligned}$$

Similarly, we can show that $\frac{1}{\pi}\int_{-\infty}^\infty S(x,i\rho)\widetilde{\Theta}_f(\rho)\mathrm{d}\rho = f(x)$. □

By putting

$$f(x) = F(x)R(P,0)(x) + \int_x^\infty F(t)K(P,0;Q)(t,x)\mathrm{d}t \tag{4.3}$$

and

$$g(x) = R(0,P)(x)G(x) + \int_x^\infty \overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(t,x)}G(t)\mathrm{d}t, \tag{4.4}$$

where $F$ and $G$ can be obtained by solving the above Volterra equations of the second kind, it follows from changing the order of integration and the transformation formulae (2.22) and (2.26) that

$$\Phi_f(\rho) = \Theta_F(\rho),\ \widetilde{\Phi}_g(\rho) = \widetilde{\Theta}_G(\rho). \tag{4.5}$$

Furthermore, we have the following lemma.

**Lemma 4.2.** *For $f, g \in \big(L^2(0,\infty)\big)^4$, it holds that*

$$\int_0^\infty f(x)g(x)\mathrm{d}x = \int_0^\infty F(x)G(x)\mathrm{d}x + \int_0^\infty\int_0^\infty F(y)\mathfrak{F}(x,y)G(x)\mathrm{d}x\mathrm{d}y, \tag{4.6}$$

*where*

$$\mathfrak{F}(x,y) = \begin{cases} R(P,0)(y)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)} \\ +\int_0^y K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}\mathrm{d}t, \quad 0 \le y \le x, \\ K(P,0;Q)(y,x)R(0,P)(x) \\ +\int_0^x K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}\mathrm{d}t, \quad 0 \le x \le y. \end{cases} \tag{4.7}$$

*Proof.* On the one hand, since $R(P,0)(\cdot) = R^{-1}(0,P)(\cdot)$, we have by changing of the order of integration

$$\begin{aligned}
&\int_0^\infty f(x)g(x)\mathrm{d}x \\
&= \int_0^\infty \Big\{F(x)R(P,0)(x) + \int_x^\infty F(t)K(P,0;Q)(t,x)\mathrm{d}t\Big\} \\
&\quad \times \Big\{R(0,P)(x)G(x) + \int_x^\infty \overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(t,x)}G(t)\mathrm{d}t\Big\}\mathrm{d}x \\
&= \int_0^\infty F(x)G(x)\mathrm{d}x + \int_0^\infty\int_x^\infty F(x)R(P,0)(x)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(t,x)}G(t)\mathrm{d}t\mathrm{d}x \\
&\quad + \int_0^\infty\int_0^t F(t)K(P,0;Q)(t,x)R(0,P)(x)G(x)\mathrm{d}x\mathrm{d}t \\
&\quad + \int_0^\infty\int_x^\infty\int_x^\infty F(t)K(P,0;Q)(t,x)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(s,x)}G(s)\mathrm{d}t\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

On the other hand, by (4.7),

$$\begin{aligned}
&\int_0^\infty\int_0^\infty F(y)\mathfrak{F}(x,y)G(x)\mathrm{d}x\mathrm{d}y \\
&= \int_0^\infty\int_0^y F(y)\Big\{K(P,0;Q)(y,x)R(0,P)(x) \\
&\quad + \int_0^x K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}\mathrm{d}t\Big\}G(x)\mathrm{d}x\mathrm{d}y \\
&\quad + \int_0^\infty\int_y^\infty F(y)\Big\{R(P,0)(y)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)} \\
&\quad + \int_0^y K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}\mathrm{d}t\Big\}G(x)\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

Therefore, proving (4.6), it is equivalent to showing that

$$\begin{aligned}
&\int_0^\infty\int_x^\infty\int_x^\infty F(t)K(P,0;Q)(t,x)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(s,x)}G(s)\mathrm{d}t\mathrm{d}s\mathrm{d}x \\
&= \int_0^\infty\int_0^y\int_0^x F(y)K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}G(x)\mathrm{d}t\mathrm{d}x\mathrm{d}y
\end{aligned}$$

$$+\int_0^\infty \int_y^\infty \int_0^y F(y)K(P,0;Q)(y,t)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,t)}G(x)\mathrm{d}t\mathrm{d}x\mathrm{d}y,$$

which can be easily proved by changing of the order of integration. $\square$

From (2.4) and Lemma 2.1 it follows that $\mathfrak{F}(\cdot,\cdot) \in \big(C^1(\overline{\Omega})\big)^4$ and $\mathfrak{F}(\cdot,\cdot) \in \big(C^1(\overline{\mathbb{R}^2_+ \setminus \Omega})\big)^4$.

**Lemma 4.3.** *For $\mathfrak{F}(x,y)$ defined by (4.7), it holds that*

$$\frac{\partial \mathfrak{F}}{\partial x}(x,y)B + B\frac{\partial \mathfrak{F}}{\partial y}(x,y) = 0, \tag{4.8}$$

$$\mathfrak{F}(x,0) = \mathcal{J}(x), \quad \mathfrak{F}(0,y) = \mathcal{L}(y), \tag{4.9}$$

*where*

$$\mathcal{J}(x) = \overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,0)}, \quad \mathcal{L}(y) = K(P,0;Q)(y,0). \tag{4.10}$$

*Moreover,*

$$\mathcal{J}(x) - B\mathcal{J}(x)B = \mathcal{L}(x) - B\mathcal{L}(x)B. \tag{4.11}$$

*Proof.* For $y \le x$, in view of (2.7)–(2.9) in Lemma 2.1, we see by integration by parts that

$$\begin{aligned}
&\frac{\partial \mathfrak{F}}{\partial x}(x,y)B + B\frac{\partial \mathfrak{F}}{\partial y}(x,y)\\
&= \Big\{R(P,0)(y)\overline{K_x^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)}\\
&\quad + \int_0^y K(P,0;Q)(y,s)\overline{K_x^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,s)}\mathrm{d}s\Big\}B\\
&\quad + B\Big\{R'(P,0)(y)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)}\\
&\quad + K(P,0;Q)(y,y)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)}\Big\}\\
&\quad + B\Big\{R(P,0)(y)\overline{K_y^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)}\\
&\quad + \int_0^y K_y(P,0;Q)(y,s)\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,s)}\mathrm{d}s\Big\}\\
&= \big\{BR'(P,0)(y) - R(P,0)(y)P(y) + BK(P,0;Q)(y,y) - K(P,0;Q)(y,y)B\big\}\\
&\quad \times \overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,y)} + K(P,0;Q)(y,0)B\overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,0)}\\
&\quad + \int_0^y \big\{BK_y(P,0;Q)(y,s) + K_s(P,0;Q)(y,s)B - K(P,0;Q)(y,s)P(s)\big\}\\
&\quad \times \overline{K^T\big(-\overline{P^T},0;\overline{Q^T}\big)(x,s)}\mathrm{d}s = 0,
\end{aligned}$$

where we have used the relation: $B = QB + BQ$. For the case $x \le y$, the proof of (4.8) is similar. On the other hand, (4.9) is obvious by (4.7).

Furthermore, it can be directly verified that the unique solution of problem (4.8) and (4.9) is

$$\mathfrak{F}(x,y) = \begin{cases} \frac{1}{2}\{\mathcal{J}(x+y) + \mathcal{J}(x-y)\} - \frac{1}{2}B\{\mathcal{J}(x+y) - \mathcal{J}(x-y)\}B, & y \le x,\\ \frac{1}{2}\{\mathcal{L}(x+y) + \mathcal{L}(y-x)\} - \frac{1}{2}B\{\mathcal{L}(x+y) - \mathcal{L}(y-x)\}B, & x \le y.\end{cases}$$

Consequently, (4.11) follows from the continuity of $\mathfrak{F}(x,y)$ at $x = y$. $\square$

Now we apply Lemma 4.3 to show the following lemma.

**Lemma 4.4.** *It holds that*

$$\Theta_{\mathcal{J}}(\rho)Q = \Theta_{\mathcal{J}}(\rho) = \widetilde{\Theta}_{\mathcal{L}}(\rho) = Q\widetilde{\Theta}_{\mathcal{L}}(\rho).$$

*Proof.* By (4.1), we have

$$\Theta_{\mathcal{J}}(\rho) = \int_0^\infty \mathcal{J}(x)S(x,i\rho)\mathrm{d}x, \quad \widetilde{\Theta}_{\mathcal{L}}(\rho) = \int_0^\infty \widetilde{S}(x,i\rho)\mathcal{L}(x)\mathrm{d}x,$$

where $S(x,i\rho)$ and $\widetilde{S}(x,i\rho)$ are given by (4.2). Since $Q^2 = Q$, it is sufficient to prove that for all $x \geq 0$,

$$\mathcal{J}(x)Q = Q\mathcal{L}(x), \quad \mathcal{J}(x)BQ = -QB\mathcal{L}(x).$$

First, multiplying right (4.11) by $Q$, we obtain by $QB+BQ = B$ and $\mathcal{L}(x)Q = \mathcal{L}(x)$ which follows from (2.24) and (4.10) that

$$\{\mathcal{J}(x) - B\mathcal{J}(x)B\}Q = \mathcal{L}(x)Q - B\mathcal{L}(x)(B - QB) = \mathcal{L}(x). \tag{4.12}$$

Second, since it follows from (2.27) and (4.10) that $Q\mathcal{J}(x) = \mathcal{J}(x)$, we have $QB\mathcal{J}(x) = (B - BQ)\mathcal{J}(x) = 0$. Consequently, it follows from (4.12) that

$$Q\mathcal{L}(x) = Q\{\mathcal{J}(x) - B\mathcal{J}(x)B\}Q = \mathcal{J}(x)Q - QB\mathcal{J}(x)BQ = \mathcal{J}(x)Q.$$

On the other hand, multiplying left (4.12) by $B$, by $B^2 = E$ we have

$$B\mathcal{J}(x)Q - \mathcal{J}(x)BQ = B\mathcal{L}(x) = (QB + BQ)\mathcal{L}(x);$$

that is,

$$\mathcal{J}(x)BQ + QB\mathcal{L}(x) = B\{\mathcal{J}(x)Q - Q\mathcal{L}(x)\} = 0.$$

Thus the proof is complete. $\square$

*Proof of Theorem 1.2.* Let $\mathcal{L}_\sigma(x) = \gamma_\sigma(x)\mathcal{L}(x)$, $\mathcal{J}_\sigma(x) = \gamma_\sigma(x)\mathcal{J}(x)$, where the scalar function $\gamma_\sigma(x)$ is defined by (3.3). It is obvious that both $\mathcal{L}_\sigma$ and $\mathcal{J}_\sigma$ are continuously differentiable matrix-valued functions with compact support. Then it follows easily from Lemma 4.4 that $\Theta_{\mathcal{J}_\sigma}(\rho) = \widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho)$. Hence, combining (1.4), (1.5), (4.2), $Q\mathcal{J}(\cdot) = \mathcal{J}(\cdot)$, $\mathcal{L}(\cdot)Q = \mathcal{L}(\cdot)$ and Lemma 4.1, we conclude easily that the following matrix-valued function

$$\mathfrak{F}_\sigma(x,y) := \frac{1}{\pi}\int_{-\infty}^\infty S(y,i\rho)\Theta_{\mathcal{J}_\sigma}(\rho)\widetilde{S}(x,i\rho)\mathrm{d}\rho = \frac{1}{\pi}\int_{-\infty}^\infty S(y,i\rho)\widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\widetilde{S}(x,i\rho)\mathrm{d}\rho$$

satisfies the equation

$$U_xB + BU_y = 0,$$

and the conditions

$$U(x,0) = \mathcal{J}_\sigma(x), \quad U(0,y) = \mathcal{L}_\sigma(y)$$

for all $x, y > 0$. Therefore, if we define $\mathfrak{F}_\sigma(0,0) = \mathcal{L}(0) = \mathcal{J}(0)$, then $\mathfrak{F}_\sigma(x,y) = \mathfrak{F}(x,y)$ in the domain $0 \leq x, y \leq \sigma$, since $\gamma_\sigma(x) \equiv 1$ on $[0,\sigma]$ and the two matrix-valued functions satisfy the same boundary problem as that in Lemma 4.3. Moreover, if $f, g \in \big(\mathbb{K}_\sigma^2(0,\infty)\big)^4$, then it follows from (4.3) and (4.4) that

$F(x) = G(x) = 0$ for $x > \sigma$. Consequently, it follows from (4.1), (4.5), Lemmas 4.1 and 4.2 that

$$\begin{aligned} \int_0^\infty f(x)g(x)\mathrm{d}x &= \int_0^\infty F(x)G(x)\mathrm{d}x + \int_0^\infty \int_0^\infty F(y)\mathfrak{F}(x,y)G(x)\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty F(x)G(x)\mathrm{d}x + \int_0^\infty \int_0^\infty F(y)\mathfrak{F}_\sigma(x,y)G(x)\mathrm{d}x\mathrm{d}y \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \Theta_F(\rho)\{Q + \Theta_{\mathcal{J}_\sigma}(\rho)\}\widetilde{\Theta}_G(\rho)\mathrm{d}\rho \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \Phi_f(\rho)\{Q + \widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\}\widetilde{\Phi}_g(\rho)\mathrm{d}\rho. \end{aligned} \tag{4.13}$$

Now we define

$$D(\rho) = \frac{1}{\pi}\lim_{\sigma\to\infty}\{Q + \Theta_{\mathcal{J}_\sigma}(\rho)\} = \frac{1}{\pi}\lim_{\sigma\to\infty}\{Q + \widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho)\} \tag{4.14}$$

where the limits exist in the sense of convergence of distributions. Indeed, by (4.1) and (4.2) we see that both $\Theta_{\mathcal{J}_\sigma}(\rho)$ and $\widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho)$ are linear combination of the Fourier cosine and sine transform of some matrix-valued function with compact support. Then it follows from the property of the Fourier transform (see e.g. Page 105 in [21]) that $\Theta_{\mathcal{J}_\sigma}(\rho) \to \Theta_{\mathcal{J}}(\rho)$ and $\widetilde{\Theta}_{\mathcal{L}_\sigma}(\rho) \to \widetilde{\Theta}_{\mathcal{L}}(\rho)$ as $\sigma \to \infty$ in the sense of distributions, whence $D(\rho) \in (Z')^4$. Therefore, by (4.14) we see

$$D(\rho) = \frac{1}{\pi}\{Q + \Theta_{\mathcal{J}}(\rho)\} = \frac{1}{\pi}\{Q + \widetilde{\Theta}_{\mathcal{L}}(\rho)\}. \tag{4.15}$$

Therefore, by (4.13), (4.14) and (4.15) we can prove the Marchenko-Parseval equality (1.7) similarly to (1.2). Moreover, if we let $g(t) = \varsigma(t)E$ or $f(t) = \varsigma(t)E$ where $\varsigma(t)$ is defined by (3.20), then we can prove (1.8) similarly to (1.3). $\square$

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