Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 145, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

INTEGRAL SOLUTIONS OF FRACTIONAL EVOLUTION EQUATIONS WITH NONDENSE DOMAIN

HAIBO GU, YONG ZHOU, BASHIR AHMAD, AHMED ALSAEDI

Communicated by Mokhtar Kirane

Abstract. In this article, we study the existence of integral solutions for two classes of fractional order evolution equations with nondensely defined linear operators. First, we consider the nonhomogeneous fractional order evolution equation and obtain its integral solution by Laplace transform and probability density function. Subsequently, based on the form of integral solution for nonhomogeneous fractional order evolution equation, we investigate the existence of integral solution for nonlinear fractional order evolution equation by noncompact measure method.

1. Introduction

We consider the nonhomogeneous fractional order evolution equation

$$\begin{gathered} {}^C D_{0+}^q u(t) = Au(t) + f(t), \quad t \in (0, b], \\ u(0) = u_0 \end{gathered} \tag{1.1}$$

and the nonlinear fractional order evolution equation

$$\begin{gathered} {}^C D_{0+}^q u(t) = Au(t) + g(t, u(t)), \quad t \in (0, b], \\ u(0) = u_0, \end{gathered} \tag{1.2}$$

where ${}^C D_{0+}^q$ is the Caputo fractional derivative of order $0 < q < 1$, the state $u(\cdot)$ takes values in a Banach space X with norm $|\cdot|$, $A : D(A) \subseteq X \to X$ is a nondensely closed linear operator on X, f and g are given functions satisfying appropriate conditions.

For the integer order evolution equation:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & t \in (0, b], \\ u(0) = u_0, \end{cases}$$

in case A is a Hille-Yosida operator and is densely defined (i.e., $\overline{D(A)} = X$), the problem has been extensively studied (see [15]). When A is a Hille-Yosida operator but its domain is nondensely defined, there have many results (see [2, 8, 18, 19]

2010 *Mathematics Subject Classification.* 26A33, 34K37, 37L05, 47J35.
Key words and phrases. Fractional evolution equation; Caputo derivative; integral solution; nondense domain.
©2017 Texas State University.
Submitted March 11, 2017. Published June 19, 2017.

and the references therein). It is noted that Da Prato and Sinestrari are the first to work on equations with nondense domains, see [5].

On the other hand, fractional order differential equations have recently been applied in various areas of engineering, physics and bio-engineering, and other applied sciences. For some fundamental results in the theory of fractional calculus and fractional differential equations, we refer the reader to the monographs by Samko et al. [17], Kilbas et al. [9], Diethelm [4] and Zhou [30], and the papers [1, 10, 29, 24, 27, 25, 26, 28] and the references cited therein.

For nonlinear fractional evolution equation (1.2) with initial data or nonlocal condition, when A is densely defined, there have been many results on the existence of mild solutions (see [11, 16, 20, 23]). In [23], by using similar methods due to El-Borai [6, 7], Zhou and Jiao proposed a suitable concept on mild solution by applying probability density function and Laplace transform, which is widely used now. When A is not densely defined, there have been some investigations (see [13, 22]). However, in [13], there was an error in transforming integral solution into an available form. Zhang et al. [22] presented a formula for integral solution by using the similar method described in [23], but the equivalency of integral equations was not proved.

Motivated by the above discussion, in this paper, we will firstly give the integral solution for nonhomogeneous fractional evolution equation (1.1) by Laplace transform and probability density function, and subsequently investigate the existence of integral solution for nonlinear fractional order evolution equation (1.2) by Ascoli-Arzela theorem and the measure of noncompactness. In what follows we do not require the C_0-semigroup (will be given later) to be compact.

The rest of this paper is organized as follows. In Section 2, notation and preliminaries are given. The integral solution of nonhomogeneous fractional evolution equation (1.1) is given in Section 3. In Section 4, the existence of integral solution for nonlinear fractional order evolution equation (1.2) is studied. The paper concludes with a problem proposed for further research.

2. Preliminaries

In this section, we recall some concepts on fractional calculus and present some lemmas and assumptions which are useful in the sequel.

Let $p > 0$, $n = \lceil p \rceil$ (the least integer greater than or equal to p) and $u \in L^1([0,b], X)$. The Riemann-Liouville fractional integral is defined by

$$I_{0+}^p u(t) = g_p(t) * u(t) = \int_0^t g_p(t-s)u(s)ds, \ t > 0,$$

where $*$ denotes convolution and $g_p(t) = t^{p-1}/\Gamma(p)$. In case $p = 0$, we set $g_0(t) = \delta(t)$, the Dirac measure concentrated at the origin. For $u \in C([0,b], X)$, the Riemann-Liouville fractional derivative is defined by

$${}^L D_{0+}^p u(t) = \frac{d^n}{dt^n}(g_{n-p}(t) * u(t))$$

and the Caputo fractional derivative can be defined by

$${}^C D_{0+}^p u(t) = g_{n-p}(t) * \frac{d^n u(t)}{dt^n}$$

for all $t > 0$. For more details, see [9].

Next, we introduce the Hausdorff measure of noncompactness $\beta(\cdot)$ defined on each bounded subset Ω of Banach space X by

$$\beta(\Omega) = \inf\{\epsilon > 0, \Omega \text{ has a finite } \epsilon\text{-net in } X\}.$$

Some basic properties of $\beta(\cdot)$ are listed in the following Lemmas.

Lemma 2.1 ([3]). *The noncompact measure $\beta(\cdot)$ satisfies:*

(i) *for all bounded subsets B_1, B_2 of X, $B_1 \subseteq B_2$ implies $\beta(B_1) \le \beta(B_2)$;*
(ii) *$\beta(\{x\} \cup B) = \beta(B)$ for every $x \in X$ and every nonempty subset $B \subseteq X$;*
(iii) *$\beta(B) = 0$ if and only if B is relatively compact in X;*
(iv) *$\beta(B_1 + B_2) \le \beta(B_1) + \beta(B_2)$, where $B_1 + B_2 = \{x + y : x \in B_1,\ y \in B_2\}$;*
(v) *$\beta(B_1 \cup B_2) \le \max\{\beta(B_1), \beta(B_2)\}$;*
(vi) *$\beta(\lambda B) \le |\lambda|\beta(B)$ for any $\lambda \in \mathbb{R}$.*

Lemma 2.2 ([14]). *Let $J = [0, b]$ and $\{u_n\}_{n=1}^{\infty}$ be a sequence of Bochner integrable functions from J into X with $|u_n(t)| \le \tilde{m}(t)$ for almost all $t \in J$ and every $n \ge 1$, where $\tilde{m} \in L(J, \mathbb{R}^+)$. Then the function $\psi(t) = \beta(\{u_n(t)\}_{n=1}^{\infty})$ belongs to $L(J, \mathbb{R}^+)$ and satisfies*

$$\beta\Big(\Big\{ \int_0^t u_n(s)ds : n \ge 1\Big\}\Big) \le 2\int_0^t \psi(s)ds.$$

Let $X_0 = \overline{D(A)}$ and A_0 be the part of A in $\overline{D(A)}$ defined by

$$D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\}, \quad A_0 x = Ax.$$

Proposition 2.3 ([15]). *The part A_0 of A generates a strongly continuous semigroup(that is, C_0-semigroup) $\{Q(t)\}_{t\ge 0}$ on X_0.*

In the forthcoming analysis, we need the following hypothesis:

(H1) The linear operator $A : D(A) \subset X \to X$ satisfies the Hille-Yosida condition, that is, there exist two constant $\omega \in \mathbb{R}$ and $\overline{M}$ such that $(\omega, +\infty) \subseteq \rho(A)$ and

$$\|(\lambda I - A)^{-k}\|_{\mathcal{L}(X)} \le \frac{\overline{M}}{(\lambda - \omega)^k}, \quad \text{for all } \lambda > \omega,\ k \ge 1.$$

(H2) $Q(t)$ is continuous in the uniform operator topology for $t > 0$, and $\{Q(t)\}_{t\ge0}$ is uniformly bounded, that is, there exists $M > 1$ such that $\sup_{t\in[0,+\infty)} |Q(t)| < M$.

3. Integral solution to nonhomogeneous Cauchy problem

Here we derive the integral solution for nonhomogeneous fractional order evolution equation (1.1) with the aid of Laplace transform and probability density function. For Cauchy problem (1.1), it is assumed that $u_0 \in X_0$ and $f : J \to X$ is continuous.

Definition 3.1. A function $u(t)$ is said to be an integral solution of (1.1) if

(i) $u : J \to X$ is continuous;
(ii) $I_{0+}^q u(t) \in D(A)$ for $t \in J$ and
(iii)
$$u(t) = u_0 + AI_{0+}^q u(t) + I_{0+}^q f(t), \quad t \in J. \tag{3.1}$$

Remark 3.2. If $u(t)$ is an integral solution of (1.1), then $u(t) \in X_0$ for $t \in J$. In fact, by $I^q_{0+}u(t) \in D(A)$, we have $I^1_{0+}u(t) = I^{1-q}_{0+}I^q_{0+}u(t) \in D(A)$ for $t \in J$. Then $u(t) = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} u(s)ds \in X_0$ for $t \in J$.

Definition 3.3 ([12])**.** The Wright function $M_q(\theta)$ defined by

$$M_q(\theta) = \sum_{n=1}^{\infty} \frac{(-\theta)^{n-1}}{(n-1)!\Gamma(1-qn)}$$

is such that

$$\int_0^\infty \theta^\delta M_q(\theta)d\theta = \frac{\Gamma(1+\delta)}{\Gamma(1+q\delta)}, \quad \text{for } \delta \geq 0.$$

Consider the auxiliary problem

$$\begin{gathered} {}^C D^q_{0+}u(t) = A_0u(t) + f(t), \quad t \in (0,b], \\ u(0) = u_0. \end{gathered} \tag{3.2}$$

By Definition 3.1, the integral solution of (3.2) can be written as

$$u(t) = u_0 + A_0 I^q_{0+}u(t) + I^q_{0+}f(t) \tag{3.3}$$

for $u_0 \in X_0$ and $t \in J$. The following Lemma gives an equivalent form of (3.3) by means of Laplace transform.

Lemma 3.4. *If f take values in X_0, then the integral equation* (3.3) *can be expressed as*

$$u(t) = \Big(I^{1-q}_{0+}K_q(t)\Big)u_0 + \int_0^t K_q(t-s)f(s)ds, \quad t \in J, \tag{3.4}$$

where

$$K_q(t) = t^{q-1}P_q(t), \quad P_q(t) = \int_0^\infty q\theta M_q(\theta)Q(t^q\theta)d\theta.$$

Proof. Let $\lambda > 0$. Applying the Laplace transform

$$\chi(\lambda) = \int_0^\infty e^{-\lambda s}u(s)ds \quad \text{and} \quad \omega(\lambda) = \int_0^\infty e^{-\lambda s}f(s)ds$$

to (3.3), we obtain

$$\begin{aligned} \chi(\lambda) &= \lambda^{-1}u_0 + \frac{1}{\lambda^q}A_0\chi(\lambda) + \frac{1}{\lambda^q}\omega(\lambda) \\ &= \lambda^{q-1}(\lambda^q I - A_0)^{-1}u_0 + (\lambda^q I - A_0)^{-1}\omega(\lambda) \\ &= \lambda^{q-1}\int_0^\infty e^{-\lambda^q s}Q(s)u_0ds + \int_0^\infty e^{-\lambda^q s}Q(s)\omega(\lambda)ds, \end{aligned} \tag{3.5}$$

provided that the integrals in (3.5) exist, where I is the identity operator defined on X.

The Laplace transform of

$$\psi_q(\theta) = \frac{q}{\theta^{q+1}}M_q(\theta^{-q}),$$

is

$$\int_0^\infty e^{-\lambda\theta}\psi_q(\theta)d\theta = e^{-\lambda^q}, \tag{3.6}$$

where $q \in (0,1)$. Using (3.6), we have

$$\begin{aligned}
\int_0^\infty e^{-\lambda^q s} Q(s) u_0 ds &= \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} Q(t^q) u_0 dt \\
&= \int_0^\infty \int_0^\infty q\psi_q(\theta) e^{-(\lambda t\theta)} Q(t^q) t^{q-1} u_0 d\theta dt \\
&= \int_0^\infty \int_0^\infty q\psi_q(\theta) e^{-\lambda t} Q\Big(\frac{t^q}{\theta^q}\Big) \frac{t^{q-1}}{\theta^q} u_0 d\theta dt \\
&= \int_0^\infty e^{-\lambda t} \Big[q \int_0^\infty \psi_q(\theta) Q\Big(\frac{t^q}{\theta^q}\Big) \frac{t^{q-1}}{\theta^q} u_0 d\theta \Big] dt \\
&= \int_0^\infty e^{-\lambda t} t^{q-1} P_q(t) u_0 dt
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda^q s} Q(s) \omega(\lambda) ds \\
&= \int_0^\infty \int_0^\infty q t^{q-1} e^{-(\lambda t)^q} Q(t^q) e^{-\lambda s} f(s) ds dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty q\psi_q(\theta) e^{-(\lambda t\theta)} Q(t^q) e^{-\lambda s} t^{q-1} f(s) d\theta ds dt \\
&= \int_0^\infty \int_0^\infty \int_0^\infty q\psi_q(\theta) e^{-\lambda(t+s)} Q\Big(\frac{t^q}{\theta^q}\Big) \frac{t^{q-1}}{\theta^q} f(s) d\theta ds dt \\
&= \int_0^\infty e^{-\lambda t} \Big[q \int_0^t \int_0^\infty \psi_q(\theta) Q\Big(\frac{(t-s)^q}{\theta^q}\Big) \frac{(t-s)^{q-1}}{\theta^q} f(s) d\theta ds \Big] dt \\
&= \int_0^\infty e^{-\lambda t} \Big[\int_0^t (t-s)^{q-1} P_q(t-s) f(s) ds \Big] dt.
\end{aligned} \tag{3.8}$$

Since the Laplace inverse transform of λ^{q-1} is

$$\mathfrak{L}^{-1}(\lambda^{q-1}) = \frac{t^{-q}}{\Gamma(1-q)} = g_{1-q}(t),$$

therefore, by (3.5), (3.7) and (3.8), for $t \in J$, we obtain

$$\begin{aligned}
u(t) &= \Big(\mathfrak{L}^{-1}(\lambda^{q-1}) * K_q(t)\Big) u_0 + \int_0^t K_q(t-s) f(s) ds \\
&= \Big(I_{0+}^{1-q} K_q(t)\Big) u_0 + \int_0^t K_q(t-s) f(s) ds.
\end{aligned} \tag{3.9}$$

This completes the proof. □

Remark 3.5. Let $S_q(t) = I_{0+}^{1-q} K_q(t)$. By the uniqueness of Laplace inverse transform, it is obvious that operators $S_q(t)$ and $P_q(t)$ (obtained here) are the same as the ones given in [23]. In addition, we also obtain the relationship between $S_q(t)$ and $K_q(t)$; that is, $S_q(t) = I_{0+}^{1-q} K_q(t)$ for $t \geq 0$. So, we can say that $\{K_q(t)\}_{t\geq 0}$ is generated by A_0.

Proposition 3.6 ([30])**.** With assumption (H2), $P_q(t)$ is continuous in the uniform operator topology for $t > 0$.

Proposition 3.7 ([23]). *With assumption* (H2), *for any fixed* $t > 0$, $\{K_q(t)\}_{t>0}$ *and* $\{S_q(t)\}_{t>0}$ *are linear operators, and for any* $x \in X_0$,

$$|K_q(t)x| \le \frac{Mt^{q-1}}{\Gamma(q)}|x| \quad \text{and} \quad |S_q(t)x| \le M|x|.$$

Proposition 3.8 ([23]). *With assumption* (H2), $\{K_q(t)\}_{t>0}$ *and* $\{S_q(t)\}_{t>0}$ *are strongly continuous, that is, for any* $x \in X_0$ *and* $0 < t' < t'' \le b$,

$$|K_q(t')x - K_q(t'')x| \to 0 \quad \text{and} \quad |S_q(t')x - S_q(t'')x| \to 0, \quad \text{as } t'' \to t'.$$

If we assume that f takes values in X_0, then (3.4) can be written as

$$u(t) = S_q(t)u_0 + \int_0^t K_q(t-s) \lim_{\lambda\to+\infty} B_\lambda f(s)ds \tag{3.10}$$

or

$$u(t) = S_q(t)u_0 + \lim_{\lambda\to+\infty} \int_0^t K_q(t-s)B_\lambda f(s)ds, \tag{3.11}$$

where $B_\lambda = \lambda(\lambda I - A)^{-1}$, since $\lim_{\lambda\to+\infty} B_\lambda x = x$ for $x \in X_0$. When f takes values in X, but not in X_0, then the limit in (3.11) exists (as we will prove). But the limit in (3.10) will no longer exist.

Lemma 3.9. *Any solution of integral equation* (3.1) *with values in* X_0 *is represented by* (3.11).

Proof. Let

$$u_\lambda(t) = B_\lambda u(t), \quad f_\lambda(t) = B_\lambda f(t), \quad u_\lambda = B_\lambda u_0.$$

By applying B_λ to (3.1), we have

$$u_\lambda(t) = u_\lambda + A_0 I^q_{0+} u_\lambda(t) + I^q_{0+} f_\lambda(t).$$

Hence, by Lemma 3.4, we obtain

$$u_\lambda(t) = S_q(t)u_\lambda + \int_0^t K_q(t-s)f_\lambda(s)ds.$$

As $u(t), u_0 \in X_0$, we have

$$u_\lambda(t) \to u(t), u_\lambda \to u_0, \quad S_q(t)u_\lambda \to S_q(t)u_0, \quad \text{as } \lambda \to +\infty.$$

Thus (3.11) holds. This completes the proof. □

Let us define

$$\Phi_q(t)x = \lim_{\lambda\to+\infty} \int_0^t K_q(t-s)B_\lambda x\, ds = \lim_{\lambda\to+\infty} \int_0^t K_q(s)B_\lambda x\, ds, \tag{3.12}$$

for $x \in X$ and $t \ge 0$.

Proposition 3.10. *For* $x \in X$ *and* $t \ge 0$, *the limit in* (3.12) *exists and defines a bounded linear operator* $\Phi_q(t)$.

Proof. Let

$$\Phi^0_q(t)x = \int_0^t K_q(t-s)x\, ds = \int_0^t K_q(s)x\, ds,$$

for $x_0 \in X_0$ and $t \ge 0$. Then, the definition

$$\Phi_q(t) = (\lambda I - A)\Phi^0(t)(\lambda I - A)^{-1},$$

for $\lambda > \omega$, extends $\Phi_q^0(t)$ from X_0 to X. This definition is independent of λ because of the resolvent identity. As $\Phi_q(t)$ maps X into X_0, we have

$$\Phi_q(t)x = \lim_{\lambda\to+\infty} B_\lambda \Phi_q(t)x = \lim_{\lambda\to+\infty} \Phi_q^0(t)B_\lambda x.$$

This completes the proof. $\square$

Proposition 3.11. *For $x \in X_0$ and $t \geq 0$, ${}^C D_{0+}^q \Phi_q^0(t)x = S_q(t)x$ and $S_q(t)x = A\Phi_q^0(t)x + x$.*

The proof of the above proposition follows directly from the definitions of $S_q(t)$ and $\Phi_q^0(t)$ for $t \geq 0$.

Lemma 3.12. *(i) For $x \in X$ and $t \geq 0$, $I_{0+}^q \Phi_q(t) \in D(A)$ and*

$$\Phi_q(t)x = A\big(I_{0+}^q \Phi_q(t)x\big) + \frac{t^q}{\Gamma(1+q)}x. \tag{3.13}$$

(ii) For $x \in D(A)$,

$$\Phi_q(t)Ax + x = S_q(t)x. \tag{3.14}$$

Proof. (i) For $x \in X$ and $t \geq 0$, let

$$V(t) = \lambda I_{0+}^q \Phi_q^0(t)(\lambda I - A)^{-1}x + \frac{t^q}{\Gamma(1+q)}(\lambda I - A)^{-1}x - \Phi_q^0(t)(\lambda I - A)^{-1}x.$$

Clearly $V(0) = 0$. By Proposition 3.11, we have

$$\begin{aligned}
&{}^C D_{0+}^q V(t)\\
&= \lambda\Phi_q^0(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - {}^C D_{0+}^q \Phi_q^0(t)(\lambda I - A)^{-1}x\\
&= \lambda\Phi_q^0(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - S_q(t)(\lambda I - A)^{-1}x\\
&= \lambda\Phi_q^0(t)(\lambda I - A)^{-1}x + (\lambda I - A)^{-1}x - A\Phi_q^0(t)(\lambda I - A)^{-1}x - (\lambda I - A)^{-1}x\\
&= \lambda\Phi_q^0(t)(\lambda I - A)^{-1}x - A\Phi_q^0(t)(\lambda I - A)^{-1}x\\
&= (\lambda I - A)\Phi_q^0(t)(\lambda I - A)^{-1}x\\
&= \Phi_q(t)x.
\end{aligned}$$

Then

$$V(t) = I_{0+}^q \Phi_q(t)x + V(0) = I_{0+}^q \Phi_q(t)x$$

and

$$(\lambda I - A)V(t) = (\lambda I - A)I_{0+}^q \Phi_q(t)x = \lambda I_{0+}^q \Phi_q(t)x + \frac{t^q}{\Gamma(1+q)}x - \Phi_q(t)x.$$

Thus

$$\Phi_q(t)x = A\big(I_{0+}^q \Phi_q(t)x\big) + \frac{t^q}{\Gamma(1+q)}x.$$

(ii) For $x \in D(A)$, it follows by Proposition 3.11 that

$$\begin{aligned}
\Phi_q(t)Ax &= \lim_{\lambda\to+\infty} \int_0^t K_q(s)B_\lambda Ax\,ds = \lim_{\lambda\to+\infty} A_0 \int_0^t K_q(s)B_\lambda x\,ds\\
&= A_0\Phi_q^0(t)x = S_q(t)x - x.
\end{aligned}$$

This completes the proof. $\square$

Theorem 3.13. *$u(t)$ is an integral solution of (1.1) if and only if*

$$u(t) = S_q(t)u_0 + \lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda f(s)ds \tag{3.15}$$

for $t \in J$ and $u_0 \in X_0$.

Proof. In view of Lemma 3.9, we only need to show that (3.15) is the integral solution of (1.1). Indeed it is sufficient to prove the theorem for $u_0 = 0$, because it can easily be proved for the special case $f = 0$. We complete the proof in two steps.

Step I. Assume that f is continuously differentiable, then for $t \in J$, we have

$$\begin{aligned}
u_\lambda(t) &= \int_0^t K_q(s)B_\lambda f(s)ds \\
&= \int_0^t K_q(s)B_\lambda\Big(f(0) + \int_0^s f'(r)dr\Big)ds \\
&= \int_0^t K_q(s)B_\lambda f(0)ds + \int_0^t K_q(s)B_\lambda\Big(\int_0^s f'(r)dr\Big)ds \\
&= \Phi_q^0(t)B_\lambda f(0) + \int_0^t \Phi_q^0(t-r)B_\lambda f'(r)dr.
\end{aligned}$$

By Lemma 3.12, for $t \in J$, we obtain

$$\begin{aligned}
u(t) &= \lim_{\lambda\to+\infty} u_\lambda(t) \\
&= \Phi_q(t)f(0) + \int_0^t \Phi_q(t-r)f'(r)dr \\
&= A\big(I_{0+}^q\Phi_q(t)f(0)\big) + \frac{t^q}{\Gamma(1+q)}f(0) \\
&\quad + \int_0^t \Big[A\big(I_{0+}^q\Phi_q(t-r)\big) + \frac{(t-r)^q}{\Gamma(1+q)}\Big]f'(r)dr \\
&= A\Big[I_{0+}^q\Phi_q(t)f(0) + \int_0^t I_{0+}^q\Phi_q(t-r)f'(r)dr\Big] \\
&\quad + \frac{t^q}{\Gamma(1+q)}f(0) + \frac{1}{\Gamma(1+q)}\int_0^t (t-r)^q f'(r)dr \\
&= A\Big[I_{0+}^q\Phi_q(t)f(0) + I_{0+}^q\Big(\int_0^t \Phi_q(t-r)f'(r)dr\Big)\Big] \\
&\quad + \frac{t^q}{\Gamma(1+q)}f(0) + \frac{1}{\Gamma(1+q)}\int_0^t (t-r)^q f'(r)dr \\
&= A\big(I_{0+}^q u(t)\big) + I_{0+}^q f(t).
\end{aligned}$$

Step II. We approximate f by continuously differentiable functions f_n such that

$$\sup_{t\in J}|f(t) - f_n(t)| \to 0, \quad \text{as } n\to\infty.$$

Letting

$$u_n(t) = \lim_{\lambda\to\infty}\int_0^t K_q(s)B_\lambda f_n(s)ds,$$

we have

$$u_n(t) = A\big(I_{0+}^q u_n(t)\big) + I_{0+}^q f_n(t).$$

Then

$$\begin{aligned}
|u_n(t) - u_m(t)| &= \Big|\lim_{\lambda\to\infty}\int_0^t K_q(s)B_\lambda\big[f_n(s) - f_m(s)\big]ds\Big| \\
&\le \frac{M\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|f_n(s) - f_m(s)|ds \\
&\le \frac{M\overline{M}b^q}{\Gamma(q)}\|f_n - f_m\|,
\end{aligned}$$

which implies that $\{u_n\}$ is a Cauchy sequence and its limit, denoted by $u(t)$, exists. Taking limit on both sides of (3), we obtain

$$u(t) = A\big(I_{0+}^q u(t)\big) + I_{0+}^q f(t), \quad \text{for } t \in J.$$

Therefore, (3.15) is the integral solution of (1.1). This completes the proof. $\square$

Remark 3.14. (i) Integrating the last term in (3.15) and using Proposition 3.8, the integral solution (3.15) can be expressed as

$$u(t) = S_q(t)u_0 + \frac{d}{dt}\int_0^t \Phi_q(t-s)f(s)ds.$$

(ii) $(\lambda^q I - A)^{-1}x = \lambda\int_0^\infty e^{-\lambda t}\Phi_q(t)x\,ds$ for $x \in X$ and $\lambda^q > \omega$. In fact, by taking Laplace transform of (3.13), we obtain

$$\begin{aligned}
\mathfrak{L}[\Phi_q(t)x] &= A\mathfrak{L}[I_{0+}^q\Phi_q(t)x] + \mathfrak{L}[\frac{t^q}{\Gamma(1+q)}x] \\
&= \lambda^{-q}A\mathfrak{L}[\Phi_q(t)x] + \lambda^{-q-1}x \\
&= \lambda^{-1}(\lambda^q I - A)^{-1}x.
\end{aligned}$$

(iii) We can say that A generates the operator $\{\Phi_q(t)\}_{t\ge0}$. When $q = 1$, $\{\Phi_q(t)\}_{t\ge0}$ degenerates into $\{S(t)\}_{t\ge0}$, which is integrated semigroup generated by A in [18].

4. Integral solution to a nonlinear Cauchy problem

In this section, we study the existence of integral solution of nonlinear fractional evolution equation (1.2). We need the following assumptions:

(H3) for each $t \in J$, the function $g(t,\cdot) : X \to X$ is continuous and for each $x \in X$, the function $g(\cdot,x) : J \to X$ is strongly measurable;

(H4) there exists a function $m \in L(J,\mathbb{R}^+)$ such that

$$I_{0+}^q m(t) \in C(J,\mathbb{R}^+), \quad \lim_{t\to0+} I_{0+}^q m(t) = 0,$$
$$|g(t,x)| \le m(t) \quad \text{for all } x \in X \text{ and almost all } t \in J;$$

(H5) there exists a constant $l > 0$ such that for any bounded $D \subseteq X$,

$$\beta(g(t,D)) \le l\beta(D), \quad \text{for a.e. } t \in J.$$

By Theorem 3.13, it is easy to see that the integral solution of (1.2) is equal to the solution of

$$u(t)=S_q(t)u_0+\frac{d}{dt}\int_0^t \Phi_q(t-s)g(s,u(s))ds \tag{4.1}$$

or

$$u(t)=S_q(t)u_0+\lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda g(s,u(s))ds. \tag{4.2}$$

For $u\in C(J,X_0)$, define an operator

$$(\mathscr{T}u)(t)=(\mathscr{T}_1u)(t)+(\mathscr{T}_2u)(t),$$

where

$$(\mathscr{T}_1u)(t)=S_q(t)u_0 \quad\text{and}\quad (\mathscr{T}_2u)(t)=\lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda g(s,u(s))ds,$$

for all $t\in J$. Let $B_r(J)=\{u\in C(J,X_0):\|u\|\le r\}$.

Lemma 4.1. *Suppose that conditions* (H1)–(H4) *hold. Then* $\{\mathscr{T}u:u\in B_r(J)\}$ *is equicontinuous.*

Proof. By Proposition 3.8, $S_q(t)u_0$ is uniformly continuous on J. Consequently, $\{\mathscr{T}_1u:\ u\in B_r(J)\}$ is equicontinuous.

For $u\in B_r(J)$, taking $t_1=0$, $0<t_2\le b$, we obtain

$$\begin{aligned}
|(\mathscr{T}_2u)(t_2)-(\mathscr{T}_2u)(0)&=\Big|\lim_{\lambda\to+\infty}\int_0^{t_2}K_q(t-s)B_\lambda g(s,u(s))ds\Big|\\
&\le\frac{M\overline{M}}{\Gamma(q)}\int_0^{t_2}(t_2-s)^{q-1}m(s)ds\to0,\quad\text{as } t_2\to0.
\end{aligned}$$

For $0<t_1<t_2\le b$, we have

$$\begin{aligned}
&|(\mathscr{T}_2u)(t_2)-(\mathscr{T}_2u)(t_1)|\\
&\le\Big|\lim_{\lambda\to+\infty}\int_{t_1}^{t_2}(t_2-s)^{q-1}P_q(t_2-s)B_\lambda g(s,u(s))ds\Big|\\
&\quad+\Big|\lim_{\lambda\to+\infty}\int_0^{t_1}(t_2-s)^{q-1}P_q(t_2-s)B_\lambda g(s,u(s))ds\\
&-\lim_{\lambda\to+\infty}\int_0^{t_1}(t_1-s)^{q-1}P_q(t_2-s)B_\lambda g(s,u(s))ds\Big|\\
&+\Big|\lim_{\lambda\to+\infty}\int_0^{t_1}(t_1-s)^{q-1}P_q(t_2-s)B_\lambda g(s,u(s))ds\\
&-\lim_{\lambda\to+\infty}\int_0^{t_1}(t_1-s)^{q-1}P_q(t_1-s)B_\lambda g(s,u(s))ds\Big|\\
&\le\frac{M\overline{M}}{\Gamma(q)}\Big|\int_{t_1}^{t_2}(t_2-s)^{q-1}m(s)ds\Big|\\
&\quad+\frac{M\overline{M}}{\Gamma(q)}\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]m(s)ds\\
&\quad+\Big|\lim_{\lambda\to+\infty}\int_0^{t_1}(t_1-s)^{q-1}[P_q(t_2-s)-P_q(t_1-s)]B_\lambda g(s,u(s))ds\Big|\\
&\le I_1+I_2+I_3,
\end{aligned}$$

where

$$I_1 = \frac{M\overline{M}}{\Gamma(q)}\Big|\int_0^{t_2}(t_2-s)^{q-1}m(s)ds - \int_0^{t_1}(t_1-s)^{q-1}m(s)ds\Big|,$$
$$I_2 = \frac{2M\overline{M}}{\Gamma(q)}\int_0^{t_1}\Big[(t_1-s)^{q-1}-(t_2-s)^{q-1}\Big]m(s)ds,$$
$$I_3 = \Big|\lim_{\lambda\to+\infty}\int_0^{t_1}(t_1-s)^{q-1}[P_q(t_2-s)-P_q(t_1-s)]B_\lambda g(s,u(s))ds\Big|.$$

By condition (H4), one can deduce that $\lim_{t_2\to t_1} I_1 = 0$. Noting that

$$\big[(t_1-s)^{q-1}-(t_2-s)^{q-1}\big]m(s) \le (t_1-s)^{q-1}m(s),$$

and $\int_0^{t_1}(t_1-s)^{q-1}m(s)ds$ exists, it follows by Lebesgue dominated convergence theorem that

$$\int_0^{t_1}[(t_1-s)^{q-1}-(t_2-s)^{q-1}]m(s)ds \to 0, \quad \text{as } t_2\to t_1,$$

which implies that $\lim_{t_2\to t_1} I_2 = 0$.

For $\varepsilon>0$ small enough, by (H4), we have

$$\begin{aligned}
I_3 &\le \overline{M}\int_0^{t_1-\varepsilon}(t_1-s)^{q-1}|P_q(t_2-s)-P_q(t_1-s)||g(s,u(s))|ds\\
&\quad + \overline{M}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{q-1}|P_q(t_2-s)-P_q(t_1-s)||g(s,u(s))|ds\\
&\le \overline{M}\int_0^{t_1}(t_1-s)^{q-1}m(s)ds \sup_{s\in[0,t_1-\varepsilon]}|P_q(t_2-s)-P_q(t_1-s)|\\
&\quad + \frac{2M\overline{M}}{\Gamma(q)}\int_{t_1-\varepsilon}^{t_1}(t_1-s)^{q-1}m(s)ds\\
&\le I_{31}+I_{32}+I_{33},
\end{aligned}$$

where

$$I_{31} = \frac{r\Gamma(q)}{M\overline{M}}\sup_{s\in[0,t_1-\varepsilon]}|P_q(t_2-s)-P_q(t_1-s)|,$$
$$I_{32} = \frac{2M\overline{M}}{\Gamma(q)}\Big|\int_0^{t_1}(t_1-s)^{q-1}m(s)ds - \int_0^{t_1-\varepsilon}(t_1-\varepsilon-s)^{q-1}m(s)ds\Big|,$$
$$I_{33} = \frac{2M\overline{M}}{\Gamma(q)}\int_0^{t_1-\varepsilon}[(t_1-\varepsilon-s)^{q-1}-(t_1-s)^{q-1}]m(s)ds.$$

By Proposition 3.6, it follows that $I_{31}\to 0$ as $t_2\to t_1$. Applying the arguments similar to the ones employed in proving that I_1, I_2 tend to zero, we obtain $I_{32}\to 0$ and $I_{33}\to 0$ as $\varepsilon\to 0$. Thus, I_3 tends to zero independently of $u\in B_r(J)$ as $t_2\to t_1$, $\varepsilon\to 0$. Therefore, $|(\mathscr{T}_2u)(t_2)-(\mathscr{T}_2u)(t_1)|\to 0$ independently of $u\in B_r(J)$ as $t_2\to t_1$, which implies that $\{\mathscr{T}_2u:\ u\in B_r(J)\}$ is equicontinuous. Therefore, $\{\mathscr{T}u:\ u\in B(J)\}$ is equicontinuous. The proof is complete. □

Lemma 4.2. *Assume that* (H1)–(H4) *hold. Then $\mathscr{T}$ maps $B_r(J)$ into $B_r(J)$, and is continuous in $B_r(J)$.*

Proof. **Claim I.** $\mathscr{T}$ maps $B_r(J)$ into $B_r(J)$. Obviously, by (H4), there exists a constant $r > 0$ such that

$$M\Big(|u_0| + \sup_{t\in J}\Big\{\frac{\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}m(s)ds\Big\}\Big) \le r.$$

For any $u \in B_r(J)$, by Proposition 3.7, we have

$$\begin{aligned} |(\mathscr{T}u)(t)| &\le |S_q(t)u_0| + \Big|\lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda g(s,u(s))ds\Big| \\ &\le M|u_0| + \frac{M\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s,u(s))|ds \\ &\le M\Big(|u_0| + \sup_{t\in J}\Big\{\frac{\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}m(s)ds\Big\}\Big) \le r. \end{aligned}$$

Hence $\|\mathscr{T}u\| \le r$ for any $u \in B_r(J)$.

Claim II. $\mathscr{T}$ is continuous in $B_r(J)$. For any $u_m, u \in B_r(J)$, $m = 1, 2, \dots$, with $\lim_{m\to\infty} u_m = u$, by (H3), we have

$$g(t, u_m(t)) \to g(t, u(t)) \quad \text{as } m \to \infty,$$

for $t \in J$. On the one hand, using (H4), for each $t \in J$, we obtain

$$(t-s)^{q-1}|g(s,u_m(s)) - g(s,u(s))| \le 2(t-s)^{q-1}m(s), \quad \text{a.e. in } [0,t).$$

As the function $s \to 2(t-s)^{q-1}m(s)$ is integrable for $s \in [0,t)$ and $t \in J$, by Lebesgue dominated convergence theorem, we obtain

$$\int_0^t (t-s)^{q-1}|g(s,u_m(s)) - g(s,u(s))|ds \to 0 \quad \text{as } m \to \infty.$$

For $t \in J$, we obtain

$$\begin{aligned} &|(\mathscr{T}u_m)(t) - (\mathscr{T}u)(t)| \\ &\le \Big|\lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda(g(s,u_m(s)) - g(s,u(s)))ds\Big| \\ &\le \frac{M\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{q-1}|g(s,u_m(s)) - g(s,u(s))|ds \to 0 \quad \text{as } m \to \infty. \end{aligned}$$

Therefore, $\mathscr{T}u_m \to \mathscr{T}u$ pointwise on J as $m \to \infty$. Hence it follows by Lemma 4.1 that $\mathscr{T}u_m \to \mathscr{T}u$ uniformly on J as $m \to \infty$ and so $\mathscr{T}$ is continuous. The proof is complete. □

Theorem 4.3. *Assume that* (H1)–(H5) *hold. Then the Cauchy problem* (1.2) *has at least one integral solution in* $B_r(J)$.

Proof. Let $y_0(t) = S_q(t)u_0$ for all $t \in J$ and $y_{m+1} = \mathscr{T}y_m$, $m = 0, 1, 2, \cdots$. Consider the set $\mathscr{H} = \{y_m : m = 0, 1, 2, \cdots\}$, and show that it is relatively compact.

By Lemmas 4.1 and 4.2, $\mathscr{H}$ is uniformly bounded and euqicontinuous on J. Next, for any $t \in J$, we just need to show that $\mathscr{H}(t) = \{y_m(t), m = 0, 1, 2, \cdots\}$ is relatively compact in X_0.

By the assumption (H5) together with Lemmas 2.1 and 2.2, for any $t \in J$, we have

$$\beta\Big(\mathscr{H}(t)\Big) = \beta\Big(\{y_m(t)\}_{m=0}^{\infty}\Big) = \beta\Big(\{y_0(t)\} \cup \{y_m(t)\}_{m=1}^{\infty}\Big) = \beta\Big(\{y_m(t)\}_{m=1}^{\infty}\Big)$$

and

$$\begin{aligned}
\beta\Big(\{y_m(t)\}_{m=1}^{\infty}\Big) &= \beta\Big(\{(\mathscr{T}y_m)(t)\}_{m=0}^{\infty}\Big)\\
&= \beta\Big(\Big\{S_q(t)u_0 + \lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda g(s,y_m(s))ds\Big\}_{m=0}^{\infty}\Big)\\
&= \beta\Big(\Big\{\lim_{\lambda\to+\infty}\int_0^t K_q(t-s)B_\lambda g(s,y_m(s))ds\Big\}_{m=0}^{\infty}\Big)\\
&\le \frac{2M\overline{M}}{\Gamma(q)}\int_0^t (t-s)^{1-q}\beta\Big(g(s,\{y_m(s)\}_{m=0}^{\infty})\Big)ds\\
&\le \frac{2M\overline{M}l}{\Gamma(q)}\int_0^t (t-s)^{1-q}\beta\Big(\{y_m(s)\}_{m=0}^{\infty}\Big)ds.
\end{aligned}$$

Thus

$$\beta(\mathscr{H}(t)) \le \frac{2M\overline{M}l}{\Gamma(q)}\int_0^t (t-s)^{1-q}\beta(\mathscr{H}(s))ds.$$

Therefore, by generalized Grownwall's inequality [21], we infer that $\beta(\mathscr{H}(t)) = 0$. In consequence, $\mathscr{H}(t)$ is relatively compact. Hence, it follows from Ascoli-Arzela theorem that $\mathscr{H}$ is relatively compact. Therefore, there exists a convergent subsequence of $\{y_m\}_{m=0}^{\infty}$. For the sake of clarity, let $\lim_{m\to\infty} y_m = y^* \in B_r(J)$. Thus, by continuity of the operator $\mathscr{T}$, we have

$$y^* = \lim_{m\to\infty} y_m = \lim_{m\to\infty}\mathscr{T}y_{m-1} = \mathscr{T}\big(\lim_{m\to\infty} y_{m-1}\big) = \mathscr{T}y^*,$$

which implies the Cauchy problem (1.2) has least an integral solution. □

5. An example

As an application of our results we consider the fractional time partial differential equation

$$\begin{gathered}
\frac{\partial^q}{\partial t^q}z(t,x) = \frac{\partial^2}{\partial x^2}z(,x) + G(t,z(t,x)), \quad x\in[0,\pi],\ t\in(0,b],\ 0<q<1,\\
z((t,0) = z(t,\pi) = 0, \quad t\in(0,b],\\
z(0,x) = z_0, \quad x\in[0,\pi],
\end{gathered} \tag{5.1}$$

where $G : [0,b]\times\mathbb{R}\to\mathbb{R}$ is a given function. Let

$$\begin{gathered}
u(t)(x) = z(t,x), \quad t\in[0,b],\ x\in[0,\pi],\\
g(t,u)(x) = G(t,u(x)), \quad t\in[0,b],\ x\in[0,\pi].
\end{gathered}$$

We choose $X = C([0,\pi],\mathbb{R})$ endowed with the uniform topology and consider the operator $A : D(A)\subset X\to X$ defined by:

$$D(A) = \{u\in C^2([0,\pi],\mathbb{R}) : u(0)=u(\pi)=0\}, \quad Au = u''.$$

It is well known that the operator A satisfies the Hille-Yosida condition with $(0,+\infty)\subset\rho(A)$, $\|(\lambda I - A)^{-1}\| \le \frac{1}{\lambda}$ for $\lambda>0$, and

$$\overline{D(A)} = \{u\in X : u(0)=u(\pi)=0\} \ne X.$$

We can show that problem (1.2) is an abstract formulation of problem (5.1). Under suitable conditions, Theorem 4.3 implies that problem (5.1) has a unique solution z on $[0,b]\times[0,\pi]$.

Concluding remarks. In this article, we have obtained the integral solution for nonhomogeneous Cauchy problem (1.1) and established the relationship between $\{S_q(t)\}_{t\geq 0}$ and $\{K_q(t)\}_{t\geq 0}$. Also sufficient conditions ensuring the existence of integral solutions to nonlinear Cauchy problem (1.2), involving a linear closed operator A of Hille-Yosida type with not densely defined domain, are presented.

For further research, we propose the following open problem: How to establish the existence of an integral solution to fractional evolution equation (1.2) when linear closed operator A is not a Hille-Yosida type and its domain is not densely defined?

Acknowledgements. The authors would like to express their thanks to the editor and anonymous referees for their suggestions and comments that improved the quality of the paper. The second author is supported by National Natural Science Foundation of China (11671339).

References

[1] A. Alsaedi, B. Ahmad, M. Kirane; *Maximum principle for certain generalized time and space fractional diffusion equations,* Quarterly of Applied Mathematics, **73.1** (2015), 163-175.
[2] N. Abada, M. Benchohra, H. Hammouche; *Existence and controllability results for nondensely defined impulsive semilinear functional differential inclusions,* J. Differential Equations, **246** (2009), 3834-3863.
[3] J. Banas, K. Goebel; *Measure of noncompactness in Banach Spaces,* in: Lecture Notes in Pure and App. Math., vol. 60, Marcel Dekker, New York, (1980).
[4] K. Diethelm; *The Analysis of Fractional Differential Equations,* Lect. Note. Math., Springer, (2010).
[5] G. Da Prato, E. Sinestrari; *Differential operators with nondense domain,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., **14** (1987), 285-344.
[6] M. M. El-Borai; *Semigroups and some nonlinear fractional differential equations,* Applied Math. Comput., **149** (2004), 823-831.
[7] M. M. El-Borai; *Some probability densities and fundamental solutions of fractional evolution equations,* Chaos Solutions & Fractals, **149** (2004), 823-831.
[8] X. L. Fu; *On solutions of neutral nonlocal evolution equations with nondense domain,* J. Math. Anal. Appl., **299** (2004), 392-410.
[9] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations,* Elsevier Science B.V., Amsterdam, (2006).
[10] M. Kirane, B. Ahmad, A. Alsaedi, et al.; *Non-existence of global solutions to a system of fractional diffusion equations,* Acta Applicandae Mathematicae, **133.1** (2014), 235-248.
[11] M. Li, C. Chen, F. B. Li; *On fractional powers of generators of fractional resolvent families,* J. Funct. Anal., **259** (2010), 2702-2726.
[12] F. Mainardi, P. Paradisi, R. Gorenflo; *Probability Distributions Generated by Fractional Diffusion Equations,* in: J. Kertesz, I. Kondor (Eds.), Econophysics: An Emerging Science, Kluwer, Dordrecht, (2000).
[13] G. M. Mophou, G. M. N'Guérékata; *On integral solutions of some nonlocal fractional differential equations with nondense domain,* Nonlinear Anal., **71** (2009), 4668-4675.
[14] H. Mönch; *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces,* Nonlinear Anal., **4** (1980), 985-999.
[15] A. Pazy; *Semigroups of Linear Operator and Applications to Partial Differential Equations,* Springer-Verlag, New York, (1983).
[16] J. G. Peng, K. X. Li; *A novel characteristic of solution operator for the fractional abstract Cauchy problem,* J. Math. Anal. Appl., **385** (2012), 786-796.
[17] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional integrals and derivatives: Theory and Applications,* Gordon and Breach, Yverdon, (1993).
[18] H. R. Thieme; *Semiflows generated by Lipschitz perturbations of non-densely defined operators,* Differential Integral Equations, **3** (1990), 1035-1066.

[19] H. R. Thieme; *Integrated semigroups and integrated solutions to abstract Cauchy problems,* J. Math. Anal. Appl., **152** (1990), 416-447.

[20] R. N. Wang, D. H. Chen, T.-J. Xiao; *Abstract fractional cauchy problems with almost sectorial operators,* J. Differential Equations, **252** (2012), 202-235.

[21] H. P. Ye, J. M. Gao, Y. S. Ding; *A generalized Gronwall inequality and its application to a fractional differential equation,* J. Math. Anal. Appl., **328** (2007).

[22] Z. F. Zhang, B. Liu; *Existence Results for nondensely defined fractional differential equations with nonlocal conditions,* Math. Appl., **25** (2012), 403-412.

[23] Y. Zhou, F. Jiao; *Nonlocal Cauchy problem for fractional evolution equations,* Nonlinear Anal. Real World Appl., **11** (2010), 4465-4475.

[24] Y. Zhou, B. Ahmad, A. Alsaedi; *Existence of nonoscillatory solutions for fractional neutral differential equations,* Applied Math. Letters, **72** (2017), 70-74.

[25] Y. Zhou, L. Peng; *On the time-fractional Navier-Stokes equations,* Comput. Math. Appl., **73.6** (2017), 874-891.

[26] Y. Zhou, L. Peng; *Weak solution of the time-fractional Navier-Stokes equations and optimal control,* Comput. Math. Appl., **73.6** (2017), 1016-1027.

[27] Y. Zhou, V. Vijayakumar, R. Murugesu; *Controllability for fractional evolution inclusions without compactness,* Evol. Equ. Control Theory, **4** (2015), 507-524.

[28] Y. Zhou, L. Zhang; *Existence and multiplicity results of homoclinic solutions for fractional Hamiltonian systems,* Comput. Math. Appl., **73.6** (2017), 1325-1345.

[29] Y. Zhou, L. Zhang, X. H. Shen; *Existence of mild solutions for fractional evolution equations,* J. Integral Equations Appl., **25** (2013), 557-586.

[30] Y. Zhou; *Basic Theory of Fractional Differential Equations,* World Scientific, Singapore, (2014).

Haibo Gu
School of Mathematics Sciences, Xinjiang Normal University, Urumqi, Xinjiang 830054, China.
Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China
E-mail address: hbgu_math@163.com

Yong Zhou (corresponding author)
Faculty of Mathematics and Computational Science, Xiangtan University, Hunan 411105, China.
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
E-mail address: yzhou@xtu.edu.cn

Bashir Ahmad
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
E-mail address: bashirahmad_qau@yahoo.com

Ahmed Alsaedi
Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
E-mail address: aalsaedi@hotmail.com