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EXISTENCE AND REGULARITY OF SOLUTIONS FOR STOKES SYSTEMS WITH NON-SMOOTH BOUNDARY DATA IN A POLYHEDRON

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Abstract. The Stokes problem is fundamental in the study of fluid flows. In the case of smooth domains and data, this problem is extensively studied in the literature. But there are only a few results for non-smooth boundary data. In [13], there are some promising results in the 2 dimensional case. The aim of this work is to extend those results to a polyhedron domain with non-regular data.

1. Introduction

Let Ω be a convex polyhedron of $\mathbb{R}^3$ with boundary Γ. The steady, creeping flow of an incompressible fluid is governed by Stokes system

$$\begin{aligned} -\operatorname{div}(2\eta \mathbf{d}(\mathbf{u}) - p\boldsymbol{\delta}) &= \mathbf{f} \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{u} &= 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{g} \quad \text{on } \Gamma. \end{aligned} \tag{1.1}$$

Where $\mathbf{u}$ the velocity field and p the pressure are the unknowns of the problem. $\mathbf{f}$ and $\mathbf{g}$ are given functions respectively defined on Ω and its boundary Γ, and respectively representing the inertia forces and boundary data. Finally, $\mathbf{d}(\mathbf{u}) = (\nabla \mathbf{u} + \nabla^t \mathbf{u})/2$ is the strain rate tensor, $\boldsymbol{\delta}$ the identity tensor and η the viscosity of the fluid, supposed to be constant (Newtonian fluid).

In the bi-dimensional framework, the authors obtained in previous work [13], some promising results on the existence and regularity of the solution to the system (1.1). More precisely, the boundary Γ is supposed to be a set of segments $\Gamma_i =]S_i, S_{i+1}[$, $i = 1, \dots, N$ and the data $\mathbf{g}_{|\Gamma_i} \in (H^s(\Gamma_i))^2$ with $-1/2 < s < +1/2$.

This work represents a generalization of the results obtained in [13]. It concerns the existence and regularity results of solutions to non-homogeneous Stokes system in a polyhedron with non enough regular data $\mathbf{g}$ on the boundary. More precisely, we assume that Ω is a convex polyhedron and it is supposed that its boundary Γ is composed of surfaces $F_i, i = 1, \dots, N$:

$$\Gamma = \cup_{i=1}^N F_i, \quad \Gamma_i = \partial F_i = \cup_{j=1}^{N_i} \Gamma_{ij}.$$

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For $j = 1, \dots, N_i$, Γ_{ij} stands for the edges of the i^{th} face F_i see Figure 1.

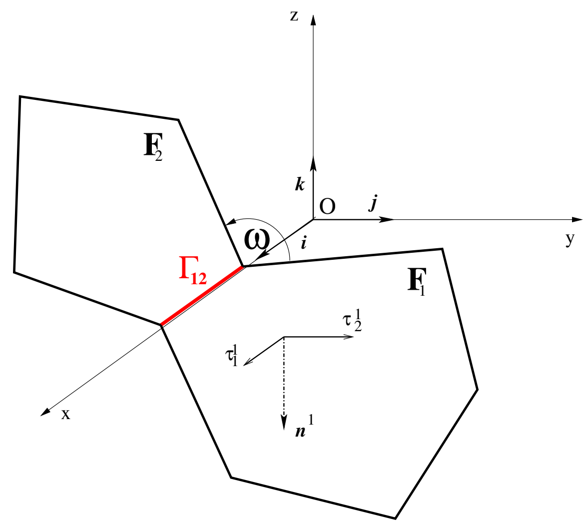

Figure 1. Boundary of the domain

For each face F_i, we define a local direct basis $(\mathbf{n}^i, \boldsymbol{\tau}_1^i, \boldsymbol{\tau}_2^i)$, $\mathbf{n}^i$ being the outward unit normal to F_i and $(\boldsymbol{\tau}_1^i, \boldsymbol{\tau}_2^i)$ a basis of the tangent plan containing the face F_i.

Note that for a Newtonian fluid, the viscosity is constant. Then, dividing by $\eta > 0$, the first equation of (1.1) becomes

$$-\Delta \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega.$$

To simplify, it is assumed that $\mathbf{f} = 0$ in Ω. Given a family of distributions $\mathbf{g}_i = (g_1^i, g_2^i, g_3^i)$ defined on F_i. Our purpose is to show existence, uniqueness and regularity results for $(\mathbf{u}, p)$ solution to the Stokes problem

$$-\Delta \mathbf{u} + \nabla p = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\operatorname{div} \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.3}$$

$$\mathbf{u} = \mathbf{g}_i, \quad \text{on } F_i,\ i = 1, \dots, N, \tag{1.4}$$

where (1.2) and (1.3) are verified in the distribution sense and (1.4) is verified in generalized sense of traces.

Let $\mathbb{H}^s(\cdot)$ denote the product space $(H^s(\cdot))^3$, where $H^s(\cdot)$ denotes the usual Sobolev space (See e.g. [9]).

Remark 1.1. Let $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \dots, \mathbf{g}_N)$. We then have the classical result:

If $\mathbf{g} \in \prod_{i=1,N} \mathbb{H}^{1/2}(F_i)$ and $\mathbf{g}$ is the trace of a function of $\mathbb{H}^1(\Omega)$ then the previous Stokes problem has a unique solution in $\mathbb{H}^1(\Omega) \times L_0^2(\Omega)$. Here $L_0^2(\Omega)$ is the sub-space of L^2-functions with zero mean.

2. Existence and uniqueness result

We consider the case where the data satisfy

$$\mathbf{g}_i \in \mathbb{H}^s(F_i), \quad \text{for } i = 1, \dots, N \text{ with } |s| < \frac{1}{2}.$$

Let $H = (H_i)_{i=1}^N$ such that $H_i = \mathbf{g}_i \cdot \mathbf{n}_i$ on the face F_i. We will assume that the H_i functions satisfy

$$\sum_{i=1}^{N} \langle H_i, 1\rangle = 0. \tag{2.1}$$

Note that (2.1) is a compatibility condition which is necessary for the existence of solutions to the Stokes problem (1.2)–(1.4).

Let us also define the tangential part of $\mathbf{g}_i$ on Γ_i by

$$\mathbf{G} = (\mathbf{G}_i)_{i=1}^N \quad \text{where } \mathbf{G}_i = (\mathbf{g}_i)_t = \mathbf{g}_i - (\mathbf{g}_i \cdot \mathbf{n}_i)\mathbf{n}_i. \tag{2.2}$$

In this case, the solution of the system (1.2)–(1.4) is not regular enough and then it can not be variational. We then use the transposition technique as in the search for very weak solutions. This contains, in particular, the interesting case with regular data on each face F_i of the boundary but without connection or compatibility at the edges Γ_{ij} of Γ.

The approach developed here may be extended, in natural way, to the case where Ω is a non convex polyhedron of $\mathbb{R}^3$ and also to the case where the data belong to $W^{s,p}$ spaces. This could be done by using the results of Dauge [6], Maz'ya-Plamenevskiĭ [10] and Grisvard [8]. The main result of this work is the following theorem.

Theorem 2.1. *Let $\mathbf{g}_i = (g_i^1, g_i^2, g_i^3)$ given in $\mathbb{H}^s(F_i)$ for $i = 1, \dots, N$ and $-1/2 < s < 1/2$. Then, there exists a unique $\mathbf{u} \in \cap_{\sigma < s+\frac{1}{2}} \mathbb{H}^\sigma(\Omega)$ and a unique (up to an additive constant distribution) $p \in H^{s-\frac{1}{2}}(\Omega)$, solution to* (1.2)–(1.4)*.*

The proof of the Theorem 2.1 is done in two steps. First, we recall some results on the traces of functions in $H^s(\Omega)$ and establish some useful lemma for the rest of the presentation. We will then use the result of Dauge [6] and transposition techniques to show that the problem (1.2)–(1.4) has a solution $(\mathbf{u}, p) \in (L^2(\Omega))^3 \times H^{-1}(\Omega)$. And then, in section 3, we will use interpolation techniques to increase the regularity of the solution, and establish the proof of Theorem 2.1.

Remark 2.2. We did not get the optimal regularity that one could hope, namely, $\mathbf{u} \in \mathbb{H}^{s+\frac{1}{2}}(\Omega)$. This is because of the following fact: Noting $H^{1/2}(\Gamma)$ the space of traces of functions $H^1(\Omega)$, it is difficult to characterize its dual $(H^{1/2}(\Gamma))'$. $H^{1/2}(\Gamma)$ is actually a dense, but non closed, sub-space of $\prod_{j=1}^N H^{1/2}(F_j)$.

2.1. **Theorem of traces.** Let us first recall some brief results on vector fields. For more details, we refer to Girault-Raviart [7]. For any $\mathbf{v} = (v_1, v_2, v_3) \in (\mathcal{D}'(\Omega))^3$, $\operatorname{curl} \mathbf{v}$ is defined by

$$\operatorname{curl} \mathbf{v} = \Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big).$$

It is then easy to see that

$$\operatorname{curl}(\operatorname{curl} \mathbf{v}) = -\Delta \mathbf{v} + \nabla(\operatorname{div} \mathbf{v}). \tag{2.3}$$

As on Figure 1, $\{\mathbf{n}, \boldsymbol{\tau}_1, \boldsymbol{\tau}_2\}$ denote the local coordinates and $\mathbf{v_t} = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n})\mathbf{n}$ the tangential part of the vector $\mathbf{v}$. In the sequel, $(\cdot, \cdot)$ denotes the usual inner product on $(L^2(\Omega))^3$ and $\langle \cdot, \cdot \rangle$ the duality pairing. Let

$$\mathbb{H}(\operatorname{curl}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3; \operatorname{curl} \mathbf{v} \in (L^2(\Omega))^3\}.$$

We recall in the following theorem the tangential trace and a Green's formula that we use later

Theorem 2.3 (Girault-Raviart [7, Prop. 1.2]). *The mapping* $\gamma_{\mathbf{t}} : \mathbf{v} \to \gamma_{\mathbf{t}}\mathbf{v} = (\mathbf{v}\times\mathbf{n})_\Gamma$ *defined on* $[\mathcal{D}(\bar{\Omega})]^3$ *extends by continuity to a linear continuous mapping, still denoted* $\gamma_{\mathbf{t}}$*, from* $\mathbb{H}(\mathrm{curl},\Omega)$ *to* $\mathbb{H}^{-1/2}(\Gamma)$*. Moreover,*

$$\int_\Omega \mathrm{curl}\,\mathbf{v}\cdot\boldsymbol{\varphi} = \int_\Omega \mathbf{v}\cdot\mathrm{curl}\,\boldsymbol{\varphi} + \langle\gamma_{\mathbf{t}}\mathbf{v},\boldsymbol{\varphi}\rangle_\Gamma, \quad \forall\mathbf{v}\in\mathbb{H}(\mathrm{curl},\Omega) \text{ and } \forall\boldsymbol{\varphi}\in\mathbb{H}^1(\Omega). \tag{2.4}$$

Now, by definition of the domain Ω, see Figure 1, each face F_i of $\Gamma = \partial\Omega$ is a convex polygon, the boundary $\Gamma_i = \partial F_i$ is Lipschitz-continuous. For $x \in F_i$, $\rho_i(x)$ denotes the distance of x to Γ_i. We then may construct the following sub-space (see Grisvard [8]):

$$\mathbb{H}^{1/2}_{0,0}(F_i) = \{\mathbf{v}\in\mathbb{H}^{1/2}(F_i) : \frac{\mathbf{v}}{\sqrt{\rho_i}} \in \mathbb{L}^2(F_i)\}.$$

Equipped with the norm

$$|||\mathbf{v}||| = \big\{\|\mathbf{v}\|^2_{1/2,F_i} + \|\frac{\mathbf{v}}{\sqrt{\rho_i}}\|^2_{0,F_i}\big\}^{1/2},$$

where $\mathbb{H}^{1/2}_{0,0}(F_i)$ is a Hilbert space.

Introducing the space

$$\mathbb{W} = \mathbb{H}^3(\Omega)\cap\mathbb{H}^2_0(\Omega),$$

we first give a result on the tangential trace of the Laplacian operator.

Theorem 2.4. *The mapping* $T_1 : \boldsymbol{\varphi} \to (\gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi}_{|_{F_i}})_{i=1}^N$ *from* $\mathbb{W}$ *on* $\prod_{j=1}^N (H^{1/2}_{0,0}(F_j))^2$ *is linear, continuous, onto and has a linear continuous right inverse.*

Proof. let us first show that for any $\boldsymbol{\varphi}\in\mathbb{W}$,

$$\Big(\gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi}_{|_{F_i}}\Big)_{i=1}^N \in \prod_{j=1}^N (H^{1/2}_{0,0}(F_j))^2.$$

Let $\boldsymbol{\varphi}\in\mathbb{W}$, by definition of the tangential trace $\gamma_{\mathbf{t}}$, we have $\gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi} = \Delta\boldsymbol{\varphi}\times\mathbf{n}$ and then $\gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi}$ belongs to the tangential plan. Therefore, it is sufficient to verify that $\Delta\boldsymbol{\varphi}_{|_{F_j}} \in (H^{1/2}_{0,0}(F_j))^2, \forall j = 1,\dots,N$. To do so, we will first show that for any $\psi\in(\mathcal{C}^2(\overline{\Omega}))^3\cap\mathbb{H}^2_0(\Omega)$, we have $\Delta\psi = 0$ on Γ_{ij}, where Γ_{ij} is an edge of the boundary Γ.

Without loss of generality, one may suppose that the boundary Γ admits a face F_1 belonging to the plan xoy and another face F_2 adjacent to F_1 having an angle $\omega < \pi$ with the plan xoy, see 1.

As, $\psi\in\mathbb{H}^2_0(\Omega)$, we have $\nabla\psi = 0$ on F_1 and F_2. Then:

On F_1, the basis of the tangent plan is $(\boldsymbol{\tau}_1,\boldsymbol{\tau}_2)$ with $\boldsymbol{\tau}_1 = \mathbf{i}$ and $\boldsymbol{\tau}_2 = \mathbf{j}$, then

$$\frac{\partial}{\partial\boldsymbol{\tau}_1}(\nabla\psi) = \frac{\partial}{\partial\mathbf{i}}(\nabla\psi) = \nabla(\nabla\psi)\cdot\mathbf{i} = \frac{\partial}{\partial x}(\nabla\psi) = 0,$$
$$\frac{\partial}{\partial\boldsymbol{\tau}_2}(\nabla\psi) = \frac{\partial}{\partial\mathbf{j}}(\nabla\psi) = \nabla(\nabla\psi)\cdot\mathbf{j} = \frac{\partial}{\partial y}(\nabla\psi) = 0.$$

Hence,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial^2\psi}{\partial x\partial y} = \frac{\partial^2\psi}{\partial x\partial z} = 0 \quad\text{and}\quad \frac{\partial^2\psi}{\partial y^2} = \frac{\partial^2\psi}{\partial y\partial x} = \frac{\partial^2\psi}{\partial y\partial z} = 0.$$

On F_2, $\boldsymbol{\tau} = \alpha\mathbf{i} + \beta\mathbf{k}$, with $\beta \neq 0$. $\frac{\partial}{\partial \boldsymbol{\tau}}(\nabla\psi) = \nabla(\nabla\psi)\boldsymbol{\tau} = 0$, then

$$\alpha \frac{\partial^2\psi}{\partial x \partial z} + \beta \frac{\partial^2\psi}{\partial z^2} = 0.$$

We get $\Delta\psi = 0$ on Γ_{12} the common edge to both faces F_1 and F_2.

To show that T_1 is onto, we use the trace theorem by Grisvard [8]. □

Let us now consider the space

$$\mathbf{E} = \{\mathbf{v} \in \mathbb{H}^2(\Omega) \cap \mathbb{H}^1_0(\Omega) : \operatorname{div}\mathbf{v} \in H^1_0(\Omega)\}.$$

We get, as a corollary of Theorem 2.4, the following result.

Corollary 2.5. *The mapping* $T_2 : \mathbf{v} \to (\gamma_{\mathbf{t}} \operatorname{curl} \mathbf{v}_{|F_i})_{i=1}^N$ *from* $\mathbf{E}$ *on* $\prod_{j=1}^N (H^{1/2}_{0,0}(F_j))^2$ *is linear, continuous and onto.*

Proof. Let $\mathbf{v} \in \mathbf{E}$, to show that $T_2(\mathbf{v}) = (\gamma_{\mathbf{t}} \operatorname{curl} \mathbf{v}_{|F_j})_{j=1}^N \in \prod_{j=1}^N (H^{1/2}_{0,0}(F_j))^2$, we use the same procedure as for the previous Theorem 2.4.

First we show that T_2 is an onto mapping: let $\mathbf{h} \in \prod_{j=1}^N (H^{1/2}_{0,0}(F_j))^2$. By Theorem 2.4, there exists $\boldsymbol{\varphi} \in \mathbb{W}$ such that $\mathbf{h} = \gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi}$. We set $\mathbf{v} = -\operatorname{curl}\boldsymbol{\varphi}$, it is easy to see that $\mathbf{v} \in \mathbf{E}$. On the other hand,

$$\gamma_{\mathbf{t}} \operatorname{curl}\mathbf{v} = -\gamma_{\mathbf{t}}(\operatorname{curl}(\operatorname{curl}\boldsymbol{\varphi})) = \gamma_{\mathbf{t}}(\Delta\boldsymbol{\varphi}) - \gamma_{\mathbf{t}}(\nabla(\operatorname{div}\boldsymbol{\varphi})).$$

As, $\boldsymbol{\varphi} \in \mathbb{H}^2_0(\Omega)$, we obtain $\operatorname{div}\boldsymbol{\varphi} = 0$ on Γ. Therefore $\gamma_{\mathbf{t}}(\nabla(\operatorname{div}\boldsymbol{\varphi})) = 0$. Then, there exists $\mathbf{v}$ in $\mathbf{E}$ such that $\mathbf{h} = \gamma_{\mathbf{t}} \operatorname{curl}\mathbf{v}$. Hence we obtain the expected result. □

In the sequel, we will need the Hilbert space

$$\mathbb{V} = \{\boldsymbol{\psi} \in \mathbb{H}^1(\Omega) : \Delta^2\boldsymbol{\psi} \in \mathbb{L}^2(\Omega)\}.$$

Equipped with the inner product

$$(\!(\mathbf{u},\mathbf{v})\!)_{\mathbb{V}} = (\!(\mathbf{u},\mathbf{v})\!)_{\mathbb{H}^1} + (\!(\Delta^2\mathbf{u}, \Delta^2\mathbf{v})\!)_{\mathbb{L}^2}\,.$$

Note that thanks to formula (2.4), one also obtains the Green's formula: for all $\boldsymbol{\varphi} \in \mathbb{W}$ and all $\boldsymbol{\psi} \in \mathbb{V}$,

$$\int_\Omega \operatorname{curl}\boldsymbol{\psi} \cdot \operatorname{curl}\Delta\boldsymbol{\varphi} = -\int_\Omega \Delta\boldsymbol{\psi}\cdot\Delta\boldsymbol{\varphi} + \langle \gamma_{\mathbf{t}}\Delta\boldsymbol{\varphi}, \operatorname{curl}\boldsymbol{\psi}\rangle.$$

Proposition 2.6.
- *$(\mathcal{D}(\overline{\Omega}))^3$ is dense in $\mathbb{V}$.*
- *The mapping $\gamma_{\mathbf{t}}$ defined on $(\mathcal{D}(\overline{\Omega}))^3$ by $\gamma_{\mathbf{t}}\boldsymbol{\psi} = \boldsymbol{\psi} \times \mathbf{n}_{|\Gamma}$ may be extended in a linear, continuous mapping from $\mathbb{V}$ on $\prod_{j=1}^N \big(H^{-1/2}(F_j)\big)^2$.*

Proof. To show that $(\mathcal{D}(\overline{\Omega}))^3$ is dense in $\mathbb{V}$, one just need to show that if $\mathcal{L}$ is a linear continuous form on $\mathbb{V}$ vanishing on $(\mathcal{D}(\overline{\Omega}))^3$ then it is identically equal to zero.

Let $\mathcal{L}(\cdot)$ be a linear continuous form on $\mathbb{V}$. Then, there exists a unique $\mathbf{u} \in \mathbb{H}^1(\Omega)$, and $\mathbf{f} \in \mathbb{L}^2(\Omega)$ such that

$$\mathcal{L}(\mathbf{v}) = \int_\Omega (\nabla\mathbf{u}\cdot\nabla\mathbf{v} + \mathbf{u}\cdot\mathbf{v})\,dx + \int_\Omega \mathbf{f}\cdot\Delta^2\mathbf{v}\,dx \quad \forall \mathbf{v} \in \mathbb{V}.$$

Suppose that $\mathcal{L}(\boldsymbol{\varphi}) = 0$ for all $\boldsymbol{\varphi} \in (\mathcal{D}(\overline{\Omega}))^3$. Noting $\tilde{\mathbf{u}}$, $\widetilde{\nabla\mathbf{u}}$, and $\tilde{\mathbf{f}}$ the extensions of $\mathbf{u}$, $\nabla\mathbf{v}$, and $\mathbf{f}$ by 0 to $\mathbb{R}^3$. Then

$$\int_{\mathbb{R}^3} (\tilde{\mathbf{u}}\cdot\boldsymbol{\varphi} + \widetilde{\nabla\mathbf{u}}\cdot\nabla\boldsymbol{\varphi} + \tilde{\mathbf{f}}\cdot\Delta^2\boldsymbol{\varphi})\,dx = 0, \quad \forall \boldsymbol{\varphi} \in (\mathcal{D}(\mathbb{R}^3))^3;$$

using a Green formula we obtain

$$\Delta^2\tilde{\mathbf{f}} - \operatorname{div}(\widetilde{\nabla\mathbf{u}}) + \tilde{\mathbf{u}} = 0.$$

We therefore deduce that $\tilde{\mathbf{f}} \in \mathbb{H}^3(\mathbb{R}^3)$. Since $\mathbf{f}$ vanishes outside of the domain Ω, we conclude that $\mathbf{f}$ is in $\mathbb{H}^3_0(\Omega)$. Let, $(\mathbf{f}_n)_{n\in\mathbb{N}}$, be a sequence of $(\mathcal{D}(\Omega))^3$ functions which converge, in $\mathbb{H}^3(\Omega)$, to $\mathbf{f}$. Using the sequence $\mathbf{f}_n$, we obtain

$$\begin{aligned}
\mathcal{L}(\mathbf{v}) &= \int_\Omega (\mathbf{u}\cdot\mathbf{v} + \nabla\mathbf{u}\cdot\nabla\mathbf{v})\,dx + \lim_{n\to\infty}\int_\Omega \mathbf{f}_n\cdot\Delta^2\mathbf{v}\,dx,\\
&= \int_\Omega (\mathbf{u}\cdot\mathbf{v} + \nabla\mathbf{u}\cdot\nabla\mathbf{v})\,dx - \lim_{n\to\infty}\int_\Omega \nabla(\Delta\mathbf{f}_n)\cdot\nabla\mathbf{v}\,dx,\\
&= \int_\Omega (\mathbf{u}\cdot\mathbf{v} + \nabla\mathbf{u}\cdot\nabla\mathbf{v})\,dx - \int_\Omega \nabla(\Delta\mathbf{f})\cdot\nabla\mathbf{v}\,dx.
\end{aligned}$$

$\mathcal{L}$ is then a linear continuous form on $\mathbb{V}$ for the $\mathbb{H}^1(\Omega)$-norm. Finally, as $\mathcal{L}$ vanishes on $(\mathcal{D}(\overline{\Omega}))^3$ which is dense in $\mathbb{H}^1(\Omega)$, it vanishes everywhere. This ends the proof of the first point.

The second statement in the proposition is a consequence of the denseness of $\mathcal{D}(\overline{\Omega}))^3$ in $\mathbb{V}$ and the trace theorem 2.4. □

3. Proof of theorem 2.1

We first introduce some additional functional spaces and some auxiliary problems. Let $\mathbf{Z}$ be a subspace of H^1 with a zero-mean functions given by

$$\mathbf{Z} = \{z \in H^1(\Omega) : \int_\Omega z = 0\}. \tag{3.1}$$

Let $\mathbf{g}_i$ be the boundary data of our initial problem (1.2)–(1.4), and consider the variational problem

$$\begin{gathered}
z \in \mathbf{Z},\\
\int_\Omega \nabla z\cdot\nabla v = \sum_{i=1}^N \langle \mathbf{g}_i\cdot\mathbf{n}_i, \gamma_0 v\rangle, \quad \forall v \in \mathbf{Z}.
\end{gathered}\tag{3.2}$$

The auxiliary problem (3.2) admits a unique solution $z \in \mathbf{Z}$. Moreover, z is the solution of the boundary-value problem

$$\begin{gathered}
\Delta z = 0,\\
\nabla z\cdot\mathbf{n} = H,\\
\int_\Omega z = 0.
\end{gathered}$$

Let us now define the vector

$$\mathbf{u}_1 = \nabla z.$$

It is easy to check that $\mathbf{u}_1$ satisfy the following properties

$$\Delta\mathbf{u}_1 = 0, \quad \operatorname{curl}\mathbf{u}_1 = 0, \quad \operatorname{div}(\mathbf{u}_1) = 0.$$

Therefore,

$$\mathbf{u}_1 \in \mathbb{H}(\operatorname{div},\operatorname{curl},\Omega) \implies (\mathbf{u}_1\cdot\mathbf{n})_{|\Gamma} \in H^{-1/2}(\Gamma).$$

Where

$$\mathbb{H}(\operatorname{div},\operatorname{curl},\Omega) = \big\{\mathbf{v} \in L^2(\Omega)^3; \operatorname{div}(\mathbf{v}) \in L^2(\Omega) \text{ and } \operatorname{curl}(\mathbf{v}) \in L^2(\Omega)^3\big\}.$$

Now, using the result in Jerison-Kenig [11, Theorem 2] and the interpolation result, one gets

$$(\mathbf{u}_1 \cdot \mathbf{n})_{|\Gamma} \in H^{-s'}(\Gamma), \quad s < s' < \frac{1}{2}.$$

Let $p_1 = 0$, then it is easy to see that the couple $(\mathbf{u}_1, p_1)$ is a solution to the Stokes problem

$$\begin{gathered} -\Delta \mathbf{u}_1 + \nabla p_1 = 0, \text{ in } \Omega, \\ \operatorname{div} \mathbf{u}_1 = 0, \quad \text{in } \Omega, \\ \mathbf{u}_1 \cdot \mathbf{n} = H, \quad \text{on } \Gamma. \end{gathered} \tag{3.3}$$

We also consider the auxiliary Stokes problem

$$\begin{gathered} -\Delta \mathbf{w} + \nabla p_0 = 0, \quad \text{in } \Omega, \\ \operatorname{div} \mathbf{w} = 0, \quad \text{in } \Omega, \\ \mathbf{w} \cdot \mathbf{n} = 0, \quad \text{on } \Gamma, \\ \mathbf{w} \times \mathbf{n} = \mathbf{G} - \mathbf{u}_1 \times \mathbf{n} = \mathbf{G}', \quad \text{on } \Gamma. \end{gathered} \tag{3.4}$$

To show the main result of this paper, given in Theorem 2.1, in addition to the space $\mathbf{Z}$ previously introduced (3.1), we will need also the use of the space

$$\mathbf{Z}_0 = \big\{ g \in H_0^1(\Omega); \ \int_\Omega g = 0 \big\}.$$

Following Dauge [6], the Stokes operator

$$\mathbf{S}(\mathbf{v}, q) = (-\Delta \mathbf{v} + \nabla q, \operatorname{div} \mathbf{v})$$

is an isomorphism from $\mathbf{E} \times \mathbf{Z}$ to $L^2(\Omega) \times \mathbf{Z}_0$. Indeed this gives, by transposition, the following result.

Proposition 3.1. *For any linear continuous form $\mathcal{L}(\cdot,\cdot)$ on $\mathbf{E} \times \mathbf{Z}$, there exists a unique $(\mathbf{u}, p)$, element of $L^2(\Omega) \times H^{-1}(\Omega)/\mathbb{R}$ satisfying*

$$\int_\Omega \mathbf{u} \cdot \big(-\Delta \mathbf{v} + \nabla q \big) dx - \big\langle p, \operatorname{div} \mathbf{v} \big\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \mathcal{L}(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{E} \times \mathbf{Z}.$$

Remark 3.2. Note that any p in $\mathbf{Z}_0'$, the dual space of $\mathbf{Z}_0$, may be written as $p = p_0 + c$ where $p_0 \in H^{-1}(\Omega)$ and c is some constant.

Taking into account the previous remark, concerning $\mathbf{Z}_0'$ the dual space of $\mathbf{Z}_0$, one may give an equivalent formulation of the proposition 3.1 as follows: If $\mathcal{L}$ is a linear continuous form on $\mathbf{E} \times \mathbf{Z}$, then there exists a unique $\mathbf{u}$ in $\mathbb{L}^2(\Omega)$ and unique (up to a constant) $p_0 \in H^{-1}(\Omega)$ such that

$$\int_\Omega \mathbf{u} \cdot (-\Delta \mathbf{v} + \nabla q) - \langle p_0, \operatorname{div} \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \mathcal{L}(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{E} \times \mathbf{Z}. \tag{3.5}$$

We now have all the ingredients to prove theorem 2.1.

Proof of Theorem 2.1. Let $s \in \mathbb{R}$ satisfying $-1/2 < s < 1/2$. Then, there exists $\varepsilon \in]0, 1[$, such that $s = -\frac{1}{2} + \varepsilon$. Consider now $\mathbf{g}$ belonging to $\prod_{j=1}^N \mathbb{H}^{-\frac{1}{2}+\varepsilon}(F_j)$. For all $i = 1, \dots, N$, we set $\mathbf{g}_i = \mathbf{g}_{|F_i}$ and recall the tangential part $\mathbf{G}_i$ of functions of $\mathbf{g}_i$ on F_i, previously defined in (2.2). Since $\mathbf{g}_i \in \mathbb{H}^{-\frac{1}{2}+\varepsilon}(F_i)$, we obtain

$$\mathbf{G}_i \in H^{-\frac{1}{2}+\varepsilon}(F_i).$$

Let us now define the linear form $\mathcal{L}$ on the space $\mathbf{E} \times \mathbf{Z}$:

$$\mathcal{L}(\mathbf{v}, q) = \sum_{i=1}^{N} \langle \mathbf{G}'_i, \gamma_{\mathbf{t}} \operatorname{curl} \mathbf{v}_{|F_i} \rangle, \tag{3.6}$$

where $\mathbf{G}'$ is given by $\mathbf{G}' = \mathbf{G} - \mathbf{u}_1 \times \mathbf{n}$, see the last equation of (3.4).

Proposition 3.3. *Let $\mathcal{L}$ be the linear form given in* (3.6)*. There exists $\mathbf{w} \in \mathbb{L}^2(\Omega)$ unique, and $p_0 \in H^{-1}(\Omega)$ unique up to a constant, satisfying*

$$\int_\Omega \mathbf{w} \cdot (-\Delta \mathbf{v} + \nabla q) - \langle p_0, \operatorname{div} \mathbf{v} \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} = \mathcal{L}(\mathbf{v}, q), \quad \forall (\mathbf{v}, q) \in \mathbf{E} \times \mathbf{Z}. \tag{3.7}$$

Moreover, $(\mathbf{w}, p_0)$ is a solution to the Stokes problem (3.4)*.*

Proof. On one hand, it is quite easy to see that the form $\mathcal{L}$ given by (3.6) is linear and continuous on $\mathbf{E} \times \mathbf{Z}$. On the other hand, following the remark 3.2 and the Proposition 3.1, there exists $(\mathbf{w}, p_0) \in \mathbb{L}^2(\Omega) \times H^{-1}(\Omega)$ satisfying the equality (3.7).

Let us show now that the above pair $(\mathbf{w}, p_0)$, obtained in the previous proposition, verify the Stokes equations (3.4). Let w_j, $j = 1, 2, 3$ denote the components of $\mathbf{w}$.

(1) Choosing $\mathbf{v} = (v_1, 0, 0) \in (\mathcal{D}(\Omega))^3$ and $q = 0$, we obtain the following equation

$$-\Delta w_1 + \frac{\partial p_0}{\partial x_1} = 0, \quad \text{in the distribution sense.}$$

and, in a same way, for v_j, $j = 2, 3$, in $\mathcal{D}(\Omega)$, $v_k = 0$, $k = 1, 2, 3$ with $k \neq j$ and $q = 0$, one gets

$$-\Delta w_j + \frac{\partial p_0}{\partial x_j} = 0, \quad j = 1, 2, \text{ in the distribution sense.}$$

Then $-\Delta \mathbf{w} + \nabla p_0 = 0$, which is precisely the first equation of (3.4).

(2) Now, let q_0 be any element of $\mathcal{D}(\Omega)$, choosing $\mathbf{v} = 0$ and $q = q_0 - \frac{1}{|\Omega|} \int_\Omega q_0$. Using (3.7) and (2.1), one gets the incompressibility condition $\operatorname{div} \mathbf{w} = 0$.

Finally,

$$\begin{gathered} \mathbf{w} \in \mathbb{L}^2(\Omega) \\ \operatorname{div} \mathbf{w} = 0 \end{gathered}$$

implies that $\mathbf{w} \cdot \mathbf{n}$ is well defined as an element of the dual $[H^{1/2}(\Gamma)]'$. Moreover, since q is an element of $H^1(\Omega)$, we have

$$\int_\Omega q \operatorname{div} \mathbf{w} + \int_\Omega \nabla q \cdot \mathbf{w} = \langle \mathbf{w} \cdot \mathbf{n}, \gamma_0 q \rangle.$$

Thus, noting that $\gamma_0(H^1(\Omega)) = \gamma_0 \mathbf{Z}$, it follows that

$$\mathbf{w} \cdot \mathbf{n} = 0 \quad \text{and then } \mathbf{w} \cdot \mathbf{n}_i = 0. \tag{3.8}$$

(3) It remains to show that

$$\mathbf{w}_t = \mathbf{G}'_i \quad \text{on } F_i, \text{for } i = 1, \dots, N.$$

On the one hand, as $\mathbf{w} \in \mathbb{L}^2(\Omega)$ satisfying $\operatorname{div} \mathbf{w} = 0$, there exists $\boldsymbol{\psi} \in \mathbb{H}^1(\Omega)$ such that

$$\mathbf{w} = \operatorname{curl} \boldsymbol{\psi} \quad \text{and} \quad \operatorname{div} \boldsymbol{\psi} = 0 \quad \text{since } \mathbf{w} \cdot \mathbf{n} = 0.$$

Consequently, $\boldsymbol{\psi} \in \mathbb{V}$.

On the other hand, $\mathbf{w}$ satisfies the first equation of the Stokes equation $-\Delta\mathbf{w} + \nabla p_0 = 0$. Thus, applying the curl operator to the Stokes equation and using (2.3), we obtain

$$-\Delta(\operatorname{curl}(\operatorname{curl}\boldsymbol{\psi})) + \operatorname{curl}(\nabla p_0) = \Delta^2\boldsymbol{\psi} - \nabla(\operatorname{div}\boldsymbol{\psi}) = \Delta^2\boldsymbol{\psi} = 0.$$

Therefore,

$$\boldsymbol{\psi} \in \mathbb{H}^1(\Omega) \quad \text{and} \quad \Delta^2\boldsymbol{\psi} = 0.$$

Then $\boldsymbol{\psi} \in \mathbb{V}$. Furthermore, we already saw that

$$\boldsymbol{\varphi} \in \mathbb{W} \Longrightarrow \mathbf{v} = -\operatorname{curl}\boldsymbol{\varphi} \in \mathbf{E}.$$

Consequently, let $\mathbf{v} \in \mathbf{E}$ such that $\operatorname{div}\mathbf{v} = 0$; thanks to equality (3.7) one obtains

$$\int_\Omega \mathbf{w}\cdot\Delta\mathbf{v} = \int_\Omega \mathbf{w}\cdot\nabla q + \sum_i \langle \mathbf{G}'_i, \gamma_{\mathbf{t}}\operatorname{curl}\mathbf{v}_{|_{F_i}}\rangle.$$

And then, thanks to (3.8), we have

$$\int_\Omega \mathbf{w}\cdot\Delta\mathbf{v} = \sum_{i=1}^N \langle \mathbf{G}'_i, \gamma_{\mathbf{t}}\operatorname{curl}\mathbf{v}_{|_{F_i}}\rangle = \langle \mathbf{w}_t, \gamma_{\mathbf{t}}\operatorname{curl}\mathbf{v}\rangle.$$

Therefore, using the surjectivity obtained at Corollary 2.5, we obtain

$$\mathbf{w}_t = \mathbf{G}'_i \text{ on } F_i,\ \forall i = 1,\dots,N$$

and then $\mathbf{w}_t = \mathbf{G}'$ on Γ. It is therefore deduced that $(\mathbf{u}, p_0) = (\mathbf{u}_1 + \mathbf{w}, p_0)$ is a solution of our initial problem. This completes the proof. □

In summary, we have the following results:

(i) considering the proposition 3.3: Let $\varepsilon > 0$, for all $(g_1, g_2, g_3) \in \mathbb{H}^{-\frac{1}{2}+\varepsilon}(\Gamma)$, there exists $\mathbf{u} \in \mathbb{L}^2(\Omega)$ unique and p in $H^{-1}(\Omega)$ unique (up to an additive constant) such that

$$\begin{gathered} -\Delta\mathbf{u} + \nabla p = 0 \\ \operatorname{div}\mathbf{u} = 0 \\ \mathbf{u} = (g_1, g_2, g_3) \quad \text{on } \Gamma. \end{gathered}$$

(ii) For $(g_1, g_2, g_3) \in \mathbb{H}^{1/2}(\Gamma)$, there exists (see Temam [14]) $(\mathbf{u}, p)$ unique in $\mathbb{H}^1(\Omega) \times L^2_0(\Omega)$ solution of the Stokes system (3.7)–(2.1).

Then Theorem 2.1 follows then by interpolation.

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