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AN ALTERNATIVE APPROACH TO CRITICAL PDES

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ABSTRACT. In this article, we use an alternative method to prove the existence of an infinite sequence of distinct non-radial nodal G-invariant solutions for critical nonlinear elliptic problems defined in the whole the Euclidean space. Our proof is via approximation of the problem on symmetric bounded domains. The base model problem of interest originating from Physics is stated below:

$$-\Delta u = |u|^{\frac{4}{n-2}} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3.$$

1. INTRODUCTION

In this article, our main motivation is based on the work by Ding [14] which proved the existence of non-radial solutions of the above problem. In this work we find both the type and the number of such solutions. The pleasant surprise is the fact that in order to answer these two questions it needed to use a new method of solving critical, (or supercritical), PDEs, the method itself seems to have particular value in that it can be used and in other types of PDEs. However, the main objective here is to prove the existence of non-radial nodal (sign-changing) solutions, for the following critical nonlinear elliptic problem:

$$-\Delta u = |u|^{\frac{4}{n-2}} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3. \tag{1.1}$$

As mentioned above the problem (1.1) owns its origin in many astrophysical and physical contexts and more precisely in the the Lane-Emden-Fowler problem,

$$\begin{gathered} -\Delta u = u^p, \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a domain with smooth boundary in $\mathbb{R}^N$ and $p > 1$. But its greatest interest lies in its relation to the Yamabe problem (for a complete and detailed study we refer to [5], nevertheless it has an autonomous presence holding an important place among the most famous nonlinear partial differential equations). Indicatively, we refer to the classical papers [20, 14, 28], which are some of the large number of very good papers that are devoted to the study of this problem.

Concerning the resolution of the problem (1.1), our proof is via approximation of the problem on symmetric bounded domains. This method is different from that previously used by other authors referred within this paper and can be used to

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solve polyharmonic equations with supercritical exponent and even in the critical of supercritical case, providing an alternative way of utilizing the best constants of the Sobolev inequalities. Furthermore, it enables us to determine the kind and the number of solutions of the problem.

In problem (1.1) the main difficulty comes from the double lack of compactness. By lack of compactness, we mean that the functionals that we consider do not satisfy the Palais-Smale condition, (i.e. there exists a sequence along which the functional remains bounded, its gradient goes to zero, and does not converge). The first main difficulty comes from the fact that the exponent $2^* = \frac{2n}{n-2} = \frac{4}{n-2} + 1$ is critical, and the second is some extra difficulty because of the lack of compactness in unbounded domains.

The first obstacle can be overcome by obtaining the solutions of the following corresponding problem

$$\begin{gathered} -\Delta u_\varepsilon + \varepsilon a(x) u_\varepsilon = f(x)|u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon, \quad n \geq 3, \\ u_\varepsilon \not\equiv 0 \quad \text{in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon, \end{gathered} \tag{1.2}$$

where $\Omega_\varepsilon, \varepsilon > 0$ is an expanding domain in $\mathbb{R}^n$, $n \geq 3$, invariant under the action of a subgroup G of the isometry group $O(n)$ and $a, f \in C^\infty(\overline{\Omega}_\varepsilon)$ are two smooth $G-$invariant functions on $\overline{\Omega}_\varepsilon$.

The main idea to overcome the second difficulty is to solve the problem (1.2) in a sequence of Ω_εs and henceforth to obtain the solutions of the limit problem (1.1) as the limits of the solutions, as $\frac{1}{\varepsilon}$ tends to ∞, of the sequence of the problem (1.2).

Problem (1.2) has been studied by many authors. We refer to [3, 4, 9, 14, 18, 21] and the references therein for a further discussion of both the problem itself and several variations of it. Some special cases also have been studied. For example, no solution can exist if Ω is starshaped, as a consequence of the Pohozaev identity (see in [30]). Furthermore, if Ω is an annulus, there are infinite solutions (see in [26]). Also, a general result of Bahri and Coron guarantees the existence of positive solutions in domains Ω having nontrivial topology (i.e. certain homology groups of Ω are non trivial) (see in [6]). The existence and multiplicity of positive or nodal solutions of critical equations on bounded domains or in some contractible domains have been determined by other authors (see for example in [15, 18, 21, 29, 33]). Some more nonexistence results in this case are available, (see in [1, 4, 11, 23]).

As we have mentioned above, in problem (1.2) the main difficulty comes because the exponent 2^* is the critical exponent for the Sobolev imbedding $H_1^2(\Omega) \hookrightarrow L^p(\Omega)$. Because the Sobolev embedding $H_1^2(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any real p such that $1 \leq p < 2^*$ while if $1 \leq p \leq 2^*$ is only continuous (see in [5]), in our case we have to solve a variational problem with lack of compactness. The symmetry property of the domain allows us to improve the Sobolev embedding in higher L^p spaces and to overcome this obstruction. More precisely, it is well known that if (M, g) is a smooth compact Riemannian $n-$manifold invariant under the action of an arbitrary compact subgroup G of $\mathrm{Isom}_g(M)$, $O_G^x = \{\sigma(x), \sigma \in G\}$, $\mathrm{Card} O_G^x = \infty$ and $k = \min_{x \in M} \dim O_G^x$, then $k \geq 1$ and the Sobolev embedding $H_{1,G}^2(\Omega) \hookrightarrow L^p(\Omega)$ is compact for any $1 \leq p < \frac{2(n-k)}{n-k-2}$ but if $1 \leq p \leq \frac{2(n-k)}{n-k-2}$ is only continuous (see in [22], [17], [13]). Thus, in our case the symmetry property of Ω_εs allows us to solve problems with subcritical exponent using the classical variation method.

As a small overview on the history and progress of the study of our problem we mention the following: Loewner and Nirenberg [27] studied problem (1.1) for

$n = 4$. Gidas, Ni and Nirenberg in their celebrated paper [20] proved symmetry and some related properties of positive solutions of a larger class of second order elliptic equations. Concerning problem (1.1) they proved that any positive solution, which has finite energy, namely $\int_{\mathbb{R}^n} |\nabla u|^2 dx < \infty$, is necessarily of the form

$$u(x) = \Big(\frac{\lambda\sqrt{n(n-2)}}{\lambda^2 + |x - x_0|^2}\Big)^{(n-2)/2},$$

where $\lambda > 0, x_0 \in \mathbb{R}^n$. These solutions yield the well-known one-instanton solutions in a regular gauge of the Yang-Mills equation. In addition, since the equation

$$-\Delta u = |u|^{\frac{4}{n-2}} u, \quad n \geq 3,$$

is invariant under the conformal transformations of $\mathbb{R}^n$, if $u(x)$ is a solution, then for any $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, $\lambda^{\frac{n-2}{2}} u(\frac{x-x_0}{\lambda})$ is also a solution. Moreover, all solutions obtained in this way have the same energy and we will say that these solutions are equivalent. In particular, all these solutions are equivalent. Ding in [14] used Ambrosetti and Rabinowitz analysis (see in [2]) to prove that this problem has infinite distinct solutions $u \in C^2(\mathbb{R}^n)$, with finite energy and which changes sign, but he did not specify the type of these solutions. Caffarelli, Gidas and Spruck in their classical paper [10] studied non-negative smooth solutions of the conformally invariant equation

$$-\Delta u = u^{\frac{n+2}{n-2}}, \quad u \geq 0, \quad n \geq 3,$$

in a punctured ball $B_1(0)\backslash\{0\} \subseteq \mathbb{R}^n$, with an isolated singularity at the origin. In this paper, the authors introduced a heuristic idea of asymptotic symmetry technique which can roughly be described as follows: After an inversion, the function u becomes defined in the complement of B_1, is strictly positive of ∂B_1, and in some sense 'goes to zero' at infinity. If the function u can be extended to B_1 as a super solution of our problem, then it can start the reflection process at infinity and moved all the way to ∂B_1. This would imply asymptotic radial symmetry at infinity. With this comprehensive report on this issue we would like, on the one hand, to emphasize the important contribution of this great article of Caffarelli, Gidas and Spruck on the study on the direction of finding the radial solutions of our problem and on the other hand, we wish to make clear that in our procedural paper we do not care about the radial solutions but we do care about the existence of non-radial solutions. Schoen in [31] built solutions of (1.1) with prescribed isolated singularities. In another paper [32], Schoen and Yau have used the geometrical meaning of problem (1.1) in order to derive, through ideas of conformal geometry, the existence of weak solutions having a singular set whose Hausdorff dimension is less than or equal to $\frac{n-2}{2}$. Let us notice that in this paper the authors explain how to build solutions of (1.1) with a singular set whose Hausdofff dimension is not necessarily an integer. Mazzeo and Smale have proved [28] the existence of singular solutions of (1.1) for a very large variety of singular sets. Bartsch and Schneider in [7] proved that for $N > 2m$ the equation

$$(-\Delta)^m = |u|^{\frac{4m}{N-2m}} u$$

on $\mathbb{R}^N$ has a sequence of nodal, finite energy solutions which is unbounded in $\mathcal{D}^{m,2}(\mathbb{R}^N)$, the completion of $\mathcal{D}(\mathbb{R}^N)$ with respect to the scalar product

$$(u,\upsilon)=\begin{cases}\int_{\mathbb{R}^N}\Delta^{m/2}u\cdot\Delta^{m/2}\upsilon, & \text{if } m \text{ is even}\\ \int_{\mathbb{R}^N}\nabla\Delta^{(m-1)/2}u\cdot\nabla\Delta^{(m-1)/2}\upsilon, & \text{if } m \text{ is odd.}\end{cases}$$

This generalizes the result of Ding for $m=1$, and provides interesting information concerning the number and the kind of the solutions of the equation (see Remark 3.5). Finally, for reasons of completeness, we refer in this point to the paper of Wang [35] where the following nonlinear Neumann elliptic problem is studied:

$$\begin{gathered}-\Delta u=u^{\frac{N+2}{N-2}},\quad u>0\quad\text{in } \mathbb{R}^N\backslash\Omega,\\ u(x)\to 0\quad\text{as } |x|\to+\infty,\\ \frac{\partial u}{\partial n}=0\quad\text{on } \partial\Omega,\end{gathered}\tag{1.3}$$

where n denotes interior unit normal vector and Ω is a smooth bounded domain in $\mathbb{R}^N$, $N\geq 4$. In this paper, it is proved that if $N\geq 4$, (Wang believes that the results will also hold in the case of $N=3$), and Ω is a smooth and bounded domain then the problem (1.3) has infinity many non-radial positive solutions, whose energy can be made arbitrarily large when Ω is convex as seen from inside (with some symmetries). We refer to the Wang's problem (1.3) due to its close relationship with our problem and as we will see later if we choose suitable Ω we can have a result on this problem in almost all the space. In particular, in both problems we have to solve the same non-linear differential equation with critical exponent with boundary conditions Dirichlet and Neumann respectively. In addition, in both cases the domain Ω presents some symmetries. However, a subsequent process in each case is completely different from that of another. In our case, our goal is to solve the problem in the whole space, starting from an open symmetric domain Ω of n-dimensional space and we extend Ω so that it remains symmetrical to fill almost all the space. In the other case is considered the corresponding Neumann problem in $\mathbb{R}^N\backslash\Omega$ where Ω is convex seen from inside with some symmetries. If we choose appropriate a such Ω with a small volume as much as we can say that the solutions of Wang satisfy the conditions of the problem in almost all the space. Finally, in both problems we take infinity many non-radial solutions, whose energy can be made arbitrary large, however in the first problem we find nodal solutions while in the second are founded positive solutions.

In this research our goal is to specify the kind and the number of solutions of the problem (1.1). We prove the existence of a sequence $\{u_k\}$ of non-radial, inequivalent, nodal G-invariant solutions of (P), such that: $\lim_{k\to\infty}\int_{\mathbb{R}^n}|\nabla u_k|^2dx=\infty$.

This article is arranged as follows: Section 2 is devoted to notation and some necessary preliminary results. In addition, in this section two examples are presented. Furthermore, in Section 2, we introduce our main tool, meaning the process through which an open symmetric domain of $n-$ dimensional space can be extended in an appropriate manner to *'fill'* eventually the entire space *'almost everywhere'*, remaining symmetric, and subsequently we solve the auxiliary problem (1.2). Section 3 is devoted to some basic definitions and to the proof of the main theorem.

2. Some notation and preliminary results

Let $C^\infty(\Omega)$ be the space of smooth functions on Ω and $\mathcal{D}(\Omega)$ be the set of infinitely differentiable functions whose support is compact in Ω. We define, also, the Sobolev space $H_1^2(\Omega)$ as the completion of $C^\infty(\Omega)$ with respect to the norm:

$$\|u\|_{H_1^2(\Omega)} = \Big(\int_\Omega (|\nabla u|^2 + |u|^2)dx\Big)^{1/2}.$$

The Sobolev space $\mathring{H}_1^2(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $H_1^2(\Omega)$.

In the following, we suppose that Ω is a bounded, smooth, domain of $\mathbb{R}^n$, $n \ge 3$, G-invariant under the action of a compact subgroup G of the isometry group $O(n)$, without finite subgroup. Such Ωs in $\mathbb{R}^n$ can be constructed as follows:

Let Ω be a bounded, smooth, domain of $\mathbb{R}^n = \mathbb{R}^k \times \mathbb{R}^{n-k}$, $k \ge 2$, $n-k \ge 1$ such that $\overline{\Omega} \subset (\mathbb{R}^k \backslash \{0\}) \times \mathbb{R}^{n-k}$. Suppose that $\overline{\Omega}$ is invariant under the action of $G_{k,n-k}$, that is $\tau(\overline{\Omega}) = \overline{\Omega}$ for all $\tau \in G_{k,n-k}$, where $G_{k,n-k} = O(k) \times Id_{n-k}$ is the subgroup of the isometry group $O(n)$ of the type:

$$(x_1, x_2) \to (\sigma(x_1), x_2), \quad \sigma \in O(k), \quad x_1 \in \mathbb{R}^k,\ x_2 \in \mathbb{R}^{n-k}.$$

Then $\overline{\Omega}$ is a bounded, smooth, domain of $\mathbb{R}^n$, invariant under the action of the subgroup $G_{k,n-k}$ of the isometry group $O(n)$. We denote by $H_{1,G}^2(\Omega)$ and $\mathring{H}_{1,G}^2(\Omega)$ the subspaces of $H_1^2(\Omega)$ and $\mathring{H}_1^2(\Omega)$ of all G-invariant functions, respectively.

We consider the functional

$$J(u) = \int_\Omega (|\nabla u|^2 + a(x)u^2)dx,$$

and suppose that the operator $L(u) = -\Delta u + a(x)u$ is *coercive*. That is, there exists a real number $\lambda > 0$, such that for all $u \in \mathring{H}_1^2(\Omega)$:

$$J(u) \ge \lambda \int_\Omega (|\nabla u|^2 + u^2)dx.$$

For example, the operator L is coercive if $a(x) \ge 0$, for all $x \in \Omega$, and more generally when $a(x)$ is greater than minus the best Poincaré constant of $\mathring{H}_1^2(\Omega)$.

We consider now the problem

$$\begin{gathered} -\Delta u + a(x)u = f(x)|u|^{\frac{4}{n-2}}u, \quad n \ge 3, \\ u \not\equiv 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{2.1}$$

where Ω is defined as above and a, f are two smooth G-invariant functions.

For any small $\varepsilon > 0$ and some $m > 0$ we consider the family of expanding domains:

$$\Omega_\varepsilon = \varepsilon^{-m}\Omega = \{\varepsilon^{-m}x : x \in \Omega\}.$$

Then, it is very simple to confirm that the Ω_εs inherit the symmetry properties of Ω for any ε. We consider, also, the transformation:

$$\phi : \Omega \to \Omega_\varepsilon :\ X = \phi(x),\ x \in \Omega,\ X \in \Omega_\varepsilon, \tag{2.2}$$

and for $l > 0$, we set

$$u(x) = \varepsilon^{-l}u_\varepsilon(X). \tag{2.3}$$

In particular, we obtain

$$|\nabla u| = \varepsilon^{-l-m}|\nabla u_\varepsilon|, \tag{2.4}$$

$$\Delta u = \varepsilon^{-l-2m}\,\Delta u_\varepsilon. \tag{2.5}$$

Applying the transformation (2.2))in the equation of problem (2.1), because of (2.3), (2.4) and (2.5) we obtain the equation

$$-\Delta u_\varepsilon + \varepsilon^{2m} a(x) u_\varepsilon = \varepsilon^{2m - l\frac{4}{n-2}} f(x) |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon,$$

where we denote again by a and f the functions $a \circ \phi^{-1}$ and $f \circ \phi^{-1}$, respectively and the independent variable by x. Since l is an arbitrary positive real, we can choose $l = 2m\frac{n-2}{4}$ and thus we have

$$-\Delta u_\varepsilon + \varepsilon^{2m} a(x) u_\varepsilon = f(x) |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon. \tag{2.6}$$

From (2.6), replacing the ε^{2m} by ε, we obtain

$$-\Delta u_\varepsilon + \varepsilon a(x) u_\varepsilon = f(x) |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon.$$

So, we have to solve the critical problem

$$\begin{gathered} -\Delta u_\varepsilon + \varepsilon a(x) u_\varepsilon = f(x) |u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon, \quad n \geq 3, \\ u_\varepsilon \not\equiv 0 \quad \text{in } \Omega_\varepsilon, \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega_\varepsilon. \end{gathered} \tag{2.7}$$

We consider the functional

$$J(u_\varepsilon) = \int_{\Omega_\varepsilon} (|\nabla u_\varepsilon|^2 + \varepsilon a(x) u_\varepsilon^2) dx.$$

Since the operator $L(u) = -\Delta u + a(x)u$ is considered to be coercive in Ω the operator $L(u_\varepsilon) = -\Delta u_\varepsilon + \varepsilon a(x) u_\varepsilon$ is coercive in Ω_ε.

Denote

$$\mathcal{H}_\varepsilon = \big\{ u_\varepsilon \in \mathring{H}^2_{1,G}(\Omega_\varepsilon) : \int_{\Omega_\varepsilon} f(x) |u_\varepsilon|^{\frac{2n}{n-2}} dx = 1 \big\},$$

and suppose that an isometry σ such as $\sigma(\Omega_\varepsilon) = \Omega_\varepsilon$ exists. Furthermore, suppose also that the functions $a(x)$ and $f(x)$ are invariant under the action of σ and that

$$\mathcal{H}^\sigma_\varepsilon = \mathcal{H} \cap \big\{ u_\varepsilon \in \mathring{H}^2_1(\Omega_\varepsilon) : u_\varepsilon \circ \sigma = -u_\varepsilon \big\} \neq \emptyset.$$

By definition, a function u which satisfies $u \circ \sigma = -u$ is called *antisymmetrical*.

Under the above considerations the following theorem holds (see in [12]).

Theorem 2.1. *Problem* (1.2)*, always, has a non-radial nodal solution u. Moreover, if $f(x) > 0$ for all $x \in \overline{\Omega}_\varepsilon$,* (1.2) *has an infinity sequence $\{u_{\varepsilon_i}\}$ of non-radial nodal solutions, such that*

$$\lim_{i\to\infty} \int_{\Omega_\varepsilon} (|\nabla u_{\varepsilon_i}|^2 + u_{\varepsilon_i}^2) dx = +\infty.$$

In addition, u and $\{u_{\varepsilon_i}\}_{i=1,2,\dots}$ are G-invariant and σ-antisymmetrical.

Remark 2.2. Theorem 2.1 holds for the supercritical case and even to the critical of the supercritical case, (see in [12]), namely for every p such that:

$$\frac{2n}{n-2} < p \leq \frac{2(n-k)}{n-k-2}.$$

3. Solution of problem (1.1)

Because of the double lack of compactness, direct variational methods are not applicable to the limit problem

$$-\Delta u = |u|^{\frac{4}{n-2}} u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3. \tag{3.1}$$

However, this method is successful in approximating a solution to the problem (1.1) by solutions in the open domains Ω_{ε_j}. Thus, a solution to (1.1) may be then obtained by the limit procedure as $\varepsilon_j \to 0$.

Before we approximate the solutions in $\mathbb{R}^n$ by solutions in bounded domains $\Omega_\varepsilon \in \mathbb{R}^n$, we note that, in the generalized setting of the problems in Ω_εs, the Dirichlet condition $u_\varepsilon(x) = 0$ on $\partial\Omega_\varepsilon$ may actually be included in the condition $u_\varepsilon \in \mathring{H}^2_1(\Omega_\varepsilon)$. Moreover, since any function $u_\varepsilon \in \mathring{H}^2_1(\Omega_\varepsilon)$ can be extended onto $\mathbb{R}^n$ by

$$\tilde{u}_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & x \in \Omega_\varepsilon \\ 0, & x \in \mathbb{R}^n \backslash \Omega_\varepsilon, \end{cases}$$

generalized solutions may be defined in Ω_εs analogously to the case in $\mathbb{R}^n$. We need now the following two definitions:

Definition 3.1. A function $u_\varepsilon \in \mathring{H}^2_1(\Omega_\varepsilon)$ is a *generalized solution* of (1.2) if the function

$$g(x, u_\varepsilon) = \varepsilon a(x) u_\varepsilon - f(x)|u_\varepsilon|^{\frac{4}{n-2}} u_\varepsilon$$

is locally integrable and for all $\varphi \in C_0^\infty(\Omega_\varepsilon)$, the following holds:

$$\int_{\Omega_\varepsilon} (\nabla u_\varepsilon, \nabla\varphi) dx + \int_{\Omega_\varepsilon} g(x, u_\varepsilon)\varphi dx = 0.$$

Definition 3.2. A function $u_\varepsilon \in C^2(\Omega_\varepsilon) \cap C(\overline{\Omega}_\varepsilon)$ is a *classical solution* to (1.2) if after substituting it into the equation of (1.2), this equation becomes an identity at each $x \in \Omega_\varepsilon$ and $u_\varepsilon(x) = 0$ provided $x \in \partial\Omega_\varepsilon$.

Consider now a sequence of real numbers $\{\varepsilon_j\}_{j=1,2,\dots}$ such that $\varepsilon_j \to 0$ as $j \to \infty$ and the sequence of problems:

$$\begin{gathered} -\Delta u_{\varepsilon_j} + \varepsilon_j a(x) u_{\varepsilon_j} = f(x)|u_{\varepsilon_j}|^{\frac{4}{n-2}} u_{\varepsilon_j}, \quad n \geq 3 \\ u_{\varepsilon_j} \not\equiv 0 \quad \text{in } \Omega_{\varepsilon_j}, \quad u_{\varepsilon_j} = 0 \quad \text{on } \partial\Omega_{\varepsilon_j}, \end{gathered} \tag{3.2}$$

where $f(x) > 0$ for all $x \in \overline{\Omega}_{\varepsilon_j}$. Then, the following theorem on approximation by bounded domains holds.

Theorem 3.3. *The problem*

$$-\Delta u = f(x)|u|^{\frac{4}{n-2}} u \quad \text{in } \mathbb{R}^n, \quad n \geq 3 \tag{3.3}$$

has a generalized non-radial nodal G-invariant and σ-antisymmetrical solution u and there is a subsequence $\{u_j\}$, such that

$$u_j \rightharpoonup u \quad \text{in } H^2_{1,G} \text{ as } j \to +\infty.$$

Proof. According to Theorem 2.1, every problem (3.2) has at least one non-radial nodal G-invariant and σ-antisymmetrical solution u_{ε_j}. Let $u_{\varepsilon_j}, j = 1, 2, \dots$ an arbitrary sequence of such solutions. Since the problem (3.2) has a nontrivial solution belonging to one of the spaces considered earlier, then for any $\lambda > 0$ the function

$$\upsilon_{\varepsilon_j} = \lambda^{\frac{n-2}{4}} u_{\varepsilon_j} \in \mathring{H}^2_1(\Omega_{\varepsilon_j})$$

is a non trivial solution to the problem

$$\begin{gathered}-\Delta \upsilon_{\varepsilon_j} + \varepsilon_j a(x)\upsilon_{\varepsilon_j} = \lambda f(x)|\upsilon_{\varepsilon_j}|^{\frac{4}{n-2}}\upsilon_{\varepsilon_j}, \quad n \geq 3, \\ \upsilon_{\varepsilon_j} \not\equiv 0 \quad \text{in } \Omega_{\varepsilon_j}, \quad \upsilon_{\varepsilon_j} = 0 \quad \text{on } \partial\Omega_{\varepsilon_j}.\end{gathered} \tag{3.4}$$

For

$$\lambda = \|u_{\varepsilon_j}\|_{H_1^2(\Omega_{\varepsilon_j})}^{\frac{-4}{n-2}}$$

we conclude that

$$\upsilon_{\varepsilon_j} = \frac{u_{\varepsilon_j}}{\|u_{\varepsilon_j}\|_{H_1^2(\Omega_{\varepsilon_j})}},$$

which means that the sequence $\{\upsilon_{\varepsilon_j}\}$ is bounded in $\mathring{H}_1^2(\Omega_{\varepsilon_j})$ for all $j = 1, 2, \dots$. Thus, there exists a constant C not dependent on j and such that

$$\|\upsilon_{\varepsilon_j}\|_{H_1^2(\Omega_{\varepsilon_j})} \leq C. \tag{3.5}$$

Because of the reflexivity of $\mathring{H}_1^2(\mathbb{R}^n)$ and condition (3.5) we may choose a subsequence $\{\upsilon_j\}$ of the sequence $\{\upsilon_{\varepsilon_j}\}$ such that

$$\upsilon_j \rightharpoonup \upsilon \quad \text{in } \mathring{H}_1^2(\mathbb{R}^n) \quad \text{as } j \to +\infty. \tag{3.6}$$

We shall show that υ is a nontrivial G-invariant generalized solution to the problem (3.2). We choose an arbitrary $\varphi \in \mathcal{D}(\mathbb{R}^n)$. Then, according to the definition of $\mathcal{D}(\mathbb{R}^n)$, the support of φ is bounded in $\mathbb{R}^n$, which means that there is an Ω_{ε_0} such that $\operatorname{supp}\varphi \subset \Omega_{\varepsilon_0}$. Since, by definition, the Ω_{ε_j}s constitute a family of expanding domains, we can choose the Ω_{ε_0} such that $\Omega_{\varepsilon_0} \subset \Omega_{\varepsilon_1}$ and so $\Omega_{\varepsilon_0} \subset \Omega_{\varepsilon_j}$ for all $j = 1, 2, \dots$. Let

$$g(x, \upsilon_j) = -\varepsilon_j a(x)\upsilon_{\varepsilon_j} + \lambda f(x)|\upsilon_{\varepsilon_j}|^{\frac{4}{n-2}}\upsilon_{\varepsilon_j}.$$

Then, because the υ_j is a generalized solution to (3.4), it holds

$$\begin{aligned}\int_{\mathbb{R}^n} \nabla\upsilon_j \nabla\varphi\, dx &= \int_{\Omega_{\varepsilon_j}} \nabla\upsilon_j \nabla\varphi dx \\ &= -\int_{\Omega_{\varepsilon_j}} g(x, \upsilon_j)\varphi dx \\ &= -\int_{\Omega_{\varepsilon_0}} g(x, \upsilon_j)\varphi dx\end{aligned} \tag{3.7}$$

for all Ω_{ε_j}. By the weak convergence (3.6), we obtain the following limit relation for the left-hand side of (3.7):

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \nabla\upsilon_j \nabla\varphi dx = \int_{\mathbb{R}^n} \nabla\upsilon \nabla\varphi dx. \tag{3.8}$$

In addition, it is well known, (see [22]), that the critical exponent of the Sobolev embedding $H_{1,G}^2(\Omega_{\varepsilon_0}) \hookrightarrow L^p(\Omega_{\varepsilon_0})$ is equal to

$$\frac{2(n-k)}{n-k-2} > \frac{2n}{n-2} = 2^*,$$

from which it follows that for any real number p, such that:

$$1 < p < \frac{2(n-k)}{n-k-2}$$

this embedding is compact and then from the Sobolev and Kondrashov theorems together and (3.6) arises that:

$$\upsilon_j \to \upsilon \quad \text{in } L^{p_0-1}(\Omega_{\varepsilon_0}),\ 2 < p_0 < \frac{2(n-k)}{n-k-2} + 1, \quad \text{as } j \to +\infty. \tag{3.9}$$

Furthermore, by definition of $a(x)$ and $f(x)$, there exists a positive constant C such that:

$$|g(x,t)| \le C(|t| + |t|^{p_0-1}), \quad 2 < p_0 < \frac{2(n-k)}{n-k-2} + 1,$$

for almost all $x \in \Omega_{\varepsilon_j}$, $j = 1, 2, \dots$ and for all $t \in \mathbb{R}$. Thus, the Vainberg-Krasnoselskii Theorem (see [24] or [34]) gives:

$$\varphi g(\cdot, \upsilon_j(\cdot)) \to \varphi g(\cdot, \upsilon(\cdot)) \quad \text{in } L^{\frac{n}{n-2}}(\Omega_{\varepsilon_0}) \quad \text{as } j \to +\infty. \tag{3.10}$$

By the Hölder inequality from (3.10) follows that

$$\varphi g(\cdot, \upsilon_j(\cdot)) \to \varphi g(\cdot, \upsilon(\cdot)) \quad \text{in } L^1(\Omega_{\varepsilon_0}) \quad \text{as } j \to +\infty. \tag{3.11}$$

By (3.11) the limit relation from the right hand-side of (3.7) yields:

$$\lim_{j\to\infty} \int_{\Omega_{\varepsilon_0}} g(x, \upsilon_j)\varphi\, dx = \int_{\Omega_{\varepsilon_0}} g(x, \upsilon)\varphi\, dx. \tag{3.12}$$

Finally, passing to the limit in (3.7) because of (3.6) and (3.12), we obtain

$$\int_{\mathbb{R}^n} \nabla \upsilon \nabla \varphi\, dx = -\int_{\Omega_{\varepsilon_0}} g(x, \upsilon)\varphi\, dx = -\int_{\mathbb{R}^n} g(x, \upsilon)\varphi dx,$$

which corresponds to the definition of a weak solution. It is generalized by the force of (3.6) and since the function f is regular enough it is a classical solution, (see [25, Secs. 1.2 and 3.1]). As convergence in L^p spaces implies a.e. convergence by (3.9) follows that the function υ will be G-invariant.

It remains to prove that this solution is nontrivial. Suppose, by contradiction, that $\upsilon \equiv 0$. Then, for any $\varepsilon > 0$ there exists a positive integer j_{01} such that

$$|\upsilon| < \frac{\varepsilon}{2} \quad \text{for all } j > j_{01}. \tag{3.13}$$

On the other hand, from (3.9) by the Hölder inequality arises that $\upsilon_j \to \upsilon$ in $L^1(\Omega_{\varepsilon_0})$, which means that for any $\varepsilon > 0$ there exists a positive integer j_{02} such that

$$|\upsilon_j - \upsilon| < \frac{\varepsilon}{2} \quad \text{for all } j > j_{02}. \tag{3.14}$$

Therefore, by the standard inequality $|\upsilon_j| \le |\upsilon_j - \upsilon| + |\upsilon|$ by (3.13) and (3.14) we obtain

$$|\upsilon_j| < \varepsilon \quad \text{for any } j \ge j_0 = \max\{j_{o1}, j_{02}\}. \tag{3.15}$$

We recall now that every solution to the problem (3.2) belongs to the set

$$\mathcal{H}^\sigma_\varepsilon = \big\{u_\varepsilon \in \overset{\circ}{H}{}^2_{1,G}(\Omega_{\varepsilon_j}) : u_{\varepsilon_j} \circ \sigma = -u_{\varepsilon_j} \text{ and } \int_{\Omega_{\varepsilon_j}} f(x)|u_{\varepsilon_j}|^{\frac{2n}{n-2}} dx = 1\big\}$$

Since every υ_j corresponds to an $u_{\varepsilon_j} \in \mathcal{H}^\sigma_\varepsilon$, and $\upsilon_{\varepsilon_j} = \lambda^{\frac{n-2}{4}} u_{\varepsilon_j}$, by definition, we have

$$1 = \int_{\Omega_{\varepsilon_j}} f(x)\lambda^{-n/2}|\upsilon_j|^{\frac{2n}{n-2}}\, dx < \int_{\Omega_{\varepsilon_j}} f(x)\lambda^{-n/2}\varepsilon^{\frac{2n}{n-2}} dx,$$

which is false by (3.15) as the $\varepsilon > 0$ can be chosen as small as we want. We have proved that the limit problem

$$-\Delta \upsilon = \lambda f(x)|\upsilon|^{\frac{4}{n-2}}\upsilon \quad \text{in } \mathbb{R}^n, \quad n \geq 3 \tag{3.16}$$

has a generalized non-radial nodal G-invariant and σ-antisymmetrical solution υ, which means that the function $u = \lambda^{\frac{2n}{n-2}}\upsilon$ is a generalized non-radial nodal G-invariant and σ-antisymmetrical solution to the limit problem

$$-\Delta u = f(x)|u|^{\frac{4}{n-2}}u \quad \text{in } \mathbb{R}^n, \quad n \geq 3. \tag{3.17}$$

This completes the proof of the theorem. □

Corollary 3.4. *The problem*

$$-\Delta u = |u|^{\frac{4}{n-2}}u, \quad u \in C^2(\mathbb{R}^n), \quad n \geq 3 \tag{3.18}$$

has a sequence $\{u_k\}$ *of non-radial nodal G-invariant and σ-antisymmetrical solutions, such that*

$$\lim_{k\to+\infty}\int_{\mathbb{R}^n}|\nabla u_k|^2dx = +\infty$$

The proof of the above corollary is obtained immediately if we put

$$f(x) = \frac{1}{|\Omega_\varepsilon|} - \varepsilon|x|^\alpha, \quad \alpha > -n$$

and follow the steps of Theorem 3.3.

Remark 3.5. The number of the sequences of non-radial nodal G-invariant and σ-antisymmetrical solutions to problem (1.1), depends on the number of all subgroups of $O(n)$ of which the cardinal of orbits with minimum volume is infinite, that are on the dimension n of the domain.

To formulate our last result which is a direct conclusion from [35], we have to repeat some assumptions about Ω. Suppose that Ω is a smooth and bounded domain of $\mathbb{R}^n = \mathbb{R}^2 \times \mathbb{R}^{n-2}$, $n \geq 4$, satisfying the following properties: Let $x = (t_1, t_2, \dots, t_n) = (x_1, x_2) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$, and let $r = |x_1|$. Then:

- (H1) $x \in \Omega$ if and only if $(t_1, t_2, \dots, -t_j, \dots, t_n) \in \Omega$ for $j = 3, 4, \dots, n$;
- (H2) $(r\cos\theta, r\sin\theta, x_2) \in \Omega$ if $(r, 0, x_2) \in \Omega$, for all $\theta \in (0, 2\pi)$;
- (H3) There exists a connected component Γ of $\partial\Omega \cap \{x_2 = 0\}$, such that $H(x) \equiv \gamma > 0$ for all $x \in \Gamma$, where $H(x)$ is the mean curvature of $\partial\Omega$ at $x \in \partial\Omega$.

Remark 3.6. From (H2) arises that Γ is a circle in the plane $t_3 = \dots = t_n = 0$, and since for $x \in \Gamma$, $H(x) = \frac{\sum_{j=1}^{n-1} k_j(x)}{n-1}$, where $k_j(x)$ are the principal curvatures and $k_1(x) = \frac{1}{\sqrt{t_1^2+t_2^2}}$, implies that $H(x) \equiv \gamma = \frac{1}{\sqrt{t_1^2+t_2^2}}$, which means that a such domain is very common, e.g. a ball or an ellipsoid.

Corollary 3.7. *Suppose that Ω is a smooth bounded domain satisfying* (H1)–(H3). *Then the problem*

$$\begin{gathered} -\Delta u = u^{\frac{n+2}{n-2}} \quad u > 0 \quad \text{in } \mathbb{R}^n\backslash\Omega, \\ u(x) \to 0 \quad \text{as } |x| \to +\infty, \\ \frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.19}$$

has infinitely many non-radial positive solutions, whose energy can be made arbitrary large.

In particular, problem (3.19) *has in* $\mathbb{R}^n$*, (apart from a set* Ω *of finite measure arbitrary small), infinity many non−radial positive solutions, whose energy can be made arbitrary large, in the sense that we can choose an* Ω *with the above refereed properties and the additional property* $|\Omega| < \varepsilon$ *for given* $\varepsilon > 0$*.*

The proof of the above Corollary follows by [35, Theorem 1.1].

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