Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 152, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

FINITE TIME BLOW-UP OF SOLUTIONS FOR A NONLINEAR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS

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Communicated by Mokhtar Kirane

Abstract. In this article we study the blow-up in finite time of solutions for the Cauchy problem of fractional ordinary equations

$$u_t + a_1\, {}^cD^{\alpha_1}_{0_+}u + a_2\, {}^cD^{\alpha_2}_{0_+}u + \dots + a_n\, {}^cD^{\alpha_n}_{0_+}u = \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} f(u(s), v(s))ds,$$

$$v_t + b_1\, {}^cD^{\beta_1}_{0_+}v + b_2\, {}^cD^{\beta_2}_{0_+}v + \dots + b_n\, {}^cD^{\beta_n}_{0_+}v = \int_0^t \frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)} g(u(s), v(s))ds,$$

for $t > 0$, where the derivatives are Caputo fractional derivatives of order α_i, β_i, and f and g are two continuously differentiable functions with polynomial growth. First, we prove the existence and uniqueness of local solutions for the above system supplemented with initial conditions, then we establish that they blow-up in finite time.

1. Introduction

In this work, we study the system of ordinary fractional differential equations

$$\begin{aligned} &u_t + a_1\, {}^cD^{\alpha_1}_{0_+}u + a_2\, {}^cD^{\alpha_2}_{0_+}u + \dots + a_n\, {}^cD^{\alpha_n}_{0_+}u \\ &= \int_0^t \frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)} f(u(s), v(s))ds, \\ &v_t + b_1\, {}^cD^{\beta_1}_{0_+}v + b_2\, {}^cD^{\beta_2}_{0_+}v + \dots + b_n\, {}^cD^{\beta_n}_{0_+}v \\ &= \int_0^t \frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)} g(u(s), v(s))ds, \end{aligned} \tag{1.1}$$

for $t > 0$, with initial data

$$u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \tag{1.2}$$

and where $0 < \alpha_i < 1$, $0 < \beta_i < 1$, $i = 1, \dots, n$, $0 < \gamma_j < 1$, $j = 1, 2$, f and g are two real continuous differentiable functions defined on $\mathbb{R} \times \mathbb{R}$, a_i, b_i $i = 1, \dots, n$ are positive constants, Γ is the Euler function and $D^{\alpha_i}_{0^+}$, $D^{\beta_i}_{0^+}$, $i = 1, \dots, n$, are Caputo fractional derivatives.

In recent years, fractional differential equations have played an important role in the study of models for many phenomena in various fields of physics, biology

2010 *Mathematics Subject Classification.* 33E12, 34K37.
Key words and phrases. Fractional differential equation; Caputo fractional derivative; blow-up in finite time.

Submitted March 14, 2017. Published June 25, 2017.

and engineering, such as aerodynamics, viscoelasticity, control of dynamic systems, electrochemistry, porous media, etc (see [1, 3, 6, 14] and the references therein); their study attracted the attention of many researchers (see for instance [8, 10, 11, 12] and the references therein). In addition, a particular attention was given for the study of the local existence and uniqueness of solutions for these systems and their properties like the blow-up in finite time, the global existence, the asymptotic behavior, etc. (see [3, 10, 11, 12]).

In [9], the profile of the blowing-up solutions has been investigated for the following nonlinear nonlocal system

$$\begin{gathered} u_t(t) + D^{\alpha}_{0_+}(u - u_0)(t) = |v(t)|^q, \quad t > 0,\ q > 1, \\ v_t(t) + D^{\beta}_{0_+}(v - v_0)(t) = |u(t)|^p, \quad t > 0,\ p > 1, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{gathered}$$

as well as for solutions of systems obtained by dropping either the usual derivatives or the fractional derivatives.

In [7], some results on the blow-up of the solutions and lower bounds of the maximal time have been established for the system

$$\begin{gathered} u_t(t) + \rho D^{\alpha}_{0_+}(u - u_0)(t) = e^{v(t)}, \quad t > 0,\ \rho > 0, \\ v_t(t) + \sigma D^{\beta}_{0_+}(v - v_0)(t) = e^{u(t)}, \quad t > 0,\ \sigma > 0, \\ u(0) = u_0 > 0, \quad v(0) = v_0 > 0, \end{gathered}$$

and the subsystem obtained by dropping the usual derivatives.

In the spirit of the interesting works [4, 7, 9], we prove that the non global existence of solutions to (1.1)-(1.2) holds for polynomial nonlinearities. For the existence of solutions for the system (1.1)-(1.2), we will use the Schauder theorem.

Our paper is organized as follows: In Section 2, we give some preliminary results for fractional derivatives. In Section 3, we will prove the local existence and uniqueness of the solutions. In Section 4, we will state and prove our main result on the blow- up in finite time of solutions for system (1.1)-(1.2).

2. Preliminaries and mathematical background

For the convenience of the reader, we shall recall some known results concerning fractional integrals and derivatives that will be useful in the sequel.

The Riemann-Liouville fractional integral of order $0 < \alpha < 1$ with lower limit 0 is defined for a locally integrable function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ by

$$J^{\alpha}_{0_+}\varphi(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

where Γ is the Euler Gamma function.

The left-handed and right-handed Riemann-Liouville fractional derivatives of order α with $0 < \alpha < 1$ of a continuous function $\psi(t)$ are defined by

$$D^{\alpha}_{0_+}\psi(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{\psi(s)}{(t-s)^{\alpha}} ds, \quad t > 0,$$

and

$$D^{\alpha}_{T^-}\psi(t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^T \frac{\psi(s)}{(s-t)^{\alpha}} ds, \quad t > 0,$$

respectively. One can see that

$$\frac{d}{dt} J_{0^+}^{1-\alpha}\psi(t) = D_{0+}^{\alpha}\psi(t), \quad t > 0.$$

The integration by parts formula (see [14]) in $[0, T]$ reads

$$\int_0^T h(t) D_{0_+}^{\alpha} k(t)dt = \int_0^T (D_{T_-}^{\alpha} h(t))k(t)dt,$$

for functions h, k in $C([0, T])$ such that $D_{0+}^{\alpha} k$ and $D_{T-}^{\alpha} h$ are continuous.

The Caputo fractional derivative of order $0 < \alpha < 1$ of an absolutely continuous function $\phi(t)$ of order $0 < \alpha < 1$ is defined by

$${}^c D_{0_+}^{\alpha}\phi(t) = J_{0_+}^{1-\alpha}\frac{d}{dt}\phi(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (s-t)^{-\alpha}\phi'(s)ds.$$

The relation between the Riemann-Liouville and the Caputo fractional derivatives for an absolutely continuous function $\phi(t)$ is given by

$${}^c D_{0_+}^{\alpha}\phi(t) = D_{0_+}^{\alpha}(\phi(t) - \phi(0)), \quad 0 < \alpha < 1.$$

3. Existence and uniqueness of solutions

In this section, we deal with the existence and uniqueness of local solutions for problem (1.1)-(1.2). We say that (u, v) is a local classical solution if it satisfies equations (1.1)-(1.2) on some interval $(0, T^*)$. Our main result in this section reads as follows.

Theorem 3.1. *Assume that the functions f and g are of class $C^1(\mathbb{R}\times\mathbb{R}, \mathbb{R})$. Then system (1.1)-(1.2) admits a unique local classical solution on a maximal interval $(0, T_{\max})$ with the alternative: either $T_{\max} = +\infty$ and the solution is global; or*

$$T_{\max} < +\infty \quad \text{and} \quad \lim_{t\to T_{\max}} (|u(t)| + |v(t)|) = +\infty.$$

Proof. For the sake of completeness, we give the proof of the existence of solutions of (1.1)-(1.2). Let $k > 0$ be a positive constant and

$$h := \min\{\sigma_1,\ \sigma_2\} > 0, \tag{3.1}$$

where

$$\sigma_1 := \min\Big\{\min_{1\le i\le n}\Big(\frac{1}{2n^2\bar{a}\max_{1\le i\le n}\big(\frac{1}{\Gamma(2-\alpha_i)}\big)}\Big)^{\frac{1}{1-\alpha_i}}, \Big(\frac{k\Gamma(2-\gamma_1)}{2M}\Big)^{\frac{1}{1-\gamma_1}}\Big\},$$

$$\sigma_2 := \min\Big\{\min_{1\le i\le n}\Big(\frac{1}{2n^2\bar{b}\max_{1\le i\le n}\frac{1}{\Gamma(2-\beta_i)})}\Big)^{\frac{1}{1-\beta_i}}, \Big(\frac{k\Gamma(2-\gamma_2)}{2M}\Big)^{\frac{1}{1-\gamma_2}}\Big\},$$

$$\bar{a} = \max_{1\le i\le n}\{a_i\}, \quad \bar{b} = \max_{1\le i\le n}\{b_i\}.$$

Let $C([0, h]) \times C([0, h])$ be the space of all continuous functions (χ, ψ) on $[0, h]$ equipped with the norm

$$\|(\chi, \psi)\|_\infty = \max(\|\chi\|_\infty,\ \|\psi\|_\infty),$$

where

$$\|\chi\|_\infty = \max_{0\le t\le h}|\chi(t)|, \quad \|\psi\|_\infty = \max_{0\le t\le h}|\psi(t)|.$$

For simplicity, we assume $\alpha_1 \le \alpha_2 \le \dots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \dots \le \beta_n$.

Now, in order to prove the existence of solutions for problem (1.1)-(1.2), we rewrite it as a system of integral equations in $C([0,h])\times C([0,h])$,

$$\begin{aligned} x(t) &= -a_1 J_{0_+}^{1-\alpha_1}x(t) - a_2 J_{0_+}^{1-\alpha_2}x(t) - \dots - a_n J_{0_+}^{1-\alpha_n}x(t) + J_{0^+}^{1-\gamma_1} f(u_0 \\ &\quad + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds) \\ y(t) &= -b_1 J_{0_+}^{1-\beta_1}y(t) - b_2 J_{0_+}^{1-\beta_2}y(t) - \dots - b_n J_{0_+}^{1-\beta_n}y(t) + J_{0^+}^{1-\gamma_2} g(u_0 \\ &\quad + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds), \end{aligned} \tag{3.2}$$

via the transformation

$$u(t) = u_0 + \int_0^t x(s)ds, \quad v(t) = v_0 + \int_0^t y(s)ds,$$

and the relation ${}^cD_{0^+}^{\alpha}\psi(t) = J_{0^+}^{1-\alpha}\frac{d}{dt}\psi(t)$, and we shall prove the existence of local solutions for (3.2).

So, let us define the operator $A: C([0,h])\times C([0,h]) \to C([0,h])\times C([0,h])$ by

$$A(x,y) = (A_1(x,y),\ A_2(x,y)),$$

where

$$\begin{aligned} A_1(x(t),y(t)) &= -\sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i}x(t) \\ &\quad + J_{0^+}^{1-\gamma_1} f\Big(u_0 + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds\Big), \\ A_2(x(t),y(t)) &= -\sum_{i=1}^{n} b_i J_{0_+}^{1-\beta_i}y(t) \\ &\quad + J_{0^+}^{1-\gamma_2} g\Big(u_0 + \int_0^t x(s)ds, v_0 + \int_0^t y(s)ds\Big). \end{aligned} \tag{3.3}$$

Let us define the set

$$D := \big\{(x,y) \in C([0,h])\times C([0,h]),\ \|(x,y)\|_\infty = \sup(\|x\|_\infty,\ \|y\|_\infty) \le k\},$$

as a domain of the operator A, which is a convex, bounded, and closed subset of the Banach space $C([0,h])\times C([0,h])$. Since f and g are continuously differentiable on $[u_0-kh, u_0+kh]\times[v_0-kh, v_0+kh]$, there exists a positive constant M such that for any t in $[0,h]$ and any (x,y) in D,

$$\big|f(u_0 + \int_0^t x(s)ds,\ v_0 + \int_0^t y(s))ds\big| \le M, \tag{3.4}$$

$$\big|g((u_0 + \int_0^t x(s)ds,\ v_0 + \int_0^t y(s))ds\big| \le M, \tag{3.5}$$

and for any (u_j,v_j) in $[u_0-kh, u_0+kh]\times[v_0-kh, v_0+kh]$, $j=1,2$, and any t in $[0,h]$, there exist two positive constants L_1 and L_2 depending on u_0, v_0, k, h and on f and g respectively such that

$$|f(u_1(t),v_1(t)) - f(u_2(t),v_2(t))| \le L_1\|(u_1(t)-u_2(t), v_1(t)-v_2(t))\|, \tag{3.6}$$

$$|g(u_1(t),v_1(t)) - g(u_2(t),v_2(t))| \le L_2\|(u_1(t)-u_2(t), u_1(t)-u_2(t))\|, \tag{3.7}$$

where $\|(u_1(t)-u_2(t), v_1(t)-v_2(t))\| = |u_1(t)-u_2(t)| + |v_1(t)-v_2(t)|$.

Now, by using (3.1) and (3.6) and (3.7), for all $z_1 = (x_1, y_1) \in D$ and $z_2 = (x_2, y_2) \in D$ satisfying $\|z_1 - z_2\|_\infty < \delta$, where δ is a positive constant which will be defined later, we obtain

$$
\begin{aligned}
&\|A_1(z_1) - A_1(z_2)\|_\infty \\
&= \sup_{0\le t\le h} | -\sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i} x_1(t) + J_{0^+}^{1-\gamma_1} f(u_0 + \int_0^t x_1(s)ds,\ v_0 + \int_0^t y_1(s)ds) \\
&\quad + \sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i} x_2(t) - J_{0^+}^{1-\gamma_1} f(u_0 + \int_0^t x_2(s)ds,\ v_0 + \int_0^t y_2(s)\,ds)\ | \\
&\le \sup_{0\le t\le h} | -\sum_{i=1}^{n} a_i J_{0_+}^{1-\alpha_i} (x_1(t) - x_2(t)) \\
&\quad + J_{0^+}^{1-\gamma_1}\{f(u_0 + \int_0^t x_1(s)ds, v_0 + \int_0^t y_1(s)ds) \\
&\quad - f(u_0 + \int_0^t x_2(s)ds, v_0 + \int_0^t y_2(s)ds)\}| \\
&\le \sum_{i=1}^{n} \frac{a_i}{\Gamma(1-\alpha_i)} \int_0^h (t-s)^{-\alpha_i} \|z_1 - z_2\|_\infty ds + \frac{L_1}{\Gamma(2-\gamma_1)} h^{2-\gamma_1} \|z_1 - z_2\|_\infty \\
&\le \Big(n\bar{a} \max_{1\le\ i\le n} \{\frac{1}{\Gamma(2-\alpha_i)}\} \sum_{i=1}^{n} h^{1-\alpha_i} + \frac{L_1}{\Gamma(2-\gamma_1)}\ h^{2-\gamma_1} \Big)\delta,
\end{aligned} \tag{3.8}
$$

and in the same way, we obtain

$$
\|A_2(z_1) - A_2(z_2)\|_\infty \le \Big(n\bar{b} \max_{1\le\ i\le n} \{\frac{1}{\Gamma(2-\beta_i)}\} \sum_{i=1}^{n} h^{1-\beta_i} + \frac{L_2}{\Gamma(2-\gamma_2)} h^{2-\gamma_2} \Big)\delta. \tag{3.9}
$$

Now, given an $\varepsilon > 0$, pick $\delta = \min\big\{\frac{\varepsilon}{\omega_1}, \frac{\varepsilon}{\omega_2}\big\}$, where

$$
\begin{aligned}
\omega_1 &:= n\bar{a} \max_{1\le\ i\le n} \big\{\frac{1}{\Gamma(2-\alpha_i)}\big\} \sum_{i=1}^{n} h_i^{1-\alpha_i} + \frac{L_1}{\Gamma(2-\gamma_1)} h^{2-\gamma_1}\,, \\
\omega_2 &:= n\bar{b} \max_{1\le\ i\le n} \big\{\frac{1}{\Gamma(2-\beta_i)}\big\} \sum_{i=1}^{n} h_i^{1-\beta_i} + \frac{L_2}{\Gamma(2-\gamma_2)} h^{2-\gamma_2}\,.
\end{aligned}
$$

One can see that $\|A(z_1) - A(z_2)\|_\infty < \varepsilon$, consequently, A is a continuous operator on D.

Next, from (3.3), (3.4), (3.5) and (3.1), for all $z=(x,y)\in D$ we have

$$\begin{aligned}
&\|A_1(z)\|_\infty \\
&\leq \sup_{0\leq t\leq h}\Big|\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^t (t-s)^{-\alpha_i}x(s)\,ds \\
&\quad+\frac{1}{\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1}f(u_0+\int_0^t x(s)ds, v_0+\int_0^t y(s))ds\Big| \\
&\leq \sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\|z\|_\infty\int_0^h (t-s)^{-\alpha_i}ds+\frac{1}{\Gamma(1-\gamma_1)}\int_0^t (t-s)^{-\gamma_1}M\,ds \\
&\leq nk\bar{a}\max_{1\leq i\leq n}\Big\{\frac{1}{\Gamma(2-\alpha_i)}\Big\}\sum_{i=1}^{n}h^{1-\alpha_i}+\frac{1}{\Gamma(2-\gamma_1)}Mh^{1-\gamma_1}\leq k.
\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}
&\|A_2(z)\|_\infty \\
&\leq \sup_{0\leq t\leq h}\Big|\sum_{i=1}^{n}\frac{b_i}{\Gamma(1-\beta_i)}\int_0^t (t-s)^{-\beta_i}x(s)\,ds+\frac{1}{\Gamma(2-\gamma_2)}Mh^{1-\gamma_1}\Big| \\
&\leq \sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\beta_i)}\|z\|_\infty\int_0^h (t-s)^{-\beta_i}ds+\frac{1}{\Gamma(2-\gamma_2)}Mh^{1-\gamma_2} \\
&\leq nk\bar{b}\max_{1\leq i\leq n}\Big\{\frac{1}{\Gamma(2-\beta_i)}\Big\}\sum_{i=1}^{n}h^{1-\beta_i}+\frac{1}{\Gamma(2-\gamma_2)}Mh^{1-\gamma_2}\leq k.
\end{aligned}\tag{3.11}$$

Inequalities (3.10) and (3.11) assert that $A(D)\subset D$. Thus, the set $A(D)$ is uniformly bounded. Now, for all $0\leq t_1\leq t_2\leq h$ with $|t_1-t_2|<\eta$, and all $z=(x,y)\in C([0,h])\times C([0,h])$, from (3.6) we have

$$\begin{aligned}
&|A_1(\ z(t_1))-A_1(\ z(t_2)\ )| \\
&=\Big|-\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^{t_1}(t_1-s)^{-\alpha_i}x(s)ds \\
&\quad+\frac{1}{\Gamma(1-\gamma_1)}\int_0^{t_1}(t_1-s)^{-\gamma_1}f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau)ds \\
&\quad+\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^{t2}(t_2-s)^{-\alpha_i}x(s)ds \\
&\quad-\frac{1}{\Gamma(1-\gamma_1)}\int_0^{t_2}(t_2-s)^{-\gamma_1}f(u_0+\int_0^s x(\tau)d\tau,\ v_0+\int_0^s y(\tau)d\tau)ds\Big| \\
&\leq\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_0^{t_1}\big((t_1-s)^{-\alpha_i}-(t_2-s)^{-\alpha_i}\big)|x(s)|ds \\
&\quad+\sum_{i=1}^{n}\frac{a_i}{\Gamma(1-\alpha_i)}\int_{t_1}^{t_2}(t_2-s)^{-\alpha_i}|x(s)|ds \\
&\quad+\frac{1}{\Gamma(1-\gamma_1)}\int_0^{t_1}\big((t_1-s)^{-\gamma_1}-(t_2-s)^{-\gamma_1}\big)
\end{aligned}$$

$$
\begin{aligned}
&\times \Big|f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau)\Big|ds\\
&+\frac{1}{\Gamma(1-\gamma_1)}\int_{t_1}^{t_2}(t_2-s)^{-\gamma_1}\Big|f(u_0+\int_0^s x(\tau)d\tau, v_0+\int_0^s y(\tau)d\tau)\Big|ds\\
\le k\bar{a}&\sum_{i=1}^n \frac{1}{\Gamma(2-\alpha_i)}(t_2-t_1)^{1-\alpha_i}+\frac{2M}{\Gamma(2-\gamma_1)}(t_2-t_1)^{1-\gamma_1}.
\end{aligned}
\tag{3.12}
$$

Similarly, we obtain

$$
\begin{aligned}
&|A_2(z(t_1))-A_2(z(t_2))|\\
&\le k\bar{b}\sum_{i=1}^n \frac{1}{\Gamma(2-\beta_i)}(t_2-t_1)^{1-\beta_i}+\frac{2M}{\Gamma(2-\gamma_2)}(t_2-t_1)^{1-\gamma_2}.
\end{aligned}
\tag{3.13}
$$

From (3.12) and (3.13) it yields that $A(D)$ is equicontinuous, and so by using Arzela-Ascoli theorem, we find that $A(D)$ is relatively compact in $C([0,h])\times C([0,h])$.

Finally, by Schauder theorem, we conclude that the operator A has at least one fixed point, this means that the system of integral equations (3.2) has at least one local continuous solution (x,y) defined on $[0,h]$. Now, since for all $t\in[0,h]$,

$$
u(t)=u_0+\int_0^t x(s)ds, \quad v(t)=v_0+\int_0^t y(s)ds,
\tag{3.14}
$$

where x and y are solutions of system (3.2) of integral equations, it follows that $u'(t)=x(t)$, $v'(t)=y(t)$ for any t in $(0,h)$.

Using the definition of Caputo fractional derivative, we find for all t in $(0,h)$,

$$
\begin{aligned}
{}^cD_{0_+}^{\alpha_i}u(t)&=J_{0_+}^{1-\alpha_i}x(t)=\frac{1}{\Gamma(1-\alpha_i)}\int_0^T(t-s)^{-\alpha_i}x(s)\,ds, \quad i=1,\dots,n,\\
{}^cD_{0_+}^{\beta_i}v(t)&=J_{0_+}^{1-\beta_i}y(t)=\frac{1}{\Gamma(1-\beta_i)}\int_0^T(t-s)^{-\beta_i}y(s)\,ds, \quad i=1,\dots,n.
\end{aligned}
\tag{3.15}
$$

Combining (3.14), (3.15) and (3.2), for all t in $(0,h)$ we obtain

$$
\begin{aligned}
u'(t)+\sum_{i=1}^n a_iJ_{0_+}^{1-\alpha_i}\frac{du(t)}{dt}&=J_{0_+}^{1-\gamma_1}f(u(s),v(s))\\
v'(t)+\sum_{i=1}^n b_iJ_{0_+}^{1-\beta_i}\frac{dv(t)}{dt}&=J_{0_+}^{1-\gamma_2}g(u(s),v(s)).
\end{aligned}
\tag{3.16}
$$

Since $(u(0),v(0))=(u_0,v_0)$, we conclude that (u,v) is a classical solution for (1.1)-(1.2) on $(0,h)$, and this solution may be extended (see [2]) to a maximal interval $(0,T_{\max})$ with the alternative: either $T_{\max}=+\infty$ and the solution is global; or

$$
T_{\max}<+\infty \quad \text{and} \quad \lim_{t\to T_{\max}}(|u(t)|+|v(t)|)=+\infty.
$$

Next, we shall prove uniqueness. Assume that the Cauchy problem (1.1)-(1.2) admits two classical solutions (u_1,v_1) and (u_2,v_2) with the same initial data (u_0,v_0) on $(0,T_{\max})$. Observe that for all $t\in(0,\rho)$ with $\rho<T_{\max}$, these solutions satisfy

the following equalities:

$$\begin{aligned} (u_1-u_2)_t+\sum_{i=1}^n a_i D_{0+}^{\alpha_i}(u_1-u_2) &= J_{0^+}^{1-\gamma_1}(f(u_1,v_1)-f(u_2,v_2)),\\ (v_1-v_2)_t+\sum_{i=1}^n b_i D_{0+}^{\beta_i}(v_1-v_2) &= J_{0^+}^{1-\gamma_2}(g(u_1,v_1)-g(u_2,v_2)). \end{aligned} \tag{3.17}$$

Integrating (3.17) over $(0,t)$ yields

$$\begin{aligned} &(u_1-u_2)(t)+\int_0^t\sum_{i=1}^n a_i D_{0_+}^{\alpha_i}(u_1-u_2)(s))\,ds\\ &=\int_0^t J_{0^+}^{1-\gamma_1}(f(u_1(s),v_1(s))-f(u_2(s),v_2(s)))ds\\ &(v_1-v_2)(t)+\int_0^t\sum_{i=1}^n b_i\ D_{0_+}^{\beta_i}(u_1-u_2)(s))\,ds\\ &=\int_0^t J_{0^+}^{1-\gamma_2}(g(u_1(s),v_1(s))-g(u_2(s),v_2(s)))ds. \end{aligned} \tag{3.18}$$

Let $\theta:=\max\{\alpha_1,\alpha_2,\dots,\alpha_n,\beta_1,\beta_2,\dots,\beta_n,\gamma_1,\gamma_2\}$. Using (3.18) and the fact that f and g are locally Lipshitz on $[0,h]$, thanks to (3.6) and (3.7), for all $t\in(0,\rho)$, we have

$$\begin{aligned} &|u_1(t)-u_2(t)|\\ &\le\int_0^t\Big(\sum_{i=1}^n\frac{a_i}{\Gamma(1-\alpha_i)}(t-s)^{-\alpha_i}\\ &\quad+L_1\frac{(t-s)^{-\gamma_1}}{\Gamma(1-\gamma_1)}\Big)\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds\\ &\le\int_0^t\Big\{\sum_{i=1}^n\frac{a_i}{\Gamma(1-\alpha_i)}(t-s)^{\theta-\alpha_i}\\ &\quad+\frac{L_1}{\Gamma(1-\gamma_1)}(t-s)^{\theta-\gamma_1}\Big\}(t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds\\ &\le d_1\int_0^t(t-s)^{-\theta}\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds, \end{aligned} \tag{3.19}$$

where

$$d_1:=n\bar a\max_{1\le i\le n}\big\{\frac{1}{\Gamma(1-\alpha_i)}\rho^{\theta-\alpha_i}\big\}+\frac{L_1}{\Gamma(1-\gamma_1)}\rho^{\theta-\gamma_1},$$

and

$$\|u_1(t)-u_2(t),\ v_1(t)-v_2(t))\|=|u_1(t)-u_2(t)|+|v_1(t)-v_2(t)|.$$

Similarly,

$$\begin{aligned} |v_1(t)-v_2(t)|\le\int_0^t\Big(&\sum_{i=1}^n\frac{b_i}{\Gamma(1-\beta_i)}(t-s)^{-\beta_i}\\ &+L_2\frac{(t-s)^{-\gamma_2}}{\Gamma(1-\gamma_2)}\Big)\|u_1(s)-u_2(s),v_1(s)-v_2(s))\|ds \end{aligned}$$

$$
\begin{aligned}
&\le \int_0^t \Big\{ \sum_{i=1}^n \frac{b_i}{\Gamma(1-\beta_i)}(t-s)^{\theta-\beta_i} + \frac{L_2}{\Gamma(1-\gamma_2)}(t-s)^{\theta-\gamma_2} \Big\} \\
&\quad \times (t-s)^{-\theta} \|u_1(s)-u_2(s), v_1(s)-v_2(s))\| ds \\
&\le d_2 \int_0^t (t-s)^{-\theta} \|u_1(s)-u_2(s), v_1(s)-v_2(s)\| ds,
\end{aligned} \tag{3.20}
$$

where

$$
d_2 := n\bar{b} \max_{1\le i\le n} \Big\{ \frac{1}{\Gamma(1-\beta_i)} \rho^{\theta-\beta_i} \Big\} + \frac{L_2}{\Gamma(1-\gamma_2)} \rho^{\theta-\gamma_2}.
$$

Then from (3.19) and (3.20), we find

$$
\begin{aligned}
&\|(u_1(t)-u_2(t), v_1(t)-v_2(t)\| \\
&\le (d_1+d_2) \int_0^t (t-s)^{-\theta} \|u_1(s)-u_2(s), v_1(s)-v_2(s)\|\, ds \quad \forall t \in (0,\rho).
\end{aligned} \tag{3.21}
$$

Finally using Gronwall's inequality (see [5, p. 6]), we deduce the uniqueness and this completes the proof. □

4. Blow up results

This section is devoted to the blow up of solutions of the system (1.1)-(1.2) whenever the nonlinear terms satisfy certain growth conditions. Our main result reads as follows.

Theorem 4.1. *Assume that the assumptions of Theorem 3.1 hold, and that the functions f and g satisfy the growth conditions:*

$$
\begin{aligned}
f(\xi,\eta) &\ge a|\eta|^q, \quad \text{for all } \xi,\eta \in \mathbb{R}, \\
g(\xi,\eta) &\ge b|\xi|^p, \quad \text{for all } \xi,\eta \in \mathbb{R},
\end{aligned}
$$

for some positive constants a, b. Then for all positive initial data, the solution of (1.1)-(1.2) *blows up in a finite time.*

Proof. We proceed by contradiction. We assume that $T_{\max} = +\infty$ and we consider the function used in [4],

$$
\phi(t) = \begin{cases} T^{-\lambda}(T-t)^{\lambda} & \text{for } t \in [0,T],\ \lambda \gg 1, \\ 0 & \text{for } t > T. \end{cases} \tag{4.1}
$$

Then by multiplying the first equation in (1.1) by ϕ and integrating over $(0,T)$, we obtain

$$
\begin{aligned}
&\int_0^T u_t(t)\phi(t)dt + \int_0^T \sum_{i=1}^n a_i (D_{0_+}^{\alpha_i}(u(t)-u_0))\phi(t)dt \\
&= \int_0^T (J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\phi(t)dt.
\end{aligned} \tag{4.2}
$$

Let

$$
\psi(t) := \int_0^t \phi(s)ds = -\frac{1}{\lambda+1} T^{-\lambda}(T-t)^{\lambda+1} \quad t \in [0,T].
$$

Integrating by parts, and since $\psi(T)=0$, yields

$$\begin{aligned}\int_0^T (J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\phi(t)dt &= -\int_0^T \frac{d}{dt}(J_{0_+}^{1-\gamma_1} f(u(t),v(t)))\psi(t)dt \\ &= -\int_0^T (D_{0_+}^{\gamma_1} f(u(t),v(t)))\psi(t)dt \\ &= -\int_0^T (D_{T_-}^{\gamma_1}\psi(t)) f((u(t),v(t))dt.\end{aligned} \tag{4.3}$$

Recall (see [4]) the formulas

$$D_{T^-}^{\gamma_j}\phi(t) = C_{\lambda,\gamma_j} T^{-\lambda}(T-t)^{\lambda-\gamma_j}, \quad \text{where} C_{\lambda,\gamma_j} = \frac{\lambda\Gamma(\lambda-\gamma_j)}{\Gamma(\lambda-2\gamma_j+1)},$$

and

$$D_{T^-}^{\gamma_j}\psi(t) = -\frac{1}{\lambda+1} C_{\lambda+1,\gamma_j} T^{-\lambda}(T-t)^{\lambda+1-\gamma_j} = -C'_{\lambda,\gamma_j}\phi(t)(T-t)^{1-\gamma_j}, \tag{4.4}$$

for $j=1,2$, where $C'_{\lambda,\gamma_j} = \frac{1}{\lambda+1} C_{\lambda+1,\gamma_j}$, $j=1,2$. Then

$$-\int_0^T (D_{0_+}^{\gamma_1}\psi(t)) f(u(t),v(t))dt = \int_0^T C'_{\lambda,\gamma_1}\phi(t)(T-t)^{1-\gamma_1} f(u(t),v(t))dt. \tag{4.5}$$

From (4.2), (4.3) and (4.5) and since u_0 is positive and ϕ is in $C^1([0,T])$, thanks to (4.1), an integration by parts yields

$$\begin{aligned}&C'_{\lambda,\gamma_1}\int_0^T \phi(t)(T-t)^{1-\gamma_1} f(u(t),v(t))dt \\ &\le -\int_0^T u(t)\phi'(t)dt + \sum_{i=1}^n \int_0^T u(t) D_{T_-}^{\alpha_i}(a_i\phi(t))dt.\end{aligned} \tag{4.6}$$

Observe that if p' is the conjugate of p, then

$$\begin{aligned}&\int_0^T u(t)(-\phi'(t))dt \\ &= \int_0^T u(t)(\phi(t))^{\frac{1}{p}}(\phi(t))^{-1/p}(T-t)^{\frac{1-\gamma_2}{p}}(T-t)^{\frac{-(1-\gamma_2)}{p}}(-\phi'(t))dt \\ &\le C'_{\lambda,\gamma_2}\frac{b}{4}\int_0^T |u(t)|^p \phi(t)(T-t)^{1-\gamma_2}\,dt \\ &\quad + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|(\phi'(t))|^{p'}dt \\ &\le C'_{\lambda,\gamma_2}\frac{1}{4}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt \\ &\quad + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)|^{p'}dt,\end{aligned} \tag{4.7}$$

and for all $1 \le i \le n$,

$$
\begin{aligned}
&\int_0^T u(t)(D^{\alpha_i}_{T_-}(a_i\phi(t))dt \\
&= \int_0^T u(t)(\phi(t))^{\frac{1}{p}}(\phi(t))^{-\frac{1}{p}}(T-t)^{\frac{1-\gamma_2}{p}}(T-t)^{\frac{-(1-\gamma_2)}{p}} D^{\alpha_i}_{T_-}(a_i\phi(t))dt \\
&\le C'_{\lambda,\gamma_2}\frac{b}{4n}\int_0^T |u(t)|^p\phi(t)(T-t)^{1-\gamma_2}dt \\
&\quad + \Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p} a_i^{p'}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|(D^{\alpha_i}_{T_-}\phi(t))|^{p'}dt \\
&\le C'_{\lambda,\gamma_2}\frac{1}{4n}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt \\
&\quad + \Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\bar{a}^{p'}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)p'/p}|(D^{\alpha_i}_{T_-}\phi(t))|^{p'}dt.
\end{aligned}
\tag{4.8}
$$

Furthermore,

$$
\begin{aligned}
&C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt \\
&\le \frac{1}{2}C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}\,dt \\
&\quad + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)|^{p'}dt \\
&\quad + \bar{a}^{p'}\Big(\frac{4n}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}\sum_{i=1}^n\int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}\ |D^{\alpha_i}_{T_-}\phi(t))|^{p'}dt.
\end{aligned}
\tag{4.9}
$$

Analogously, if q' is the conjugate of q, we obtain

$$
\begin{aligned}
&C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt \\
&\le -\int_0^T v(t)\phi'(t)dt + \sum_{i=1}^n\int_0^T v(t)D^{\beta_i}_{T_-}(\ b_i(t)\phi(t))dt \\
&\le \frac{1}{2}C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt \\
&\quad + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|\phi'(t)|^{q'}dt \\
&\quad + \bar{b}^{q'}\Big(\frac{4n}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\sum_{i=1}^n\int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|D^{\beta_i}_{T-}\phi(t)|^{q'}dt.
\end{aligned}
\tag{4.10}
$$

Denote

$$
\begin{gathered}
A := C'_{\lambda,\gamma_1}\int_0^T f(u(t),v(t))\phi(t)(T-t)^{1-\gamma_1}dt, \\
B := C'_{\lambda,\gamma_2}\int_0^T g(u(t),v(t))\phi(t)(T-t)^{1-\gamma_2}dt,
\end{gathered}
$$

$$
\begin{aligned}
C &:= \int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}|\phi'(t)\ |^{p'}dt,\\
D &:= \int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}|\phi'(t)|^{q'}dt,\\
E &:= \int_0^T (\phi(t))^{-p'/p}(T-t)^{-(1-\gamma_2)\frac{p'}{p}}\sum_{i=1}^n |D^{\alpha_i}_{T-}\phi(t)|^{p'}dt,\\
F &:= \int_0^T (\phi(t))^{-q'/q}(T-t)^{-(1-\gamma_1)q'/q}\sum_{i=1}^n |D^{\beta_i}_{T-}\phi(t)|^{q'}dt.
\end{aligned}
$$

From (4.9) and (4.10) we have

$$
\begin{aligned}
A &\le \frac{1}{2}B + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{p'/p}\bar{a}^{p'}E),\\
B &\le \frac{1}{2}A + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F),
\end{aligned}
$$

then

$$
\begin{aligned}
A &\le \frac{1}{2}\Big(\frac{1}{2}A + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F)\Big) + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)\\
&= \frac{1}{4}A + \frac{1}{2}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E);
\end{aligned}
$$

thus

$$
A \le \frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)
$$

and

$$
\begin{aligned}
B &\le \frac{1}{2}\Big(\frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big)\\
&\quad + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F)\\
&\le \frac{4}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \frac{2}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E).
\end{aligned}
$$

Taking into account (4.2), (4.7) and (4.8), we deduce that

$$
\begin{aligned}
&u_0\int_0^T D^{\alpha_1}_{T_-}\phi(t)dt\\
&= \frac{u_0}{a_1}\int_0^T D^{\alpha_1}_{T_-}(a_1\phi(t))dt\\
&\le \frac{1}{a_1}\Big(-\int_0^T u(t)\ \phi'(t)dt + \int_0^T\sum_{i=1}^n u(t)\ D^{\alpha_i}_{T_-}(a_i\phi(t))\,dt\Big)\\
&\le \frac{1}{a_1}\Big(\frac{1}{2}B + \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big)\\
&\le \frac{1}{a_1}\Big(\frac{2}{3}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \frac{4}{3}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big).
\end{aligned}
$$

For $\lambda > \max\{\frac{p'}{p} + p' - 1, \frac{q'}{q} + q' - 1\}$, it holds

$$\int_0^T D_{T_-}^{\alpha_i} \phi(t)dt = C_{\alpha_i,\lambda} T^{1-\alpha_i}, \tag{4.11}$$

where

$$C_{\alpha_i,\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda-\alpha_i+2)}, \quad \forall 1 \le i \le n\,.$$

Also there exists a positive constant K such that

$$\begin{gathered} C \le KT^{(\gamma_2-1)\frac{p'}{p}+1-p'}, \quad D \le KT^{(\gamma_1-1)\frac{q'}{q}+1-q'}, \\ E \le K\sum_{i=1}^n T^{(\gamma_2-1)\frac{p'}{p}+1-p'\alpha_i}, \quad F \le K\sum_{i=1}^n T^{(\gamma_1-1)\frac{q'}{q}+1-q'\beta_i}, \quad \forall 1 \le i \le n. \end{gathered} \tag{4.12}$$

Consequently,

$$\begin{aligned} &u_0 \int_0^T D_{T_-}^{\alpha_1} \phi(t)dt \\ &\le \frac{2}{3a_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q} K\Big(T^{(\gamma_1-1)\frac{q'}{q}+1-q'} + n^{q'/q}\bar{b}^{q'}\sum_{i=1}^n T^{(\gamma_1-1)\frac{q'}{q}+1-q'\beta_i}\Big) \\ &\quad + \frac{4}{3a_1}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p} K\Big(T^{(\gamma_2-1)\frac{p'}{p}+1-p'} + n^{\frac{p'}{p}}\bar{a}^{p'}\sum_{i=1}^n T^{(\gamma_2-1)\frac{p'}{p}+1-p'\alpha_i}\Big). \end{aligned} \tag{4.13}$$

Using (4.11) and (4.13), we obtain

$$\begin{aligned} u_0 &\le C_{\alpha_1,\lambda}^{-1} K\Big\{\frac{2}{3a_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}\Big(T^{(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'} \\ &\quad + n^{q'/q}\bar{b}^{q'}\sum_{i=1}^n T^{(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i}\Big)\Big\} \\ &\quad + C_{\alpha_1,\lambda}^{-1} K\Big\{\frac{4}{3a_1}\Big(\frac{4}{bC'_{\lambda,\gamma_1}}\Big)^{p'/p}\Big(T^{(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'} \\ &\quad + n^{\frac{p'}{p}}\bar{a}^{p'}\sum_{i=1}^n T^{(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'\alpha_i}\Big)\Big\}. \end{aligned} \tag{4.14}$$

Similarly we obtain

$$\begin{aligned} &v_0 \int_0^T D_{T_-}^{\beta_1} \phi(t)dt \\ &\le \frac{1}{b_1}\Big(\frac{1}{2}A + \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F)\Big) \\ &\le C_{\beta,\lambda}^{-1}\Big(\frac{4}{3b_1}\Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q}(D + n^{q'/q}\bar{b}^{q'}F) + \frac{2}{3b_1}\Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p}(C + n^{\frac{p'}{p}}\bar{a}^{p'}E)\Big), \end{aligned}$$

which yields

$$
\begin{aligned}
v_0 \le C_{\beta_1,\lambda}^{-1} K \Big\{ \frac{4}{3b_1} \Big(\frac{4}{aC'_{\lambda,\gamma_1}}\Big)^{q'/q} \Big(T^{(\gamma_1-1)\frac{q'}{q}+\beta_1-q'} \\
&+ n^{q'/q} \bar{b}^{q'} \sum_{i=1}^{n} T^{(\gamma_1-1)\frac{q'}{q}+\beta_1-q'\beta_i} \Big) \Big\} \\
&+ C_{\beta_1,\lambda}^{-1} K \Big\{ \frac{2}{3b_1} \Big(\frac{4}{bC'_{\lambda,\gamma_2}}\Big)^{p'/p} \Big(T^{(\gamma_2-1)\frac{p'}{p}+\beta_1-p'} \\
&+ n^{\frac{p'}{p}} \bar{a}^{p'} \sum_{i=1}^{n} T^{(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i} \Big) \Big\}.
\end{aligned}
\tag{4.15}
$$

One can observe that

$$
\begin{gathered}
(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'<0, \quad (\gamma_2-1)\frac{p'}{p}+\alpha_1-p'<0, \\
(\gamma_1-1)\frac{q'}{q}+\beta_1-q'<0, \quad (\gamma_2-1)\frac{p'}{p}+\beta_1-p'<0, \\
(\gamma_2-1)\frac{p'}{p}+\alpha_1-p'\alpha_i<0, \quad (\gamma_1-1)\frac{q'}{q}+\beta_1-q'\beta_i<0, \quad \forall 1\le i\le n, \\
(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i<\beta_1-\alpha_1, \quad (\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i<\alpha_1-\beta_1, \\
\forall 1\le i\le n.
\end{gathered}
\tag{4.16}
$$

Inequalities (4.16) reduce to

$$
(\gamma_2-1)\frac{p'}{p}+\beta_1-p'\alpha_i<0, \quad \forall 1\le i\le n,
$$

or

$$
(\gamma_1-1)\frac{q'}{q}+\alpha_1-q'\beta_i<0, \quad \forall 1\le i\le n.
$$

Taking the limit when T approaches infinity in (4.14) and (4.15), we find

$$
0<u_0\le 0 \quad \text{or} \quad 0<v_0\le 0. \tag{4.17}
$$

This leads to a contradiction and consequently the maximal time of existence for the solution to (1.1)-(1.2) is finite and this completes the proof. □

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