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HARMONIC-HYPERBOLIC GEOMETRIC FLOW

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Abstract. In this article we study a coupled system for hyperbolic geometric flow on a closed manifold M, with a harmonic flow map from M to some closed target manifold N. Then we show that this flow has a unique solution for a short-time. After that, we find evolution equations for Riemannian curvature tensor, Ricci curvature tensor, and scalar curvature of M under this flow. In the final section we give some examples of this flow on closed manifolds.

1. Introduction

Let (M^m, g) and (N^n, γ) be smooth closed Riemannian manifolds. Suppose that N is isometrically embedded into Euclidean space $e_N : (N^n, \gamma) \hookrightarrow \mathbb{R}^d$ for a sufficiently large d. We identify maps $\varphi : M \to N$ with $e_N \circ \varphi :\hookrightarrow \mathbb{R}^d$. Harmonic maps $\varphi : (M, g) \to (N, \gamma)$ are critical point of the energy functional $E(\varphi) = \int_M |\nabla\varphi|^2 d\mu$, where $d\mu$ is the volume form on M with respect to the metric g and

$$|\nabla\varphi|^2 := \frac{1}{2} g^{ij} (\gamma_{\alpha\beta})_\varphi \frac{\partial\varphi^\alpha}{\partial x^i} \frac{\partial\varphi^\beta}{\partial x^j}.$$

Harmonic maps are generalizations of harmonic functions. For example the identity and constant maps are harmonic maps, also, geodesics as the map $S^1 \to M$ are harmonic maps. The first major study of harmonic mapping between Riemannian manifolds was made by Eells and Sampson [4]. They study the harmonic map flow

$$\frac{\partial\varphi}{\partial t} = \tau_g \varphi, \quad \varphi(0) = \varphi_0. \tag{1.1}$$

where $\tau_g\varphi$ denotes the tension field of φ, and showed, under suitable metric and curvature assumptions on the target manifold, flow (1.1) has unique solution. The harmonic map flow is a nonlinear heat flow in geometric analysis. Another, nonlinear heat flow and wave flow in geometric analysis are geometric flows. Geometric flows are important problem in differential geometry, because by these flow we can find canonical metrics on Riemannian manifolds. A geometric flow is an evolution of a geometric structure under a differential equation with a functional on a manifold.

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Let M be an n-dimensional complete Riemannian manifold with the Riemannian metric $g = (g_{ij})$. The Levi-Civita connection is given by the Christoffel symbols

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \Big\{ \frac{\partial g_{jl}}{\partial x^i} + \frac{\partial g_{il}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^l} \Big\} \tag{1.2}$$

and Riemannian curvature tensor, Ricci curvature tensor, scalar curvature of (M, g) as follows

$$R^k_{ijl} = \frac{\partial \Gamma^k_{jl}}{\partial x^i} - \frac{\partial \Gamma^k_{il}}{\partial x^j} + \Gamma^k_{ip}\Gamma^p_{jl} - \Gamma^k_{jp}\Gamma^p_{il}, \quad R_{ijkl} = g_{kp} R^p_{ijl},$$
$$R_{ik} = g^{jl} R_{ijkl}, \quad R = g^{ij} R_{ij}.$$

The first important geometric flow is Ricci flow, defined as follows,

$$\frac{\partial}{\partial t} g = -2 \operatorname{Ric}, \quad g(0) = g_0 \tag{1.3}$$

where Ric denotes the Ricci curvature. The Ricci flow was introduced by Hamilton in 1982 [5] and evolves a Riemannian metric by its Ricci curvature, is a natural analogue of the heat equation for metrics. The existence solution of Ricci flow studied by Hamilton (see [5]) and DeTurck (see [3]) on closed Riemannian manifolds. Also evolution equation for geometric structures dependant to metric investigated by some researcher (see [1]).

The second geometric flow is hyperbolic geometric flow which is a system of nonlinear evolution partial differential equations of second order, it is very similar to wave equation flow metrics, defined as follows,

$$\frac{\partial^2}{\partial t^2} g = -2 \operatorname{Ric}, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t}(0) = k_0. \tag{1.4}$$

where k_0 is a symmetric tensor on M and this flow is similar to Einstein equation

$$\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} - \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} + g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t}.$$

The existences and uniqueness of (1.4) studied in [2] on closed Riemannian manifold.

Another important geometric flow is the harmonic-Ricci flow, defined as follows,

$$\begin{gathered} \frac{\partial}{\partial t} g = -2 \operatorname{Ric} + 2\alpha \nabla\varphi \otimes \nabla\varphi, \quad g(0) = g_0, \\ \frac{\partial}{\partial t} \varphi = \tau_g \varphi, \quad \varphi(0) = \varphi_0. \end{gathered} \tag{1.5}$$

where α is positive coupling constant, φ is a map from M to some closed target manifold N, and this flow studied in [8].

Motivated by the above works, in this article we consider an m-dimensional, closed smooth, Riemannian manifold M whose metric $g = g(t)$ is evolving according to the flow equation

$$\begin{gathered} \frac{\partial^2}{\partial t^2} g = -2 \operatorname{Ric} + 2\alpha \nabla\varphi \otimes \nabla\varphi, \quad g(0) = g_0, \quad \frac{\partial g}{\partial t}(0) = k_0 \\ \frac{\partial}{\partial t} \varphi = \tau_g \varphi, \quad \varphi(0) = \varphi_0. \end{gathered} \tag{1.6}$$

where k_0 is a symmetric tensor on M, Ric is the Ricci tensor of the manifold, α is positive coupling constant, $\varphi(t)$ a family of smooth maps from M to N and $\tau_g \varphi$ denotes the tension field of the map φ with respect to the evolving metric

g. Finally, $(\nabla\varphi\otimes\nabla\varphi)_{ij}=\nabla_i\varphi^\lambda\nabla_j\varphi^\lambda$ is components of $\nabla_i\varphi^\lambda$. This flow called harmonic-hyperbolic geometric flow and after this, in short, we will display it with $(HG)_\alpha$ flow.

2. Short-time existence and uniqueness for the $(HG)_\alpha$ flow

In this section we study the existence and uniqueness of the $(HG)_\alpha$ flow. We use a process similar to the one in the existence and uniqueness of geometric flow, for the Ricci flow, hyperbolic geometric flow, and harmonic-Ricci flow.

Theorem 2.1. *Let (M,g_0) and (N,γ) be compact Riemannian manifolds and k_0 be a symmetric tensor on M. Then there exists a constant $T>0$ such that the initial value problem* (1.6) *has a unique smooth solution metric g and map φ on $M\times[0,T]$.*

Proof. Using the gauge fixing idea as in the Ricci flow (see [6]) and the push-forward of a solution of (1.6) we can find a system of nonlinear strictly-hyperbolic partial differential equations of second order and then the short-time existence and uniqueness result on a compact manifold, show that the existence and uniqueness for this system and in finally similar to the proof of existence and uniqueness for Ricci flow (see [6]) the pull-back of this solution complete the proof of theorem. For this end, let $(g(t),\varphi(t))_{t\in[0,T)}$ is a solution of the $(HG)_\alpha$ flow with initial data $(g(0),\varphi(0))=(g_0,\varphi_0)$, $\frac{\partial g_{ij}}{\partial t}(0)=k_{ij}(0)$. Let $\psi_t:(M,\hat g(t))\to(M,g_0)$ be solution of the harmonic map heat flow $\frac{\partial}{\partial t}\psi=\tau_g\psi$, with $\psi(0)=id_M$. Let

$$\hat g_{ij}(t)=\psi_* g_{ij},\quad \hat\varphi(t)=\psi_*\varphi(t) \tag{2.1}$$

be the push-forward of g_{ij} and φ respectively. We now find the evolution for $(\hat g_{ij}(t),\hat\varphi(t))$. Denote by $y(x,t)=\psi_t(x)=(y^1(x,t),\dots,y^n(x,t))$ in locally coordinates. Then

$$\hat g_{ij}(x,t)=\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}g_{\alpha\beta}(y,t) \tag{2.2}$$

by direct computations, we have

$$\begin{aligned}
\frac{\partial^2\hat g_{ij}}{\partial t^2}(x,t)&=\frac{\partial^2 g_{\alpha\beta}}{\partial t^2}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}+\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma\partial y^\lambda}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}\frac{\partial y^\lambda}{\partial t}\\
&\quad+2\frac{\partial^2 g_{\alpha\beta}}{\partial y^\gamma\partial t}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}\frac{\partial y^\gamma}{\partial t}+\frac{\partial}{\partial x^i}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^j}\frac{\partial^2 y^\alpha}{\partial t^2})+\frac{\partial}{\partial x^j}(g_{\alpha\beta}\frac{\partial y^\beta}{\partial x^i}\frac{\partial^2 y^\alpha}{\partial t^2})\\
&\quad+\Big[\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}-\frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j})-\frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i})\Big]\frac{\partial^2 y^\gamma}{\partial t^2}\\
&\quad+2\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial y^\beta}{\partial x^j}(\frac{\partial g_{\alpha\beta}}{\partial t}+\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t})\\
&\quad+2\frac{\partial y^\alpha}{\partial x^i}\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t})(\frac{\partial g_{\alpha\beta}}{\partial t}+\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\gamma}{\partial t})+2g_{\alpha\beta}\frac{\partial}{\partial x^i}(\frac{\partial y^\alpha}{\partial t})\frac{\partial}{\partial x^j}(\frac{\partial y^\beta}{\partial t}).
\end{aligned}$$

For the normal coordinates $\{x^i\}$ around a fixe point $p\in M$, we have $\frac{\partial g_{ij}}{\partial x^k}(p)=0$ and

$$\frac{\partial g_{\alpha\beta}}{\partial y^\gamma}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}-\frac{\partial}{\partial x^i}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^j})-\frac{\partial}{\partial x^j}(g_{\beta\gamma}\frac{\partial y^\beta}{\partial x^i})=0,\quad\forall i,j,\gamma=1,2,\dots,n. \tag{2.3}$$

Let $y(x,t)$ be a solution of the equation

$$\begin{aligned} \frac{\partial^2 y^\alpha}{\partial t^2} &= \frac{\partial y^\alpha}{\partial x^k} g^{il}(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}) \\ y^\alpha(x,0) = x^\alpha, &\quad \frac{\partial}{\partial t} y^\alpha(x,0) = y^\alpha_1(x) \end{aligned} \tag{2.4}$$

and define the vector field

$$V_i = g_{ik} g^{jl}(\hat{\Gamma}^k_{jl} - \mathring{\Gamma}^k_{jl}) \tag{2.5}$$

where $\hat{\Gamma}^k_{jl}$ and $\mathring{\Gamma}^k_{jl}$ are the connection coefficients corresponding to the metrics $\hat{g}_{ij}(x,t)$ and $g_{ij}(x,0)$, respectively, $y^\alpha_1(x) \in C^\infty(M)$. Since $\frac{\partial^2}{\partial t^2} g_{ij} = -2R_{ij} + 2\alpha \nabla_i \varphi \nabla_j \varphi$, therefore the evolution equation for $\hat{g}_{ij}$ is

$$\frac{\partial^2}{\partial t^2} \hat{g}_{ij} = -2\hat{R}_{ij} + 2\alpha \nabla_i \hat{\varphi} \nabla_j \hat{\varphi} + \hat{\nabla}_i V_j + \hat{\nabla}_j V_i + F(Dy, D_t D_x y), \tag{2.6}$$

where

$$Dy = \big(\frac{\partial y^\alpha}{\partial t}, \frac{\partial y^\alpha}{\partial x^i}\big), \quad D_t D_x y = \big(\frac{\partial^2 y^\alpha}{\partial x^i \partial t}\big), \quad \alpha,\, i = 1, 2, \dots, n.$$

The relation

$$\hat{\Gamma}^k_{jl} = \frac{\partial y^\alpha}{\partial x^j} \frac{\partial y^\beta}{\partial x^i} \frac{\partial x^k}{\partial y^\gamma} \Gamma^\gamma_{\alpha\beta} + \frac{\partial x^k}{\partial y^\alpha} \frac{\partial^2 y^\alpha}{\partial x^j \partial x^i}$$

implies

$$\frac{\partial^2 y^\alpha}{\partial t^2} = g^{jl}\big(\frac{\partial^2 y^\alpha}{\partial x^j \partial x^i} - \mathring{\Gamma}^k_{jl} \frac{\partial y^\alpha}{\partial x^j} + \Gamma^\gamma_{\alpha\beta} \frac{\partial y^\beta}{\partial x^j} \frac{\partial y^\gamma}{\partial x^i}\big) \tag{2.7}$$

and

$$\frac{\partial^2}{\partial t^2} \hat{g}_{ij} = \hat{g}^{kl} \frac{\partial^2 \hat{g}_{ij}}{\partial x^k \partial x^l} + 2\alpha \nabla_i \hat{\varphi} \nabla_j \hat{\varphi} + G(\hat{g}, D_x \hat{g}) + F(Dy, D_t D_x y), \tag{2.8}$$

where $\hat{g} = (\hat{g}_{ij})$, $D_x \hat{g} = (\frac{\partial \hat{g}_{ij}}{\partial x^k})$ for $i, j, k = 1, 2, \dots, n$. Hence, both (2.7) and (2.8) are clearly strictly hyperbolic system. On the other hand,

$$\frac{\partial \hat{\varphi}}{\partial t} = \psi_*(\frac{\partial \varphi}{\partial t}) + L_V \hat{\varphi} = \tau_{\hat{g}} \hat{\varphi} + \langle \nabla \hat{\varphi}, V \rangle = \tau_{\hat{g}} \hat{\varphi} + d\hat{\varphi}(V).$$

Using normal coordinates on (N, γ) results that ${}^N\Gamma^\lambda_{\mu\nu} = 0$ at the base point and hence $\tau_{\hat{g}} \hat{\varphi} = \Delta_{\hat{g}} \hat{\varphi}$ which implies that

$$\begin{aligned} \frac{\partial \hat{\varphi}}{\partial t} &= \Delta_{\hat{g}} \hat{\varphi} + d\hat{\varphi}(V) = \hat{g}^{kl}(\partial_k \partial_l \hat{\varphi}^\lambda - \hat{\Gamma}^j_{kl} \nabla_j \hat{\varphi}^\lambda) + \nabla_j \hat{\varphi}^\lambda \hat{g}^{kl}(\hat{\Gamma}^j_{kl} - \mathring{\Gamma}^k_{jl}) \\ &= \hat{g}^{kl}(\partial_k \partial_l \hat{\varphi}^\lambda - \mathring{\Gamma}^k_{jl} \nabla_j \hat{\varphi}^\lambda) \end{aligned} \tag{2.9}$$

and it is strictly hyperbolic equation. Since the equations (2.7), (2.8) and (2.9) are strictly hyperbolic and the manifold M is compact, it follows from the standard theory of hyperbolic equations (see [7]) that the system (1.6) has a unique smooth solution for a short time. So, the proof of the theorem is complete. □

3. Evolution equations of curvature tensor along the $(HG)_\alpha$ flow

Next, we consider the techniques and ideas used by Brendle [1] for evolution equation along the Ricci flow, and by Dai and et al [2] for the evolution equation along the hyperbolic geometric flow. We find the evolution formula for Riemannian curvature tensor, Ricci curvature tensor and scalar curvature of (M, g) under the $(HG)_\alpha$ flow.

Theorem 3.1. *Under the $(HG)_\alpha$ flow, the Riemannian curvature tensor R_{ijkl} of (M, g) satisfies the evolution equation*

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R_{ijkl}\\
&= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl})\\
&\quad - g^{pq}(R_{pjkl}R_{qi} + R_{ipkl}R_{qj} + R_{ijpl}R_{qk} + R_{ijkp}R_{ql})\\
&\quad + 2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk} - \frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\
&\quad + \alpha\Big[\frac{\partial^2(\nabla_k\varphi\nabla_j\varphi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_j\varphi\nabla_l\varphi)}{\partial x^i\partial x^k} - \frac{\partial^2(\nabla_k\varphi\nabla_i\varphi)}{\partial x^j\partial x^l} - \frac{\partial^2(\nabla_i\varphi\nabla_l\varphi)}{\partial x^j\partial x^k}\Big]
\end{aligned}\tag{3.1}$$

where $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$ and Δ is the Laplacian with respect to the evolving metric g.

Proof. The Christoffel symbol of metric g is $\Gamma^h_{jl} = \frac{1}{2}g^{hm}(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m})$, therefore by direct computations,

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}\Gamma^h_{jl} &= \frac{1}{2}\frac{\partial^2 g^{hm}}{\partial t^2}\Big(\frac{\partial g_{mj}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^j} - \frac{\partial g_{jl}}{\partial x^m}\Big) + \frac{\partial g^{hm}}{\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t} + \frac{\partial^2 g_{ml}}{\partial x^j\partial t} - \frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big)\\
&\quad + \frac{1}{2}g^{hm}\Big(\frac{\partial}{\partial x^l}(\frac{\partial^2 g_{mj}}{\partial t^2}) + \frac{\partial}{\partial x^j}(\frac{\partial^2 g_{ml}}{\partial t^2}) - \frac{\partial}{\partial x^m}(\frac{\partial^2 g_{jl}}{\partial t^2})\Big).
\end{aligned}$$

On the other hand, $R^h_{ijl} = \frac{\partial\Gamma^h_{jl}}{\partial x^i} - \frac{\partial\Gamma^h_{il}}{\partial x^j} + \Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il}$ and the Riemannian curvature tensor of (M, g) is $R_{ijkl} = g_{hk}R^h_{ijl}$, thus with a double differentiation respect to t we have

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R_{ijkl}\\
&= g_{hk}\Big[\frac{\partial}{\partial x^i}(\frac{\partial^2\Gamma^h_{jl}}{\partial t^2}) - \frac{\partial}{\partial x^j}(\frac{\partial^2\Gamma^h_{il}}{\partial t^2}) + \frac{\partial^2}{\partial t^2}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\Big]\\
&\quad + 2\frac{\partial g_{hk}}{\partial t}\Big[\frac{\partial}{\partial x^i}(\frac{\partial\Gamma^h_{jl}}{\partial t}) - \frac{\partial}{\partial x^j}(\frac{\partial\Gamma^h_{il}}{\partial t}) + \frac{\partial}{\partial t}(\Gamma^h_{ip}\Gamma^p_{jl} - \Gamma^h_{jp}\Gamma^p_{il})\Big] + R^h_{ijl}\frac{\partial^2 g_{hk}}{\partial t^2}.
\end{aligned}\tag{3.2}$$

We choose the normal coordinates around a fixed point p on M, then $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$ and $\Gamma^k_{ij}(p) = 0$. Since $\frac{\partial^2}{\partial t^2}g = -2\,\mathrm{Ric} + 2\alpha\nabla\varphi\otimes\nabla\varphi$, then we can rewrite (3.2) as follows:

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}R_{ijkl}\\
&= \frac{1}{2}\Big[\frac{\partial^2}{\partial x^i\partial x^l}(-2R_{kj}+2\alpha\nabla_k\varphi\nabla_j\varphi)-\frac{\partial^2}{\partial x^i\partial x^k}(-2R_{jl}+2\alpha\nabla_j\varphi\nabla_l\varphi)\Big]\\
&\quad-\frac{1}{2}\Big[\frac{\partial^2}{\partial x^j\partial x^l}(-2R_{ki}+2\alpha\nabla_k\varphi\nabla_i\varphi)-\frac{\partial^2}{\partial x^j\partial x^k}(-2R_{il}+2\alpha\nabla_i\varphi\nabla_l\varphi)\Big]\\
&\quad-g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^j\partial t}-\frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big)\\
&\quad+g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^i\partial t}-\frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big)\\
&\quad+2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}\frac{\partial}{\partial t}\Gamma^p_{jl}-\frac{\partial}{\partial t}\Gamma^h_{jp}\frac{\partial}{\partial t}\Gamma^p_{il}\Big).
\end{aligned}
\tag{3.3}
$$

For the other side, we have

$$
\frac{\partial^2}{\partial x^i\partial x^l}R_{jk}=\nabla_i\nabla_l R_{jk}+R_{jp}\nabla_i\Gamma^p_{lk}+R_{kp}\nabla_i\Gamma^p_{lj},
\tag{3.4}
$$

and

$$
\begin{aligned}
&-g^{pm}\frac{\partial^2 g_{kp}}{\partial x^i\partial t}\Big(\frac{\partial^2 g_{mj}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^j\partial t}-\frac{\partial^2 g_{jl}}{\partial x^m\partial t}\Big)\\
&+g^{pm}\frac{\partial^2 g_{kp}}{\partial x^j\partial t}\Big(\frac{\partial^2 g_{mi}}{\partial x^l\partial t}+\frac{\partial^2 g_{ml}}{\partial x^i\partial t}-\frac{\partial^2 g_{il}}{\partial x^m\partial t}\Big)\\
&+2g_{hk}\Big(\frac{\partial}{\partial t}\Gamma^h_{ip}\frac{\partial}{\partial t}\Gamma^p_{jl}-\frac{\partial}{\partial t}\Gamma^h_{jp}\frac{\partial}{\partial t}\Gamma^p_{il}\Big)\\
&=2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk}-\frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\Big).
\end{aligned}
\tag{3.5}
$$

Plugging (3.4) and (3.5) in (3.3) leads to

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}R_{ijkl}\\
&=-\nabla_i\nabla_l R_{jk}+\nabla_i\nabla_k R_{jl}+\nabla_j\nabla_l R_{ki}-\nabla_j\nabla_k R_{il}\\
&\quad-g^{pq}(R_{ijql}R_{kp}+R_{ijkq}R_{kp})+2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk}-\frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\
&\quad+\alpha\Big[\frac{\partial^2(\nabla_k\varphi\nabla_j\varphi)}{\partial x^i\partial x^l}-\frac{\partial^2(\nabla_j\varphi\nabla_l\varphi)}{\partial x^i\partial x^k}-\frac{\partial^2(\nabla_k\varphi\nabla_i\varphi)}{\partial x^j\partial x^l}-\frac{\partial^2(\nabla_i\varphi\nabla_l\varphi)}{\partial x^j\partial x^k}\Big]\\
&=\Delta R_{ijkl}+2(B_{ijkl}-B_{ijlk}-B_{iljk}+B_{ikjl})\\
&\quad-g^{pq}(R_{pjkl}R_{qi}+R_{ipkl}R_{qj}+R_{ijpl}R_{qk}+R_{ijkp}R_{ql})\\
&\quad+2g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{jk}-\frac{\partial}{\partial t}\Gamma^p_{jl}\frac{\partial}{\partial t}\Gamma^q_{ik}\Big)\\
&\quad+\alpha\Big[\frac{\partial^2(\nabla_k\varphi\nabla_j\varphi)}{\partial x^i\partial x^l}-\frac{\partial^2(\nabla_j\varphi\nabla_l\varphi)}{\partial x^i\partial x^k}-\frac{\partial^2(\nabla_k\varphi\nabla_i\varphi)}{\partial x^j\partial x^l}-\frac{\partial^2(\nabla_i\varphi\nabla_l\varphi)}{\partial x^j\partial x^k}\Big]
\end{aligned}
\tag{3.6}
$$

where $B_{ijkl}=g^{pr}g^{qs}R_{piqj}R_{rksl}$, so the proof is complete. □

Theorem 3.2. *The evolution equation for Ricci curvature tensor under the* $(HG)_\alpha$ *flow is as follows:*

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R_{ij}\\
&= \Delta R_{ij} + 2g^{pr}g^{qs}R_{piqj}R_{rs} - 2g^{pq}R_{pi}R_{qj}\\
&\quad + 2g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}\frac{\partial}{\partial t}\Gamma^q_{ij}\Big)\\
&\quad + \alpha g^{kl}\Big[\frac{\partial^2(\nabla_j\varphi\nabla_k\varphi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_k\varphi\nabla_l\varphi)}{\partial x^i\partial x^j} - \frac{\partial^2(\nabla_j\varphi\nabla_i\varphi)}{\partial x^k\partial x^l} - \frac{\partial^2(\nabla_i\varphi\nabla_l\varphi)}{\partial x^k\partial x^j}\Big]\\
&\quad - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl}\\
&\quad - 2\alpha g^{kp}g^{lq}\nabla_p\varphi\nabla_q\varphi R_{ikjl}.
\end{aligned} \tag{3.7}$$

Proof. We have

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ij} &= \frac{\partial^2}{\partial t^2}(g^{kl}R_{ikjl})\\
&= g^{kl}\frac{\partial^2}{\partial t^2}R_{ikjl} + 2\frac{\partial g^{kl}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + R_{ikjl}\frac{\partial^2 g^{kl}}{\partial t^2}.
\end{aligned}$$

Since $\frac{\partial g^{kl}}{\partial t} = -g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}$ and $\frac{\partial^2 g^{kl}}{\partial t^2} = -g^{kp}g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2} + 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}$, we have

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}R_{ij} = g^{kl}\frac{\partial^2}{\partial t^2}R_{ikjl} - 2g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} - g^{kp}g^{lq}\frac{\partial^2 g_{pq}}{\partial t^2}R_{ikjl}\\
+ 2g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ikjl}
\end{aligned} \tag{3.8}$$

by replacing (3.1) and $\frac{\partial^2}{\partial t^2}g_{ij} = -2R_{ij} + 2\alpha\nabla_i\varphi\nabla_j\varphi$ in (3.8) the proof is complete. ☐

From $R = g^{ij}R_{ij}$ and using (3.7) we have the following result.

Corollary 3.3. *Under the* $(HG)_\alpha$ *flow, the evolution equation of the scalar curvature satisfies*

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R\\
&= \Delta R + 2|\operatorname{Ric}|^2 + 2g^{ij}g^{kl}g_{pq}\Big(\frac{\partial}{\partial t}\Gamma^p_{il}\frac{\partial}{\partial t}\Gamma^q_{kj} - \frac{\partial}{\partial t}\Gamma^p_{kl}\frac{\partial}{\partial t}\Gamma^q_{ij}\Big)\\
&\quad + \alpha g^{ij}g^{kl}\Big[\frac{\partial^2(\nabla_j\varphi\nabla_k\varphi)}{\partial x^i\partial x^l} - \frac{\partial^2(\nabla_k\varphi\nabla_l\varphi)}{\partial x^i\partial x^j} - \frac{\partial^2(\nabla_j\varphi\nabla_i\varphi)}{\partial x^k\partial x^l} - \frac{\partial^2(\nabla_i\varphi\nabla_l\varphi)}{\partial x^k\partial x^j}\Big]\\
&\quad - 2g^{ij}g^{kp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ikjl}}{\partial t} + 4g^{kp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{kl}\\
&\quad - 4\alpha g^{ij}g^{kp}g^{lq}\nabla_p\varphi\nabla_q\varphi R_{ikjl} - 2g^{ip}g^{jq}\frac{\partial g_{pq}}{\partial t}\frac{\partial R_{ij}}{\partial t}.
\end{aligned}$$

4. Examples

In this section, we give some examples of $(HG)_\alpha$ flows.

Example 4.1. Let $(M, g(0))$ be a round two-sphere of constant Gauss curvature 1. Consider, the $(HG)_\alpha$ flow, assuming that $(N,\gamma) = (M, g(0))$ and $\varphi(0)$ is the identity map, with $g(t)=c(t)g(0)$, $c(0)=1$, $c'(0)=0$ and the fact the $\varphi(t)=\varphi(0)$ is harmonic map for all $g(t)$. The $(HG)_\alpha$ flow on $(M,g(0))$ reduces to

$$\frac{\partial^2 c(t)}{\partial t^2}=-2+2\alpha \tag{4.1}$$

and it has solution $c(t)=(-1+\alpha)t^2+1$ where for $\alpha<1$, $c(t)$ goes to zero in finite time i.e. $(M,g(t))$ shrinks to a point, while the scalar curvature R and the energy density $|\nabla\varphi|^2$ both go to infinity. For $\alpha=1$, the solution is stationary. For $\alpha>1$, $c(t)$ increasing.

Example 4.2. Let $(M^4,g(t))=(S^2\times L, c(t)g_{S^2}\oplus d(t)g_L)$ where (S^2,g_{S^2}) is a round sphere with Gauss curvature 1 and (L,G_L) is a surface with constant Gauss curvature -1. Consider, the $(HG)_\alpha$ flow, assuming that $(N,\gamma)=(M,g(0))$ and $\varphi(0)$ is the identity map. Then $\varphi(t)=\varphi(0)$ and $(HG)_\alpha$ flow results that

$$\begin{aligned}&\frac{\partial^2}{\partial t^2}c(t)=-2+2\alpha,\quad c(0)=1,\quad c'(0)=0,\\ &\frac{\partial^2}{\partial t^2}d(t)=2+2\alpha,\quad d(0)=1,\quad d'(0)=0.\end{aligned} \tag{4.2}$$

If $0<\alpha<1$, then $\frac{\partial^2}{\partial t^2}c(t)<0$ implies that $c(t)$ is decreasing and $\frac{\partial^2}{\partial t^2}d(t)>0$ results that $d(t)$ is increasing. If $\alpha=1$, then $c(t)$ is stationary and $d(t)=2t^2+1$.

Example 4.3. Let $(M,g(0))$ be a arbitrary closed Riemannian manifold, $(N,\gamma)=(M,g(0))$ and $\varphi(0)$ is the identity map. If the initial metric $g_{ij}(x,0)$ is Ricci flat, i.e. $R_{ij}(x,0)=0$, then $g_{ij}(x,t)=(\alpha t^2+t+1)g_{ij}(x,0)$ is obviously a solution to the evolution equation $(HG)_\alpha$ flow with $\frac{\partial g}{\partial t}(x,0)=g(x,0)$, therefore any Ricci flat metric is a stationary solution of the $(HG)_\alpha$ flow (1.6).

Example 4.4. A Riemannian metric g_{ij} is called Einstein if $R_{ij}=\lambda g_{ij}$ for some constant λ. A smooth manifold M with an Einstein metric is called Einstein manifold. Let $(M,g(0))$ be a closed Riemannian manifold, the initial metric $g(0)$ is Einstein that is for some constant λ it holds

$$R_{ij}(0)=\lambda g_{ij}(0) \tag{4.3}$$

and $(N,\gamma)=(M,g(0))$ and $\varphi(0)$ is the identity map. The evolving metric under the $(HG)_\alpha$ flow will be steady state, or will expand homothetically for all time, or shrink in a finite time. Since, the initial metric is Einstein for some constant λ, let $g_{ij}(t,x)=\rho(t)g_{ij}(0)$. By the definition of the Ricci tensor, we obtain

$$R_{ij}(t)=R_{ij}(0)=\lambda g_{ij}(0). \tag{4.4}$$

In the present situation, equation (1.6) becomes

$$\frac{\partial^2(\rho(t)g_{ij}(0))}{\partial t^2}=-2\lambda g_{ij}(0)+2\alpha g_{ij}(0), \tag{4.5}$$

this gives an ODE of second order

$$\frac{d^2\rho(t)}{\partial t^2}=-2\lambda+2\alpha,\quad \rho(0)=1,\quad \rho'(0)=\nu, \tag{4.6}$$

if α is constant, then the solution of the initial value problem is given by

$$\rho(t)=(\alpha-\lambda)t^2+\nu t+1. \tag{4.7}$$

Therefore the solution of the $(HG)_\alpha$ flow remains Einstein.

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