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# VANISHING VISCOSITY LIMIT FOR THE 3D NONHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES EQUATION WITH SPECIAL SLIP BOUNDARY CONDITION

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Abstract. In this article we consider the three-dimensional nonhomogeneous incompressible Navier-Stokes equation with special slip boundary conditions in a bounded domain. We discuss the problem of the vanishing viscosity limit and provide a rate of convergence estimates for the strong solution.

## 1. Introduction

Let $\Omega \subset R^3$ be a bounded smooth domain, the initial boundary value problem of the nonhomogeneous incompressible Navier-Stokes equation is given by

$$\rho\partial_t u - \nu\Delta u + \rho u \cdot \nabla u + \nabla p = 0, \quad \text{in } \Omega, \tag{1.1}$$

$$\partial_t \rho + u \cdot \nabla\rho = 0, \quad \text{in } \Omega, \tag{1.2}$$

$$\nabla \cdot u = 0, \quad \text{in } \Omega, \tag{1.3}$$

$$u(0,x) = u_0, \rho(0,x) = \rho_0, \quad \text{in } \Omega, \tag{1.4}$$

equipped with the vorticity boundary conditions

$$u \cdot n = 0, \quad \omega \cdot n = 0, \quad n \times (\Delta u) = 0 \quad \text{on } \partial\Omega. \tag{1.5}$$

Here the constant $\nu > 0$, n, $\rho, u, p$ represent the viscosity coefficient, the outward unit normal vector, the mass density, the velocity field and the pressure of the fluids, respectively. The initial density $\rho_0(x)$ is assumed to satisfy the condition $m \le \rho_0(x) \le M$ with $m$ and $M$ are given positive constants.

The vanishing viscosity limit for the nonhomogeneous incompressible Navier-Stokes equation with the cauchy problem and the periodic boundary conditions has been investigated by Itoh [14], Itoh and Tani [15] and Danchin [10], respectlvely. In the presence of a physical boundary, the vanishing viscosity limit problems become more challenging and significance because of the emergence of the boundary layer. Formally, when the viscous term is vanishing, system (1.1)-(1.4) degenerates into the nonhomogeneous incompressible Euler equation

$$\rho^0\partial_t u^0 + \rho^0 u^0 \cdot \nabla u^0 + \nabla p^0 = 0, \quad \text{in } \Omega, \tag{1.6}$$

$$\partial_t \rho^0 + u^0 \cdot \nabla\rho^0 = 0, \quad \text{in } \Omega, \tag{1.7}$$

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$$\nabla \cdot u^0 = 0, \quad \text{in } \Omega, \tag{1.8}$$

$$u^0(0,x) = u_0, \rho^0(0,x) = \rho_0, \quad \text{in } \Omega, \tag{1.9}$$

with the slip boundary conditions

$$u^0 \cdot n = 0, \quad \text{on } \partial\Omega. \tag{1.10}$$

The initial boundary value problem of the equation (1.6)-(1.10) has a smooth solution at least local in time, it has been addressed by several authors, see, e.g. [3, 15, 22]. Concerning the nonhomogeneous incompressible Navier-Stokes equation, one of the most common physical boundary conditions is the classical no-slip boundary conditions

$$u = 0, \quad \text{on } \partial\Omega, \tag{1.11}$$

which means that fluid particles are adherent to the boundary because of the positive viscosity, it was proposed by Stokes in [20]. This Dirichlet type problem has been addressed in [9, 21] and references therein. However, the asymptotic convergence of the solution is one of the major open problem except some special cases, the main challenging is a discrepancy between the no-slip boundary conditions for the nonhomogeneous incompressible Navier-Stokes equation and the tangential boundary conditions for the nonhomogeneous incompressible Euler equation.

Another class of familiar boundary conditions is the Navier-slip boundary conditions, which can be shown as follows

$$u \cdot n = 0, \quad 2(S(u)n)_\tau = -\gamma u_\tau, \quad \text{on } \partial\Omega, \tag{1.12}$$

it was first introduced in [19], where $2S(u)n = (\nabla u + (\nabla u)^\top)$ is the viscous stress tensor, $\gamma$ is a given smooth function on the boundary. We can also write the equivalently form as the following vorticity-slip condition

$$u \cdot n = 0, n \times \omega = \beta u, \quad \text{on } \partial\Omega. \tag{1.13}$$

The result of weak convergence have been considered by Ferreira and Planas [11]. As $\beta = 0$, the special vorticity-slip conditions have initially been applied to three-dimensional incompressible Navier-Stokes equation in [24]. Based on the above works, the author and coauthor found an additional condition for the density to obtain the strong convergence rate for the nonhomogeneous Navier-Stokes equation on the flat domain in [7]. However, to our best knowledge, it is still unknown if the similar strong convergence results can be established in a general bounde domain. There are many references on inviscid limit for Navier-Stokes equation with Navier-slip boundary conditions, the readers can be referred in [4, 5, 6, 8, 12, 13, 16, 17, 26].

Our main goal in this paper is to show the vanishing viscosity limit problem with the vorticity boundary condition (1.5). This type of boundary condition, which was initially established in [25] for the homogeneous incompressible Navier-Stokes equation, where the author established the mathematical result on rate of convergence for strong solution. Our approach here is motivated by the ideas [25] to study the problem for the nonhomogeneous incompressible Navier-Stokes equation and is based on the following observations: First, we need to add the some additional boundary conditions for the density, which is described by

$$\nabla \rho = 0, \quad \text{on } \partial\Omega. \tag{1.14}$$

The boundary condition (1.14) can balance well the momentum equation (3.2) with boundary conditions (3.4), we can obtain the strong solutions local in time. Second, we need to construct a new system (3.1)-(3.7), which can be regarded as a relaxed vorticity system of nonhomogeneous incompressible Navier-Stokes equation. The fact shows that the pressure vanishes in the new system, yet the new system is indeed the vorticity system of the equations (1.1)-(1.5). Our first main result is concerned with the local well-posedness of the initial boundary value problem for the equations (1.1)-(1.5).

**Theorem 1.1.** *Let $\Omega$ be the bounded smooth domain, denote $H$ by the space $\{u \in L^2(\Omega); \nabla \cdot u = 0,\ in\ \Omega, u \cdot n = 0\ on\ \partial\Omega\}$, $u_0 \in H^1(\Omega) \cap H$, $\rho_0 \in H^2(\Omega)$, $\omega_0 \in H^1(\Omega) \cap H$. Then there exists $T^\nu = T^\nu(\omega_0) > 0$, such that the initial boundary value problem (1.1)-(1.2) has a unique solution $(\rho, u, p)$ satisfying*

$$u \in L^2(0,T;H^3(\Omega)) \cap C([0,T^\nu);H^2(\Omega)),$$
$$\rho \in C([0,T^\nu);H^2(\Omega)),\ u' \in L^2(0,T;V),$$

*for any $T \in (0,T^\nu)$, and*

$$-\Delta p = \rho \partial_i u_j \partial_j u_i,$$
$$\partial_n p = (\Delta u - \rho u \cdot \nabla u) \cdot n,$$
$$\int_\Omega p = 0,$$

*for $t \in [0,T^\nu)$.*

**Remark 1.2.** To obtain the results above, we need to construct a new initial boundary value problem (3.1)-(3.7). Since there is one more condition in (1.5) than that normally Navier-slip boundary conditions, thus it is non-trivial to show the consistency of the boundary conditions to get the well-posedness.

As the viscosity coefficient $\nu$ tends to be zero, we show the following convergence of rate.

**Theorem 1.3.** *Let $\rho_0 \in H^4(\Omega), u_0 \in H \cap H^4(\Omega)$ satisfy $\nabla\rho_0 \cdot n = 0, \nabla \times u_0 \in H$, $\rho^0(t), u^0(t)$ be the solution to the Euler equations for nonhomogeneous fluids on $[0,T]$ with initial data $\rho_0, u_0$, $\rho(t), u(t)$ be the solution in Theorem 1.1. Then, we have the following*

$$\|\rho - \rho^0\|_2^2 + \|u - u^0\|_2^2 + \nu \int_0^t \|u - u^0\|_3^2 dt \le c\nu^{1-s} \tag{1.15}$$

*on the interval $[0,T]$ with $T = T(\sigma, s) > 0$ independent of $\nu \in (0,\sigma)$ for $s > 0$ and $\nu \in (0,\sigma)$.*

**Remark 1.4.** Under the vorticity boundary conditions, we can get a result mathematically of strong convergence estimate to the solutions. The rate of convergence (1.15) is better than those for the Navier-slip boundary conditions cases in [11]. Compared with the case of co-normal uniform estimate as in [18, 23], our problem here does not so tedious and complicated, it can be proved only by standard energy estimates.

The rest of this article is organized as follows: Section 2, we recall some notations, definitions, and preliminary facts. Section 3, we give the local well-posedness to the initial boundary value problem for the nonhomogeneous Navier-Stokes equations (1.1)-(1.5). Section 4, we establish the rate of convergence to the solutions.

## 2. Preliminaries

Let us start by recalling the standard notation of some function spaces and operators which are familiar in the mathematical theory of fluids modelled by Navier-Stokes system, see [24, 25]. For convenience, note the inner product by $(\cdot,\cdot)$ and the norm of the standard Hilbert space $L^2(\Omega)$, $H^s(\Omega)$ by $\|\cdot\|$, $\|\cdot\|_s$, respectively. We also denote $[A,B]=AB-BA$, the commutator between two operators $A$ and $B$. Set

$$\begin{gathered} H=\{u\in L^2(\Omega);\nabla\cdot u=0,\ \text{in } \Omega,\ u\cdot n=0 \text{ on } \Omega\},\\ V=H^1(\Omega)\cap H,\\ W=\{u\in H^2(\Omega); n\times(\nabla\times u)=0 \text{ on } \Omega\}. \end{gathered}$$

Let $\psi,\phi$ be two vector function, the following formula is shown by direct calculations:

$$\nabla\times(\psi\times\phi)=\phi\cdot\nabla\psi-\psi\cdot\nabla\phi+\psi\nabla\cdot\phi-\phi\nabla\cdot\psi, \tag{2.1}$$

$$\nabla\times(\psi\cdot\nabla\phi)=\psi\cdot\nabla(\nabla\times\phi)+\nabla\psi^\perp\cdot\nabla\phi, \tag{2.2}$$

where $\nabla\psi^\perp$ is expressed in components by

$$(\nabla\psi^\perp\cdot\nabla\phi)_j=(-1)^{j+1}\partial_{j+1}\psi\cdot\nabla\phi_{j+1}+(-1)^{j+2}\partial_{j+2}\psi\cdot\nabla\phi_{j+2}$$

with the index modulated by 3. We denote by $A=-\Delta$ the Stokes operator with $D(A)=W\subset V$ is the self-adjoint extension of the positive closed with its inverse being compact, and there is a countable eigenvalues $\{\lambda_j\}$ such that

$$0<\lambda_1\le\lambda_2\cdots\to\infty,$$

the corresponding eigenvector $\{e_j\}\subset W\cap C^\infty(\Omega)$ makes an orthogonal complete basis of $H$. We first show the following estimate.

**Lemma 2.1** ([24]). *Let $s\ge0$ be an integer. Let $u\in H^s$ be a vector-valued function, then*

$$\begin{gathered} \|u\|_s\le C(\|\nabla\times u\|_{s-1}+\|\nabla\cdot u\|_{s-1}+|n\cdot u|_{s-\frac12}),\\ \|u\|_s\le C(\|\nabla\times u\|_{s-1}+\|\nabla\cdot u\|_{s-1}+|n\times u|_{s-\frac12}+\|u\|_{s-1}). \end{gathered}$$

Assuming that $\phi(t),\psi(t),f(t)$ are smooth non-negative functions defined for all $t\ge0$, we show the following differential inequality.

**Lemma 2.2** ([21]). *Suppose $\phi(0)=\phi_0$ and $\frac{d\phi(t)}{dt}+\psi(t)\le g(\phi(t))+f(t)$ for $t\ge0$, where $g$ is a non-negative Lipschitz continuous function defined for $\phi\ge0$. Then $\phi(t)\le F(t;\phi_0)$ for $t\in[0,T(\phi_0))$ where $F(\cdot;\phi_0)$ is the solution of the initial value problem $\frac{dF(t)}{dt}=g(F(t))+f(t)$; $F(0)=\phi_0$ and $[0,T(\phi_0))$ is the largest interval to which it can be continued. Also, if $g$ is nondecreasing, then*

$$\int_0^t\psi(\tau)d\tau\le\widetilde F(t;\phi_0)$$

*with*

$$\widetilde F(t;\phi_0)=\phi_0+\int_0^t[g(F(\tau;\phi_0))+f(\tau)]d\tau.$$

## 3. Local well-posedness results

Our main purpose in this section is to solve the initial boundary value problem (1.1)-(1.5). Firstly, we give the following additional boundary condition for density:

**Lemma 3.1.** *Let the initial density satisfy the condition $\nabla\rho_0 = 0$ on the boundary, then the density have the persistence property that $\nabla\rho(t,\cdot) = 0$ on the boundary.*

*Proof.* Applying the gradient operator $\nabla$ to the transport equation (1.2), it follows that

$$\frac{D}{dt}(\nabla\rho) + \nabla u \cdot \nabla\rho = 0,$$

the ordinary differential equations is linear and the initial data satisfies $\nabla\rho_0 = 0$, we can prove the lemma. □

On the other hand, to obtain the strong solution, we need to construct the following system, which is called a relaxed vorticity equation of (1.1)-(1.5):

$$\rho_t + u\cdot\nabla\rho = 0, \quad \text{in } \Omega, \tag{3.1}$$

$$\rho(\partial_t\omega + u\cdot\nabla\omega - \omega\cdot\nabla u) + \nabla\rho\times(\partial_t u + u\cdot\nabla u) - \nu\Delta\omega + \nabla q = 0, \quad \text{in } \Omega, \tag{3.2}$$

$$\nabla\cdot\omega = 0, \quad \text{in } \Omega, \tag{3.3}$$

$$\omega\cdot n = 0, n\times(\nabla\times\omega) = 0, \quad \text{on } \partial\Omega, \tag{3.4}$$

with $u = T\omega$ given by

$$\nabla\times u = \omega, \quad \text{in } \Omega, \tag{3.5}$$

$$\nabla\cdot u = 0, \quad \text{in } \Omega, \tag{3.6}$$

$$u\cdot n = 0, \quad \text{on } \partial\Omega, \tag{3.7}$$

Where the linear operator satisfy $T : H \to V$ with $u = T\omega$, which is the unique solution of equations (3.5)-(3.7), is continuous. We claim that the initial boundary value problem (3.1)-(3.7) possesses exactly one strong solution in a maximal time interval. Let $P_k$ the orthogonal project of H onto the space $H_k$ spanned by the k first eigenfunctions $e_1,\cdots e_k$ of A. Then the solutions of system (3.1)-(3.7) can be obtained by using a Semi-Galerkin approximations method determined by the spaces $H_k$ and the operators $P_k$. For each fixed k, we consider the following finite dimensional problem: Find $T_k \in (0,T]$ such that

$$\begin{gathered}
P_m(\rho^{(m)}\partial_t\omega^{(m)} + \rho^{(m)}T\omega^{(m)}\cdot\nabla\omega^{(m)} - \rho^{(m)}\omega^{(m)}\cdot\nabla T\omega^{(m)}) \\
+ P_m(\nabla\rho^{(m)}\times(\partial_t T\omega^{(m)} + T\omega^{(m)}\cdot\nabla T\omega^{(m)})) - \nu\Delta P_m\omega^{(m)} = 0, \\
\rho_t^{(m)} + T\omega^{(m)}\cdot\nabla\rho^{(m)} = 0, \\
\omega^{(m)}(0,x) = P_m\omega_0(x),\ \rho^{(m)}(0,x) = \rho_0(x), \\
e_m\cdot n = 0, n\times(\nabla\times e_m) = 0.
\end{gathered}$$

We have an initial boundary value problem for a system of ordinary differential equations coupled to a transport equation. By using the characteristics method, it can prove the system possesses exactly one solution $(\rho^{(m)},\omega^{(m)})$ defined in a time interval $[0,T_k)$. The kth approximated problem can also be written in the form

$$\begin{aligned}
&(\rho^{(m)}\partial_t\omega^{(m)} + \rho^{(m)}T\omega^{(m)}\cdot\nabla\omega^{(m)} - \rho^{(m)}\omega^{(m)}\cdot\nabla T\omega^{(m)}, v) \\
&+ (\nabla\rho^{(m)}\times(\partial_t T\omega^{(m)} + T\omega^{(m)}\cdot\nabla T\omega^{(m)}), v) - \nu(\Delta\omega^{(m)}, v) = 0,
\end{aligned}$$

$$\rho_t^{(m)} + T\omega^{(m)} \cdot \nabla \rho^{(m)} = 0,$$
$$\omega^{(m)}(0,x) = P_m \omega_0(x),\ \rho^{(m)}(0,x) = \rho_0(x),$$
$$e_m \cdot n = 0, n \times (\nabla \times e_m) = 0.$$

Through the Semi-Galerkin approximation method, the rest of the process to estimate the solutions of (3.1)-(3.7) is rather standard. We do not give the detailed proof, the reader can be referred to Chapter 3 in [2]. The main theorem in this section is the following.

**Theorem 3.2.** *Let $\rho_0 \in H^2(\Omega)$ and $\omega_0 \in V$, then there exists $T^\nu = T^\nu(\rho_0, \omega_0) > 0$, such that problem (3.1)-(3.7) has a unique solution $(\rho, \omega, q)$ on the interval $[0, T^\nu)$ satisfying*

$$\rho \in C([0, T^\nu); W),$$
$$\omega \in L^2(0, T^\nu; W) \cap C([0, T^\nu); V), \omega' \in L^2(0, T^\nu; H),$$

*and the energy equation*

$$\|\rho(t)\|_2^2 + \|\nabla \times \omega(t)\|^2 + \nu \int_0^t \|\partial_t \omega\|^2 dx + \nu \int_0^t \|\omega(s)\|_2^2 ds \le c \tag{3.8}$$

*hold on $[0,t]$ for any $t \in (0, T^\nu)$, and $q$ is given uniquely by*

$$\Delta q = 0, \tag{3.9}$$
$$\partial_n q = -\rho(u \cdot \nabla \omega - \omega \cdot \nabla u) \cdot n, \tag{3.10}$$
$$\int_{\partial\Omega} q = 0, \tag{3.11}$$

*for a.e. $t \in (0, T^\nu)$.*

**Lemma 3.3.** *Let $\omega \in V$, $\nabla \rho = 0$ on boundary. Then*

$$\rho(T\omega \cdot \nabla \omega - \omega \cdot \nabla (T\omega)) \in H.$$

*Proof.* Since $\omega \in V$, it follows that $T\omega \in H^2(\Omega) \cap V$. Then $\omega \times T\omega \in H^1(\Omega)$. The boundary condition $T\omega \cdot n = 0$ and $\omega \cdot n = 0$ implies

$$n \times (\omega \times T\omega) = 0, \text{ on } \partial\Omega.$$

This completes the proof. □

From Lemma 3.3 we have the following corollary.

**Corollary 3.4.** *The solution $q$ in theorem 3.2 satisfies $q = 0$, for a.e. $t \in (0, T^\nu)$.*

From the analysis above, it follows that (3.2) is the curl of the equation (1.1). Thus Theorem 1.1 is proved.

**Remark 3.5.** It should be noted that constructing system (3.1)-(3.6) is necessary. If the boundary condition is replaced by the non slip boundary $\omega = 0$, then $(\Delta\omega) \cdot n$ may not be zero, from equations (3.9)-(3.11), hence $\nabla q$ may not be zero. Then the momentum equation should be of the form

$$\rho \partial_t u - \nu \Delta u + \rho u \cdot \nabla u + F(q) + \nabla p = 0, \quad \text{in } \Omega,$$

for some vector function $F$ of $q$.

## 4. Convergence of solutions

In this section we prove Theorem 1.3. Let us show the following lemma before giving the convergence estimate.

**Lemma 4.1.** *Let $\rho, u$ be a smooth solution to the nonhomogeneous incompressible Euler equations on the interval $[0,T]$ with initial $\rho_0 \in H^3(\Omega)$, $u_0 \in H^3(\Omega) \cap H$ and $\nabla\rho_0 = 0$, $\nabla \times u_0 \in H$. Then $(\nabla \times u^0) \cdot n = 0, \quad$ on $\partial\Omega$ for all $t \in [0,T]$.*

*Proof.* Note that the particle path forms a diffeomorphism on the boundary. The vorticity equations of the nonhomogeneous incompressible Navier-Stokes equation is

$$\rho^0_t + u^0 \cdot \nabla\rho^0 = 0, \quad \text{in } \Omega, \tag{4.1}$$

$$\rho^0(\partial_t\omega^0 + u^0 \cdot \nabla\omega^0 - \omega^0 \cdot \nabla u^0) + \nabla\rho^0 \times (\partial_t u^0 + u^0 \cdot \nabla u^0) = 0, \quad \text{in } \Omega, \tag{4.2}$$

From Lemma 3.1, it follows that $\nabla\rho^0 \times (\partial_t u^0 + u^0 \cdot \nabla u^0)$ vanishes on the boundary. Multiplying (4.2) by the unit outward norm vector yields

$$\frac{D(\omega^0 \cdot n)}{dt} = (\omega^0 \cdot \nabla)u^0 \cdot n + \omega^0 \cdot (u^0 \cdot \nabla)n.$$

From [25, Lemma 3.1], there exist $\alpha, \beta$ such that

$$\frac{D(\omega^0 \cdot n)}{dt} = (\alpha + \beta)(\omega^0 \cdot n).$$

Since $\omega_0 \cdot n = 0 \quad$ on $\partial\Omega$, one has $\omega^0(x,t) \cdot n = 0$ on $\partial\Omega$. This complete the proof. □

**Remark 4.2.** To obtain the asymptotic convergence of the solutions, we need some additional conditions for nonhomogeneous Euler equation to overcome the boundary layer. If the nonhomogeneous Euler equation match the boundary conditions $\omega^0 \cdot n = 0$ in mathematical structure, it can coincide with that of nonhomogeneous Navier-Stokes equation in the tangential directions. Hence, we restrict the initial data condition of the density satisfy $\nabla\rho_0 = 0$.

*Proof of Theorem 1.3.* First, we denote $a = \rho - \rho^0$, $v = u - u^0$, $w = \omega - \omega^0$. From the transport equations, it follows that

$$\frac{d}{dt}a + u^0 \cdot \nabla a = -v \cdot \nabla\rho. \tag{4.3}$$

Applying the operate $D^2$ and taking the inner product of (4.3) with $D^2a$, we have

$$\frac{d}{dt}\|a(t)\|_2^2 + (u^0 \cdot \nabla D^2 a, D^2 a) + ([D^2, u^0 \cdot \nabla]a, D^2 a) = -(D^2(v \cdot \nabla\rho), D^2 a).$$

Hence, by Young's inequality, it is easy to obtain

$$\frac{d}{dt}\|a(t)\|_2^2 \le c\delta\nu\|\Delta w\|^2 + \nu^{-1}\|a\|_2^4 + \|a\|_2^2 + \|v\|_2^2 + c\nu. \tag{4.4}$$

Secondly, we estimate the difference system between the vorticity equation of (1.1) and the vorticity equation of (1.6):

$$aw_t + (\rho v + au^0) \cdot \nabla w + \rho^0 w_t + \rho^0 u^0 \cdot \nabla w + \Phi - \nu\Delta w = \nu\Delta\omega^0, \tag{4.5}$$

with the boundary conditions

$$u \cdot n = 0, w \cdot n = 0 \quad \text{on } \partial\Omega, \tag{4.6}$$

where $\Phi = A + B$,

$$\begin{aligned} A &= aw_t^0 + av\cdot\nabla\omega^0 + \rho^0 v\cdot\nabla\omega^0 + au^0\cdot\nabla\omega^0 + aw\cdot\nabla v \\ &\quad + aw\cdot\nabla u^0 + a\omega^0\cdot\nabla v + a\omega^0\cdot\nabla u^0 + \rho^0 w\cdot\nabla v + \rho^0 w\cdot\nabla u^0 + \rho^0\omega^0\cdot\nabla v \\ &\quad + \nabla a\times(\partial_t v + \partial_t u^0 + v\cdot\nabla v + v\cdot\nabla u^0 + u^0\cdot\nabla v + u^0\cdot\nabla u^0) \\ &\quad + \nabla\rho^0\times(\partial_t v + v\cdot\nabla v + v\cdot\nabla u^0 + u^0\cdot\nabla v), \end{aligned}$$

and

$$\begin{aligned} B &= \nabla a\times(\partial_t v + \partial_t u^0 + v\cdot\nabla v + v\cdot\nabla u^0 + u^0\cdot\nabla v + u^0\cdot\nabla u^0) \\ &\quad + \nabla\rho^0\times(\partial_t v + v\cdot\nabla v + v\cdot\nabla u^0 + u^0\cdot\nabla v). \end{aligned}$$

Taking the inner product of (4.5) with $-\Delta w$, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\sqrt{a}\nabla\times w\|^2 + \|\sqrt{\rho^0}\nabla\times w\|^2)ds + \nu\|\Delta w\|^2 - (\Phi,\Delta w) \\ &= \int_{\partial\Omega}((aw)_t + (\rho_0 w)_t)\cdot(n\times(\nabla\times w))ds + \nu(\Delta\omega^0,\Delta w). \end{aligned} \tag{4.7}$$

Integrating by part yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\sqrt{a}\nabla\times w + \|\sqrt{\rho^0}\nabla\times w\|^2 \\ &+ 2\int_{\partial\Omega}(aw + \rho_0 w)\cdot(n\times(\nabla\times\omega^0)))ds + \nu\|\Delta w\|^2 \\ &= (\Phi,\Delta w) + \int_{\partial\Omega}(aw + \rho_0 w)\cdot(n\times\partial_t(\nabla\times\omega^0))ds + \nu(\Delta\omega^0,\Delta w). \end{aligned} \tag{4.8}$$

Here we use the property that $n\times(\nabla\times\omega) = 0, v\cdot n = 0, w\cdot n = 0$ on $\partial\Omega$, it follows that

$$\begin{aligned} (\Phi,-\Delta w) &= \int_{\partial\Omega}\Phi\cdot n\times(\nabla\times\omega^0) - (\nabla\times\Phi,\nabla\times w) \\ &= (\Phi,-\Delta\omega^0) - (\nabla\times\Phi,\nabla\times\omega^0) - (\nabla\times\Phi,\nabla\times w). \end{aligned} \tag{4.9}$$

Next, we list some basic facts to be used later. The unit out normal vector $n$ has been extended as follows:

$$n(x) = \frac{\nabla\varphi(r(x))}{|\nabla\varphi(r(x))|},\ x\in\Omega$$

and

$$r(x) = \min_{y\in\partial\Omega} d(x,y) = d(x,y_0), \quad y_0\in\partial\Omega,$$

which is unique when $r(x)\le\sigma$ for some $\sigma>0$, and the function is smooth and compact supported in $[0,\sigma)$ such that

$$\varphi(0) = 1, \quad \varphi'(0) = 1.$$

First we estimate on $(\nabla\times\Phi,\nabla\times w)$ from (4.9), recall that

$$(\nabla\times\Phi,\nabla\times w) = (\nabla\times(A+B),\nabla\times w).$$

It follows from the definition of $A$ that

$$\begin{aligned} |(\nabla\times(aw_t^0),\nabla\times w)| &= |(\nabla a\times w_t^0 + a\nabla\times w_t^0),\nabla\times w)| \\ &\le c\|a\|_2\|\nabla\times w_t^0\|\|\nabla\times w\| \\ &\le c(\|\nabla\times w\|^2 + \|a\|_2^2), \end{aligned}$$

and

$$\begin{aligned}
&|(\nabla \times (aw \cdot \nabla v + \rho^0 w \cdot \nabla v), \nabla \times w)| \\
&= |(\nabla (aw)^{\perp} \cdot \nabla v + aw \cdot \nabla w + \nabla(\rho^0 w)^{\perp} \cdot \nabla v + \rho^0 w \cdot \nabla w, \nabla \times w)| \\
&\le c(\|a\|_2 \|\nabla \times w\|_1^{1/2} \|\nabla \times w\|^{\frac{5}{2}} + \|\nabla \times w\|_1^{1/2} \|\nabla \times w\|^{\frac{5}{2}}) \\
&\le c\delta\nu \|\Delta w\|^2 + c(\nu^{-1/3} \|\nabla \times w\|^{\frac{10}{3}} + \|a\|_2^2 + \nu^{-1} \|\nabla \times w\|^{10}) + c\nu.
\end{aligned}$$

similarly, it obtains that

$$\begin{aligned}
&|(\nabla \times (av \cdot \nabla \omega^0 + \rho^0 v \cdot \nabla \omega^0 + au^0 \cdot \nabla \omega^0 + aw \cdot \nabla u^0 \\
&+ a\omega^0 \cdot \nabla v + a\omega^0 \cdot \nabla u^0 + \rho^0 w \cdot \nabla u^0 + \rho^0 \omega^0 \cdot \nabla v, \nabla \times w)| \\
&\le c(\|a\|_2^2 + \|\nabla \times w\|^2 + \|\nabla \times w\|^4).
\end{aligned}$$

Next, we calculate the term B, note that

$$\begin{aligned}
&|(\nabla \times B, \nabla \times w)| = |(\nabla \times (\nabla a \times (\partial_t v + \partial_t u^0 + v \cdot \nabla v + v \cdot \nabla u^0 \\
&+ u^0 \cdot \nabla v + u^0 \cdot \nabla u^0) + \nabla \rho^0 \times (\partial_t v + v \cdot \nabla v + v \cdot \nabla u^0 + u^0 \cdot \nabla v)), \nabla \times w)|.
\end{aligned}$$

it follows that

$$\begin{aligned}
&|(\nabla \times (\nabla a \times (\partial_t v + v \cdot \nabla v)), \nabla \times w)| = |(\nabla a \cdot \nabla(\partial_t v + v \cdot \nabla v) \\
&- (\partial_t v + v \cdot \nabla v) \cdot \nabla(\nabla a) + \nabla a \nabla \cdot (v \cdot \nabla v) - (\partial_t v + v \cdot \nabla v)\Delta a, \nabla \times w)| \\
&\le c(\|a\|_2 \|\nabla \times w\|_1^{1/2} \|\nabla \times w\|^{\frac{5}{2}} + \|a\|_2 \|\partial_t w\| \|\nabla \times w\|_1^{1/2} \|\nabla \times w\|^{1/2}) \\
&\le c\delta(\nu \|\Delta w\|^2 + \epsilon \|\partial_t w\|^2 + \|a\|_2^2 + \nu^{-3/2} \|a\|_2^8 + \nu^{-1} \|\nabla \times w\|^{10} \\
&\quad + \nu^{-1/2} \|\nabla \times w\|^4 + \nu).
\end{aligned}$$

Similarly, we can get

$$\begin{aligned}
&|(\nabla \times (\nabla a \times (\partial_t u^0 + v \cdot \nabla u^0 + u^0 \cdot \nabla v + u^0 \cdot \nabla u^0) \\
&+ \nabla \rho^0 \times (\partial_t v + v \cdot \nabla v + v \cdot \nabla u^0 + u^0 \cdot \nabla v)), \nabla \times w)| \\
&\le c(\|a\|_2 \|\nabla \times w\| + \|\nabla \times w\|_2^2 + c\nu),
\end{aligned}$$

and

$$\begin{aligned}
&|(\nabla \rho^0 \times (\partial_t v + v \cdot \nabla v + v \cdot \nabla u^0 + u^0 \cdot \nabla v)), \nabla \times w)| \\
&\le c(\|\partial_t v\|^2 + \|\nabla \times w\|^2 + \|\nabla \times w\|^4).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
(\nabla \times \Phi, \nabla \times w) \le c(&\delta\nu \|\Delta w\|^2 + \epsilon \|\partial_t w\|^2 + \nu^{-1} \|a\|_2^8 \\
&+ \nu^{-1/3} \|\nabla \times w\|^{\frac{10}{3}} + \nu^{-1} \|\nabla \times w\|^{10} + \nu^{-1} \|\nabla \times w\|^4 \\
&+ \|a\|_2^2 + \|\nabla \times w\|^2 + \|\nabla \times w\|^4) + \epsilon \|\partial_t w\|^2 + c\nu.
\end{aligned} \tag{4.10}$$

Second, we estimate on the term $(\nabla \times \Phi, \nabla \times \omega^0)$:

$$(\nabla \times \Phi, \nabla \times \omega^0) = (\nabla \times (A + B), \nabla \times \omega^0).$$

Recall that

$$\begin{aligned}
&(\nabla \times (\rho^0 v \cdot \nabla \omega^0 + au^0 \cdot \nabla \omega^0 + a\omega^0 \cdot \nabla u^0 + \rho^0 w \cdot \nabla u^0 + \rho^0 \omega^0 \cdot \nabla v), \nabla \times \omega^0) \\
&= \int_{\partial\Omega} n \times \Big(\rho^0 v \cdot \nabla \omega^0 + au^0 \cdot \nabla \omega^0 + a\omega^0 \cdot \nabla u^0 + \rho^0 w \cdot \nabla u^0
\end{aligned}$$

$$+\rho^0\omega^0\cdot\nabla v\Big)\nabla\times\omega^0 ds-\Big(\rho^0 v\cdot\nabla\omega^0+au^0\cdot\nabla\omega^0+a\omega^0\cdot\nabla u^0+\rho^0 w\cdot\nabla u^0$$
$$+\rho^0\omega^0\cdot\nabla v,\nabla\times\omega^0,-\Delta\omega^0\Big),$$

then, it follows from the trace theorem that

$$\begin{aligned}&\nu\int_{\partial\Omega} n\times(aw_t^0+\rho^0 v\cdot\nabla\omega^0+au^0\cdot\nabla\omega^0+a\omega^0\cdot\nabla u^0\\&+\rho^0 w\cdot\nabla u^0+\rho^0\omega^0\cdot\nabla v),\nabla\times\omega^0)\nabla\times\omega^0 ds\\&\le c\nu(\|\nabla v\|_s+\|w\|_s+\|a\|_s)\|\nabla\times\omega^0\|_1\\&\le c\nu(\|\omega\|^{1-s}\|\nabla\times\omega\|^s+\|a\|^{1-s}\|\nabla a\|^s)\\&\le c\nu(\|\nabla\times\omega\|^2+\|\nabla a\|_1^2+\nu^{2-s}).\end{aligned}$$

At the same time, the remaining term of $A$ is estimated as

$$\begin{aligned}&(\nabla\times(av\cdot\nabla\omega^0+aw\cdot\nabla v+aw\cdot\nabla u^0+a\omega^0\cdot\nabla v+\rho^0 w\cdot\nabla v,\nabla\times\omega^0)\\&\le c(\|a\|_2^2+\|\nabla\times\omega\|^2+\|\nabla\times\omega\|^4),\end{aligned}$$

By the definition of $B$, it follows that

$$\begin{aligned}(\nabla\times B,\nabla\times\omega^0)&=\int_\Omega n\times B\nabla\times\omega^0 ds-(B,\Delta\omega^0)\\&\le c(\|\nabla\times\omega\|^2+\|a\|_2^2+\epsilon\|\partial_t w\|^2+\nu^{1-s}).\end{aligned}$$

Therefore, we can deduce that

$$|(\nabla\times\Phi,\nabla\times\omega^0)|\le c(\|\nabla\times\omega\|^2+\|a\|_2^2+\epsilon\|\partial_t w\|^2+\nu^{1-s}).\tag{4.11}$$

Finally, we estimate on $(\Phi,-\Delta\omega^0)$:

$$\begin{aligned}&|(aw_t^0+\rho^0 v\cdot\nabla\omega^0+au^0\cdot\nabla\omega^0+a\omega^0\cdot\nabla u^0+\rho^0\omega^0\cdot\nabla v+\rho^0 w\cdot\nabla u^0,-\Delta\omega^0)|\\&\le c(\|\nabla\times\omega\|^2+\|a\|_2^2+\nu^{1-s}),\end{aligned}$$

$$\begin{aligned}&|(av\cdot\nabla\omega^0+aw\cdot\nabla v+aw\cdot\nabla u^0+a\omega^0\cdot\nabla v+\rho^0 w\cdot\nabla v,-\Delta\omega^0)|\\&\le c(\|a\|_2^2+\|\nabla\times\omega\|^4+\nu),\end{aligned}$$

and

$$|(B,-\Delta\omega^0)|\le c(\|\nabla\times\omega\|^2+\|a\|_2^2+\epsilon\|\partial_t w\|^2+\nu^{1-s}).$$

So

$$|(\Phi,-\Delta\omega^0)|\le c(\|\nabla\times\omega\|^2+\|\nabla\times\omega\|^4+\|a\|_2^2+\epsilon\|\partial_t w\|^2+\nu^{1-s}).\tag{4.12}$$

The remaining terms in(4.8) can be estimated as follows:

$$|\int_{\partial\Omega}(aw+\rho_0 w)\cdot(n\times\partial_t(\nabla\times\omega^0))ds|\le\|\nabla\times\omega\|^2+\|a\|_2^2+c\nu^{1-s},\tag{4.13}$$

$$\nu|(\Delta\omega^0,\Delta w)|\le c\delta\nu\|\Delta w\|^2+c\nu^{1-s},\tag{4.14}$$

$$|\int_{\partial\Omega}(aw+\rho_0 w)\cdot(n\times(\nabla\times\omega^0)))ds|\le\frac14\|\nabla\times\omega\|^2+\|a\|_2^2+c\nu^{1-s}.\tag{4.15}$$

In order to estimate $\|\partial_t w\|^2$, taking the inner product (4.5) with $\partial_t w$, it follows that

$$\begin{aligned}&\int_\Omega a|w_t|^2+\rho^0|w_t|^2+\nu\frac{d}{dt}\|\nabla\times w\|^2\\&=\int_\Omega((\rho v+au^0)\cdot\nabla w+\rho^0u^0\cdot\nabla w+\Phi+\nu\Delta\omega^0)\partial_t w\\&\quad+\int_{\partial\Omega}n\times(\nabla\times w)w_t.\end{aligned}\tag{4.16}$$

From the boundary condition $n\times(\nabla\times\omega)=0$, we have

$$\int_{\partial\Omega}n\times(\nabla\times w)w_t ds=-\frac{d}{dt}\int_{\partial\Omega}n\times(\nabla\times\omega^0)wds+\int_{\partial\Omega}n\times(\nabla\times\omega_t^0)wds.\tag{4.17}$$

It follows from the formula (4.16) and (4.17) that

$$\begin{aligned}&\int_\Omega\rho|w_t|^2dx+\nu\frac{d}{dt}(\|\nabla\times w\|^2+\int_{\partial\Omega}n\times(\nabla\times\omega^0)wds)\\&=\int_\Omega((\rho v+au^0)\cdot\nabla w+\rho^0u^0\cdot\nabla w)\partial_t w+\int_\Omega\Phi\partial_t w+\int_{\partial\Omega}n\times(\nabla\times\omega_t^0)w\\&=I+II+III.\end{aligned}$$

Hence,

$$I\le c(\|a\|_2^2+\|\nabla\times w\|^2+\|\nabla\times w\|^4+\|a\|_2^4)+\frac{m}{4}\|\partial_t w\|^2,$$

and

$$II\le c\|\Phi\|^2+\frac{m}{4}\|\partial_t w\|^2\le c(\|a\|_2^2+\|\nabla\times w\|^2+\|\partial_t v\|^2)^3+\frac{m}{4}\|\partial_t w\|^2$$

It follows from the trace theorem that

$$III\le c\|\omega\|^{1-s}\|\nabla\times\omega\|^s\le\frac14\|\nabla\times\omega\|^2+c\nu^{1-s}.$$

It follows that

$$\begin{aligned}&m\|w_t\|^2+\nu\frac{d}{dt}(\|\nabla\times w\|^2+\int_{\partial\Omega}n\times(\nabla\times\omega^0)wds)\\&\le c(\|a\|_2^2+\|\nabla\times w\|^2+\|\partial_t v\|^2)^3+\frac{m}{2}\|\partial_t w\|^2+c\nu^{1-s}.\end{aligned}\tag{4.18}$$

Through the estimates (4.4), (4.10)-(4.15), (4.18) we obtain

$$\begin{aligned}&\frac{d}{dt}(\|\sqrt{a}\nabla\times w\|^2+\|\nabla\times w\|^2+\nu\|\nabla\times w\|^2+\|a\|_2^2)+\nu\|\Delta w\|^2+m\|w_t\|^2\\&=c((\|a\|_2^2+\|\nabla\times w\|^2+\|\partial_t v\|^2)^3+\nu^{-1}\|a\|_2^4+\nu^{-3/2}\|a\|_2^8\\&+\nu^{-1/3}\|\nabla\times w\|^{\frac{10}{3}}+\nu^{-1}\|\nabla\times w\|^{10}+c\nu^{-1/2}\|\nabla\times w\|^4+\nu+\nu^{1-s}).\end{aligned}\tag{4.19}$$

If $s\in(0,1/2)$ and

$$\|a\|_2^2\le c\nu^{1-s},\quad\|\nabla\times\omega\|^2\le c\nu^{1-s}.$$

So we deduce that

$$\nu^{-3/2}\|a\|_2^4+\nu^{-1/3}\|\nabla\times w\|^{\frac{10}{3}}+\nu^{-1}\|\nabla\times w\|^{10}+c\nu^{-1/2}\|\nabla\times w\|^4=o(\nu^{1-s}),$$

and there exists some constant $c$ such that

$$\nu^{-1}\|a\|_2^4\le c\nu^{1-s}.$$

Using the initial data $a(0)=0, w(0)=0$, by the lemma 2.2, we obtain

$$\|a\|_2^2+\|\sqrt{a}\nabla\times w(t)\|^2+\|\nabla\times w(t)\|^2+\int_\Omega\|w_t\|^2dx+\nu\int_\Omega\|\Delta w(s)\|^2dx\le c\nu^{1-s}.$$

on the interval $[0,T_1]$ for $s\in(0,\frac{1}{2})$ and $\nu\in(0,\nu_1)\subset(0,\nu_0)$, where $T_1=T_1(\nu_1,s)>0$ is independent of $\nu\in(0,\nu_0)$. If $s\ge\frac{1}{2}$, we can chose a $s'\in(0,1/3)$ such that $\nu^{s'}\le c\nu^s$. The proof is complete. □

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