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SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR A LINEAR PARTIAL DIFFERENCE EQUATION

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Abstract. In this article we consider the two-dimensional boundary-value problem

$$\begin{gathered} d_{m,n} = d_{m-1,n} + f_n d_{m-1,n-1}, \quad 1 \le n < m, \\ d_{m,0} = a_m, \quad d_{m,m} = b_m, \quad m \in \mathbb{N}, \end{gathered}$$

where a_m, b_m, $m \in \mathbb{N}$ and f_n, $n \in \mathbb{N}$, are complex sequences. Employing recently introduced method of half-lines, it is shown that the boundary-value problem is solvable, by finding an explicit formula for its solution on the domain, the, so called, combinatorial domain. The problem is solved for each complex sequence f_n, $n \in \mathbb{N}$, that is, even if some of its members are equal to zero. The main result here extends a recent result in the topic.

1. Introduction

Let $\mathbb{N}$ be the set of all natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}$ the set of all integers. If $k, l \in \mathbb{Z}$ and $k \le l$, then by $\overline{k,l}$, we will denote the set of all integers j, such that $k \le j \le l$.

It is well-known that the binomial coefficients C_n^m, where $n, m \in \mathbb{N}_0$ are such that $0 \le n \le m$, satisfy the following relations:

$$C_0^m = C_0^{m-1}, \quad C_m^m = C_{m-1}^{m-1}, \tag{1.1}$$

for $m \ge 2$ and

$$C_n^m = C_n^{m-1} + C_{n-1}^{m-1}, \tag{1.2}$$

for every $m, n \in \mathbb{N}$ such that $1 \le n \le m-1$ and $m \ge 2$, (see for example the books [11, 14, 15, 22, 31], where these and many other relations connected to the binomial coefficients can be found). In other words, the three relations in (1.1) and (1.2) mean that the sequence C_n^m (with two independent variables m and n) is the solution to the following boundary-value problem for partial difference equations

$$\begin{gathered} c_{m,n} = c_{m-1,n} + c_{m-1,n-1}, \quad 1 \le n < m, \\ c_{m,0} = 1, \quad c_{m,m} = 1, \quad m \in \mathbb{N}. \end{gathered} \tag{1.3}$$

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This fact along with the existence of a closed form formula for the sequence C_n^m, that is, the formula

$$C_n^m = \frac{m!}{n!(m-n)!}, \quad 0 \le n \le m,$$

where by definition is regarded that $0! = 1$, has suggested us that the values of the quantities C_0^m and C_m^m, $m \in \mathbb{N}$, that is, the values of $c_{m,0}$ and $c_{m,m}$, $m \in \mathbb{N}$, essentially do not influence on the *solvability* of the following boundary-value problem for partial difference equations

$$\begin{gathered} c_{m,n} = c_{m-1,n} + c_{m-1,n-1}, \quad 1 \le n < m, \\ c_{m,0} = u_m, \quad c_{m,m} = v_m, \quad m \in \mathbb{N}, \end{gathered} \tag{1.4}$$

and, moreover, that these quantities need not be only integers or real numbers, but can be even complex numbers, that is, that given sequences $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$ can be complex.

It is known that the partial difference equation appearing in (1.3) and (1.4), which we call the *binomial partial difference equation*, is "solvable" (see for example [12] for a method for solving the equation). However, the notion of solvability of partial difference equations highly depends on the domain in which the equation is treated (see for example [8]), which is why we have written the word *solvable* under the signs of quotations.

The following formula for "general solution" to the binomial partial difference equation

$$c_{m,n} = \sum_{j=0}^{m} C_j^m c_{0,n-j},$$

can be found in the literature (see for example [12]). However, since the last formula depends on the values of the quantities $c_{0,l}$, $l \in \mathbb{Z}$, only, which lie on the same line, the formula can be regarded as a general solution to the equation only on a half plane.

All above mentioned have motivated us to show the solvability of boundary-value problems for the binomial partial difference equation on its natural domain, that is, on the domain

$$\mathcal{C} = \{(m,n) : 0 \le n \le m,\ m,n \in \mathbb{N}_0\} \setminus \{(0,0)\},$$

which we call, the *combinatorial domain*. The solvability of the boundary-value problem for the equation on the domain was shown in our paper [25], where we devised a method, which we call, the *method of half-lines*. There are several ideas behind the method. One of the main ideas is to slice the combinatorial domain on half-lines and consider a partial difference equation on them, but as one-dimensional (scalar) difference equations. The other idea is to try to solve the scalar equations, but if they are not solvable then we will write them in a form that looks like as solvable ones, then "solve" them and by using posed boundary conditions to get a solution on the half-lines. Finally, based on such obtained formulas on the half-lines, it should be concluded the form of the general solution for the boundary-value problem on the domain.

Studying difference equations of various types is an area of considerable interest, especially in the last few last decades (see for example [1]-[13], [15]-[30] and the references therein).

Some recent investigations of solvable difference equations show that the nonhomogeneous linear first order difference equation (with variable coefficients), that is, the equation

$$x_n = a_n x_{n-1} + b_n, \quad n \in \mathbb{N}, \tag{1.5}$$

where coefficients a_n and b_n, $n \in \mathbb{N}$, and initial value x_0 are real or complex numbers, plays a very important role in the solvability of many classes of differences equations. The method of transformation has been used successfully and developed recently by several authors in numerous papers on difference equations (see for example [19, 27, 29]), as well as on papers on systems of difference equations (see for example [5, 23, 24, 28, 30], see also numerous related references therein; a considerable interest to concrete symmetric-type difference equations started after the publication of papers [16]-[18] by Papaschinopoulos and Schinas). The most important thing connected to equation (1.5) is that it is solvable in closed form. For some methods for solving the equation see, for example, [1, 7, 15]. For periodic solutions to the equation, see [2]. For some classical results on solvability see, for example, [1, 6, 7, 10, 11, 12, 15]. Recall, that the general solution to the difference equation is

$$x_n = x_0 \prod_{j=1}^{n} a_j + \sum_{i=1}^{n} b_i \prod_{j=i+1}^{n} a_j, \tag{1.6}$$

for $n \in \mathbb{N}_0$.

One of the methods for solving the equation corresponds to the method of integrating factors for solving the linear first-order differential equation. It is interesting to note that the form of general solution to equation (1.5) given in (1.6) does not exclude the case when some of a_n-s are equal to zero, which is important here, since we will not have any restrictions in dealing with the main partial difference equation in this paper. Namely, if

$$a_{n_0} = 0 \quad \text{for some } n_0 \in \mathbb{N}, \tag{1.7}$$

then from (1.5) with $n = n_0$, we have $x_{n_0} = b_{n_0}$, and consequently

$$x_{n_0+1} = a_{n_0+1} b_{n_0} + b_{n_0+1},$$

which, on the first site, looks quite different from formula (1.6) (x_{n_0}, that is, b_{n_0} here looks like a new (shifted) initial value). However, by using formula (1.6), it follows that

$$\begin{aligned} x_{n_0+1} &= x_0 \prod_{j=1}^{n_0+1} a_j + \sum_{i=1}^{n_0+1} b_i \prod_{j=i+1}^{n_0+1} a_j, \\ &= \sum_{i=n_0}^{n_0+1} b_i \prod_{j=i+1}^{n_0+1} a_j \\ &= b_{n_0} a_{n_0+1} + b_{n_0+1}, \end{aligned}$$

since

$$\prod_{j=i+1}^{n_0+1} a_j = 0, \quad \text{for } i = \overline{1, n_0 - 1},$$

by assumption (1.7).

So, whether or not some of a_n-s are equal to zero, formula (1.6) holds, unlike the following formula

$$x_n = \prod_{j=1}^{n} a_j \Big(x_0 + \sum_{i=1}^{n} \frac{b_i}{\prod_{j=1}^{i} a_j} \Big), \quad n \in \mathbb{N}_0,$$

which holds only if

$$a_n \neq 0, \quad \text{for every } n \in \mathbb{N}_0.$$

Assume that $(n_l)_{l\in I} \subseteq \mathbb{N}$, $\operatorname{card}(I) \le \aleph_0$, is the set of all indices such that $a_{n_l} = 0$, $l \in I$, and $n_l < n_{l+1}$, for every $l \in I$. Then compact formula (1.6) can be written as follows

$$x_{n_l+k} = b_{n_l} \prod_{j=1}^{k} a_{n_l+j} + \sum_{j=1}^{k} b_{n_l+j} \prod_{i=j+1}^{k} a_{n_l+i},$$

for every $l \in I$, and $k = \overline{0, n_{l+1} - n_l - 1}$.

Our aim here is to show, by using the method of half-lines, that there is a class of partial difference equations, which includes the binomial partial difference equation, which is also solvable on the combinatorial domain, extending the main results in [25]. A problem of this type has been recently treated in [26]. However, the present paper can be regarded as the first one which applies the method of half-lines in a full generality, in the sense that is applied the general form of the solution to the linear difference equation in (1.6), unlike the ones in our previous papers in the topic ([25, 26]), where essentially some sorts of summing by using the telescoping method is employed. Our results can be regarded also as a continuation of investigation of the problem of solvability of difference equations, including partial difference equations ([5], [23]-[30]). For some classical results on the solvability of partial difference equations see, for example, [8, 10, 12], while some results up to 2003, can be found in monograph [8] (see also the related references therein, such as [9]). Some other types of partial difference equations can be found, e.g., in [3, 20, 21]. Some partial difference equations can be find also in [1, 11, 14, 15, 22, 31], but usually in passing, and they are presented and treated more as some exotic recurrent relations.

2. Main results

This section proves the main result in this article and gives some interesting consequences. Namely, we show that the boundary-value problem for partial difference equations

$$d_{m,n} = d_{m-1,n} + f_n d_{m-1,n-1}, \quad 1 \le n < m, \tag{2.1}$$

$$d_{m,0} = a_m, \quad d_{m,m} = b_m, \quad m \in \mathbb{N}, \tag{2.2}$$

where a_m, b_m and f_m, $m \in \mathbb{N}$, are complex sequences, is solvable.

To do this we present the first several steps of the method of half-lines, for the benefit of the reader and since it is not so immediately clear how to guess the formula for the boundary-value problem (2.1)-(2.2), to avoid presenting a relatively complicated formula on the spot.

If $m = n + 1$, then equation (2.1) is

$$d_{n+1,n} = d_{n,n} + f_n d_{n,n-1}, \tag{2.3}$$

for $n \in \mathbb{N}$.

If we use the change of variables $x_n = d_{n+1,n}$, then (2.3) can be regarded as an equation of type (1.5) with

$$a_n = f_n \quad \text{and} \quad b_n = d_{n,n}, \quad n \in \mathbb{N}.$$

If we solve it by using formula (1.6), we obtain

$$d_{n+1,n} = \sum_{j=1}^{n} d_{j,j} \prod_{i=j+1}^{n} f_i + d_{1,0} \prod_{i=1}^{n} f_i, \tag{2.4}$$

for $n \in \mathbb{N}_0$.

If $m = n + 2$, then (2.1) is

$$d_{n+2,n} = d_{n+1,n} + f_n d_{n+1,n-1}, \tag{2.5}$$

for $n \in \mathbb{N}$. Using the change of variables $x_n = d_{n+2,n}$, equation (2.5) can be regarded as an equation of type (1.5) with

$$a_n = f_n \quad \text{and} \quad b_n = d_{n+1,n}, \quad n \in \mathbb{N}.$$

By using formula (1.6), we obtain

$$d_{n+2,n} = \sum_{j=1}^{n} d_{j+1,j} \prod_{i=j+1}^{n} f_i + d_{2,0} \prod_{i=1}^{n} f_i, \tag{2.6}$$

for $n \in \mathbb{N}_0$. Using (2.4) with $n = j$ in (2.6), we obtain

$$d_{n+2,n} = \sum_{j=1}^{n} \Big(\sum_{l=1}^{j} d_{l,l} \prod_{s=l+1}^{j} f_s + d_{1,0} \prod_{i=1}^{j} f_i \Big) \prod_{i=j+1}^{n} f_i + d_{2,0} \prod_{i=1}^{n} f_i, \tag{2.7}$$

for $n \in \mathbb{N}_0$. We have

$$\sum_{j=1}^{n} \prod_{i=1}^{j} f_i \prod_{i=j+1}^{n} f_i = \sum_{j=1}^{n} \prod_{i=1}^{n} f_i = n \prod_{i=1}^{n} f_i, \tag{2.8}$$

and

$$\begin{aligned} \sum_{j=1}^{n} \prod_{i=j+1}^{n} f_i \sum_{l=1}^{j} d_{l,l} \prod_{s=l+1}^{j} f_s &= \sum_{l=1}^{n} d_{l,l} \sum_{j=l}^{n} \prod_{s=l+1}^{j} f_s \prod_{i=j+1}^{n} f_i \\ &= \sum_{l=1}^{n} d_{l,l} \sum_{j=l}^{n} \prod_{s=l+1}^{n} f_s \\ &= \sum_{l=1}^{n} d_{l,l} (n - l + 1) \prod_{s=l+1}^{n} f_s, \end{aligned} \tag{2.9}$$

for $n \in \mathbb{N}_0$. Employing (2.8) and (2.9) in (2.7), we obtain

$$d_{n+2,n} = \sum_{l=1}^{n} d_{l,l} (n - l + 1) \prod_{s=l+1}^{n} f_s + d_{1,0} n \prod_{i=1}^{n} f_i + d_{2,0} \prod_{i=1}^{n} f_i, \tag{2.10}$$

for $n \in \mathbb{N}_0$.

If $m = n + 3$, then (2.1) is

$$d_{n+3,n} = d_{n+2,n} + f_n d_{n+2,n-1}, \tag{2.11}$$

for $n \in \mathbb{N}$. Using the change of variables $x_n = d_{n+3,n}$, equation (2.11) can be regarded as an equation of type (1.5) with

$$a_n = f_n \quad \text{and} \quad b_n = d_{n+2,n}, \quad n \in \mathbb{N}.$$

By using formula (1.6), we obtain

$$d_{n+3,n} = \sum_{j=1}^{n} d_{j+2,j} \prod_{i=j+1}^{n} f_i + d_{3,0} \prod_{i=1}^{n} f_i, \tag{2.12}$$

for $n \in \mathbb{N}_0$. Using (2.10) with $n = j$ in (2.12), we obtain

$$\begin{aligned} d_{n+3,n} = \sum_{j=1}^{n} \Big(\sum_{l=1}^{j} d_{l,l}(j-l+1) \prod_{s=l+1}^{j} f_s + d_{1,0} j \prod_{i=1}^{j} f_i + d_{2,0} \prod_{i=1}^{j} f_i \Big) \\ \times \prod_{i=j+1}^{n} f_i + d_{3,0} \prod_{i=1}^{n} f_i, \end{aligned} \tag{2.13}$$

for $n \in \mathbb{N}_0$. We have

$$\sum_{j=1}^{n} j \prod_{i=1}^{j} f_i \prod_{i=j+1}^{n} f_i = \sum_{j=1}^{n} j \prod_{i=1}^{n} f_i = \frac{n(n+1)}{2} \prod_{i=1}^{n} f_i, \tag{2.14}$$

and

$$\begin{aligned} &\sum_{j=1}^{n} \prod_{i=j+1}^{n} f_i \sum_{l=1}^{j} d_{l,l}(j-l+1) \prod_{s=l+1}^{j} f_s \\ &= \sum_{l=1}^{n} d_{l,l} \sum_{j=l}^{n} (j-l+1) \prod_{i=j+1}^{n} f_i \prod_{s=l+1}^{j} f_s \\ &= \sum_{l=1}^{n} d_{l,l} \prod_{s=l+1}^{n} f_s \sum_{j=l}^{n} (j-l+1) \\ &= \sum_{l=1}^{n} d_{l,l} \prod_{s=l+1}^{n} f_s \sum_{s=1}^{n-l+1} s \\ &= \sum_{l=1}^{n} d_{l,l} \prod_{s=l+1}^{n} f_s \frac{(n-l+1)(n-l+2)}{2}, \end{aligned} \tag{2.15}$$

for $n \in \mathbb{N}_0$. By using (2.8), (2.14) and (2.15) in (2.13), we obtain

$$d_{n+3,n} = \sum_{l=1}^{n} d_{l,l} C_2^{n-l+2} \prod_{s=l+1}^{n} f_s + \big(d_{1,0} C_2^{n+1} + d_{2,0} C_1^{n} + d_{3,0} \big) \prod_{i=1}^{n} f_i, \tag{2.16}$$

for $n \in \mathbb{N}_0$.

Based on the formulas (2.4), (2.10) and (2.16), we may assume that

$$\begin{aligned} d_{n+k,n} = \sum_{j=1}^{n} C_{k-1}^{n-j+k-1} d_{j,j} \prod_{s=j+1}^{n} f_s \\ + \big(C_{k-1}^{n+k-2} d_{1,0} + C_{k-2}^{n+k-3} d_{2,0} + \cdots + C_0^{n-1} d_{k,0} \big) \prod_{i=1}^{n} f_i, \end{aligned} \tag{2.17}$$

for every $k, n \in \mathbb{N}$. Formulas given in (2.4), (2.10) and (2.16) show that the equality in (2.17) holds for $k = \overline{1,3}$.

If $m = n + k + 1$, then we have

$$d_{n+k+1,n} = d_{n+k,n} + f_n d_{n+k,n-1}, \tag{2.18}$$

for $n \in \mathbb{N}$. Using the change of variables $x_n = d_{n+k+1,n}$, equation (2.18) can be regarded as an equation of type (1.5) with

$$a_n = f_n \quad \text{and} \quad b_n = d_{n+k,n}, \quad n \in \mathbb{N}.$$

By using formula (1.6), we obtain

$$d_{n+k+1,n} = \sum_{j=1}^{n} d_{j+k,j} \prod_{s=j+1}^{n} f_s + d_{k+1,0} \prod_{s=1}^{n} f_s, \tag{2.19}$$

for $n \in \mathbb{N}_0$. Employing (2.17) with $n = j$ in (2.19), and by some simple calculations, it follows that

$$\begin{aligned} d_{n+k+1,n} &= \sum_{j=1}^{n} \prod_{s=j+1}^{n} f_s \sum_{i=1}^{j} C_{k-1}^{j-i+k-1} d_{i,i} \prod_{s=i+1}^{j} f_s \\ &\quad + \sum_{j=1}^{n} \big(C_{k-1}^{j+k-2} d_{1,0} + C_{k-2}^{j+k-3} d_{2,0} + \cdots \\ &\quad + C_0^{j-1} d_{k,0}\big) \prod_{s=1}^{n} f_s + d_{k+1,0} \prod_{s=1}^{n} f_s \\ &= \sum_{i=1}^{n} d_{i,i} \sum_{j=i}^{n} C_{k-1}^{j-i+k-1} \prod_{s=i+1}^{n} f_s \\ &\quad + \Big(\sum_{r=2}^{k+1} d_{r-1,0} \sum_{j=1}^{n} C_{k-r+1}^{j+k-r} + d_{k+1,0} \Big) \prod_{s=1}^{n} f_s, \end{aligned} \tag{2.20}$$

for $n \in \mathbb{N}_0$. By using the relation (1.2), we have

$$\begin{aligned} \sum_{j=1}^{n} C_{k-r+1}^{j+k-r} &= \sum_{j=1}^{n} \Big(C_{k-r+2}^{j+k-r+1} - C_{k-r+2}^{j+k-r} \Big) \\ &= C_{k-r+2}^{n+k-r+1} - C_{k-r+2}^{k-r+1} = C_{k-r+2}^{n+k-r+1}, \end{aligned} \tag{2.21}$$

for every $2 \le r \le k+1$, and

$$\sum_{j=i}^{n} C_{k-1}^{j-i+k-1} = \sum_{j=i}^{n} \Big(C_k^{j-i+k} - C_k^{j-i+k-1} \Big) = C_k^{n-i+k} - C_k^{k-1} = C_k^{n-i+k}, \tag{2.22}$$

for every $1 \le i \le n$. Using (2.21) and (2.22) into (2.20), it follows that

$$\begin{aligned} d_{n+k+1,n} &= \sum_{j=1}^{n} C_k^{n-j+k} d_{j,j} \prod_{s=j+1}^{n} f_s \\ &\quad + \big(C_k^{n+k-1} d_{1,0} + C_{k-1}^{n+k-2} d_{2,0} + \cdots + C_0^{n-1} d_{k+1,0} \big) \prod_{s=1}^{n} f_s, \end{aligned} \tag{2.23}$$

from which along with the method of induction it follows that formula (2.17) holds for every $k, n \in \mathbb{N}$.

The above described process leads us into a position to formulate and prove the main result in this note.

Theorem 2.1. *If $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$, are given sequences of complex numbers. Then the solution to partial difference equation* (2.1) *on domain $\mathcal{C}$, with the boundary conditions*

$$d_{k,0} = a_k, \quad d_{k,k} = b_k, \quad k \in \mathbb{N}, \tag{2.24}$$

is given by

$$d_{m,n} = \sum_{j=1}^{n} C_{m-n-1}^{m-j-1} b_j \prod_{s=j+1}^{n} f_s + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} a_j \prod_{s=1}^{n} f_s. \tag{2.25}$$

Proof. If we put $m = n + k$ in (2.17) and use the conditions in (2.24), we obtain formula (2.25). $\square$

Remark 2.2. Note that the hypothesis for the solution to boundary-value problem (2.1)-(2.2) has become clearer after three steps, that is, after finding the "solutions" to the corresponding first-order linear difference equations on the half-lines $m = n + j$, $j = \overline{1,3}$, $n \in \mathbb{N}_0$. Hence, to get a correct hypothesis for the general form of the solution to a boundary-value problem for a partial difference equation we have to solve first several first-order linear difference equations. First three or four equations seems will be always enough for doing this.

In the following two corollaries we have that $f_n = n$ for $n \in \mathbb{N}$, which is one of the cases that is naturally appeared in several subareas of mathematics.

Corollary 2.3. *The boundary-value problem*

$$\begin{gathered} d_{m,n} = d_{m-1,n} + n d_{m-1,n-1}, \quad 1 \le n < m, \\ d_{m,0} = 1, \quad d_{m,m} = m!, \quad m \in \mathbb{N} \end{gathered} \tag{2.26}$$

has solution

$$d_{m,n} = \prod_{j=1}^{n} (m - n + j), \tag{2.27}$$

for every $(m,n) \in \mathcal{C}$.

Proof. Using formula (2.25) we have

$$\begin{aligned} d_{m,n} &= \sum_{j=1}^{n} C_{m-n-1}^{m-j-1} j! \prod_{s=j+1}^{n} s + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} \prod_{s=1}^{n} s \\ &= n!\Big(\sum_{j=1}^{n} C_{m-n-1}^{m-j-1} + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} \Big), \end{aligned} \tag{2.28}$$

for every $(m,n) \in \mathcal{C}$. We have

$$\sum_{j=1}^{n} C_{m-n-1}^{m-j-1} = \sum_{j=1}^{n} \big(C_{m-n}^{m-j} - C_{m-n}^{m-j-1} \big) = C_{m-n}^{m-1} - C_{m-n}^{m-n-1} = C_{m-n}^{m-1}, \tag{2.29}$$

and

$$\sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} = \sum_{j=1}^{m-n} C_{n-1}^{m-1-j} = \sum_{j=1}^{m-n} \big(C_{n}^{m-j} - C_{n}^{m-1-j} \big) = C_{m-n-1}^{m-1}. \tag{2.30}$$

From (2.28)-(2.30) we obtain

$$d_{m,n} = n!\big(C_{m-n}^{m-1} + C_{m-n-1}^{m-1}\big) = n!C_{m-n}^{m} = n!C_{n}^{m} = m(m-1)\cdots(m-n+1),$$

which is nothing but formula (2.27). □

Corollary 2.4. *The boundary-value problem*

$$\begin{gathered} d_{m,n} = d_{m-1,n} + nd_{m-1,n-1}, \quad 1 \le n < m, \\ d_{m,0} = 0, \quad d_{m,m} = m!, \quad m \in \mathbb{N}. \end{gathered} \tag{2.31}$$

has solution

$$d_{m,n} = n\prod_{j=1}^{n-1}(m-n+j). \tag{2.32}$$

Proof. Using formula (2.25), then (2.29), and finally the symmetry of the binomial coefficients, we have

$$\begin{aligned} d_{m,n} &= \sum_{j=1}^{n} C_{m-n-1}^{m-j-1} j! \prod_{s=j+1}^{n} s + \sum_{j=1}^{m-n} C_{m-n-j}^{m-1-j} \cdot 0 \cdot \prod_{s=1}^{n} s \\ &= n!\sum_{j=1}^{n} C_{m-n-1}^{m-j-1} \\ &= n!C_{m-n}^{m-1} = n!C_{n-1}^{m-1}. \end{aligned} \tag{2.33}$$

From (2.33), formula (2.32) easily follows. □

Remark 2.5. By choosing the sequences $(a_m)_{m\in\mathbb{N}}$, $(b_m)_{m\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ at will, it can be obtained various other interesting formulas for solutions to the boundary-value problem (2.1)-(2.2). As we have already mentioned, sequence f_n can be chosen to have (arbitrary many) zeros. We leave it to the imagination of the reader.

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