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EXISTENCE OF CONJUGACIES AND STABLE MANIFOLDS VIA SUSPENSIONS

LUIS BARREIRA, DAVOR DRAGIČEVIĆ, CLAUDIA VALLS

Abstract. We obtain in a simpler manner versions of the Grobman-Hartman theorem and of the stable manifold theorem for a sequence of maps on a Banach space, which corresponds to consider a nonautonomous dynamics with discrete time. The proofs are made short by using a suspension to an infinite-dimensional space that makes the dynamics autonomous (and uniformly hyperbolic when originally it was nonuniformly hyperbolic).

1. Introduction

We consider two fundamental problems in the study of the asymptotic behavior of a map. The first one goes back to Poincaré and asks whether there exists an appropriate change of variables, called a conjugacy, taking the system to a linear one. The second one is related with the existence of stable invariant manifolds and goes back to Hadamard and Perron.

More precisely, we consider the general case of a nonautonomous dynamics with discrete time such that at each time m we apply a map

$$F_m(v) = A_m v + f_m(v), \tag{1.1}$$

where f_m is sufficiently regular and $f_m(0) = 0$. In the first problem we look for a sequence of homeomorphisms h_m such that

$$A_m \circ h_m = h_{m+1} \circ F_m$$

for each $m \in \mathbb{Z}$. In the second problem we look for a sequence of smooth manifolds $\mathcal{V}_m$ that are tangent to the stable spaces and that are invariant under the maps F_m, in the sense that $F_m(\mathcal{V}_m) \subset \mathcal{V}_{m+1}$ for each $m \in \mathbb{Z}$. Both problems have been studied substantially, also in the nonautonomous setting (see, for instance, [1, 4, 5, 7, 8, 9] and the references therein in the case of conjugacies and [1, 3, 6, 8, 10, 11] and the references therein in the case of stable manifolds).

We emphasize that we consider a general nonuniformly hyperbolic dynamics (see Section 2 for the definition), instead of only the uniform case that corresponds to the existence of a uniform exponential dichotomy. In a certain sense, the former is the most general notion of hyperbolic behavior, in which case the expansion and contraction may be spoiled exponentially along a given trajectory. We refer to

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[3, 5] for details on the notion of nonuniform hyperbolicity and for its ubiquity in the context of ergodic theory.

The main novelty of our paper is the method of proof, which allows for short proofs of the Grobman-Hartman and stable manifold theorems in the non-autonomous setting. The idea is to make a suspension to an infinite-dimensional space with the advantage of making the dynamics autonomous and uniform. More precisely, after the suspension the hyperbolicity is transformed into the hyperbolicity of a fixed point and this allows us to use the corresponding classical results (for an autonomous and uniformly hyperbolic dynamics). In this infinite-dimensional space we have an autonomous dynamics generated by a map

$$F(v) = Av + f(v)$$

and one can apply the well-known Grobman-Hartman and stable manifold theorems for an autonomous dynamics. Afterwards, we can descend to the original Banach space to obtain the desired results for the nonuniform nonautonomous dynamics in (1.1).

2. Strong nonuniform exponential dichotomies

Let $(A_m)_{m\in\mathbb{Z}}$ be a (two-sided) sequence of invertible bounded linear operators on a Banach space X. For each $m, n \in \mathbb{Z}$ we define

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n,\\ \mathrm{Id} & \text{if } m = n,\\ A_m^{-1}\cdots A_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

We say that the sequence $(A_m)_{m\in\mathbb{Z}}$ has a *strong nonuniform exponential dichotomy* if there exist projections P_m, for $m \in \mathbb{Z}$, satisfying

$$\mathcal{A}(m,n)P_n = P_m\mathcal{A}(m,n) \tag{2.1}$$

for $m, n \in \mathbb{Z}$ and there exist constants

$$\underline{\lambda} \le \overline{\lambda} < 0 < \underline{\mu} \le \overline{\mu}, \quad \varepsilon \ge 0 \quad \text{and} \quad D > 0 \tag{2.2}$$

such that for $m \ge n$ we have

$$\|\mathcal{A}(m,n)P_n\| \le De^{\overline{\lambda}(m-n)+\varepsilon|n|}, \quad \|\mathcal{A}(m,n)Q_n\| \le De^{\overline{\mu}(m-n)+\varepsilon|n|},$$
$$\|\mathcal{A}(n,m)Q_m\| \le De^{-\underline{\mu}(m-n)+\varepsilon|m|}, \quad \|\mathcal{A}(n,m)P_m\| \le De^{-\underline{\lambda}(m-n)+\varepsilon|m|}.$$

Now assume that $(A_m)_{m\in\mathbb{Z}}$ is a sequence of invertible linear operators with a strong nonuniform exponential dichotomy. We introduce a corresponding sequence of Lyapunov norms. For each $x \in X$ and $n \in \mathbb{Z}$, let

$$\|x\|_n = \max\big\{\|x\|_{1n}, \|x\|_{2n}\big\}, \tag{2.3}$$

where

$$\|x\|_{1n} = \sup_{m\ge n}\big(\|\mathcal{A}(m,n)P_nx\|e^{-\overline{\lambda}(m-n)}\big) + \sup_{m<n}\big(\|\mathcal{A}(n,m)P_mx\|e^{\underline{\lambda}(m-n)}\big),$$
$$\|x\|_{2n} = \sup_{m>n}\big(\|\mathcal{A}(n,m)Q_mx\|e^{\underline{\mu}(m-n)}\big) + \sup_{m\le n}\big(\|\mathcal{A}(m,n)Q_nx\|e^{\overline{\mu}(m-n)}\big).$$

Then there exists $C > 0$ such that

$$\|x\| \le \|x\|_n \le Ce^{\varepsilon|n|}\|x\|, \tag{2.4}$$

$$\|A_nx\|_{n+1} \le C\|x\|_n \quad \text{and} \quad \|A_n^{-1}x\|_n \le C\|x\|_{n+1} \tag{2.5}$$

for $x \in X$ and $n \in \mathbb{Z}$ (see [2]). Moreover, for $x \in X$ and $m \geq n$ we have

$$\|\mathcal{A}(n,m)Q_m x\|_n \leq e^{-\underline{\mu}(m-n)}\|x\|_m, \quad \|\mathcal{A}(m,n)Q_n x\|_m \leq e^{\overline{\mu}(m-n)}\|x\|_n, \tag{2.6}$$

$$\|\mathcal{A}(m,n)P_n x\|_m \leq e^{\overline{\lambda}(m-n)}\|x\|_n, \quad \|\mathcal{A}(n,m)P_m x\|_n \leq e^{-\underline{\lambda}(m-n)}\|x\|_m. \tag{2.7}$$

Now we introduce some Banach spaces. For each $1 \leq p < \infty$, let

$$Y_p = \big\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sum_{n\in\mathbb{Z}} \|x_n\|_n^p < +\infty\big\}.$$

Moreover, let

$$Y_\infty = \big\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sup_{n\in\mathbb{Z}} \|x_n\|_n < +\infty\big\}.$$

These are Banach spaces when equipped, respectively, with the norms

$$\|\mathbf{x}\|_p = \Big(\sum_{n\in\mathbb{Z}} \|x_n\|_n^p\Big)^{1/p} \quad \text{and} \quad \|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}} \|x_n\|_n.$$

We also define a bounded linear operator $\mathbb{A} \colon Y_p \to Y_p$ by

$$(\mathbb{A}\mathbf{x})_n = A_{n-1}x_{n-1}, \quad \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_p, \quad n \in \mathbb{Z}, \tag{2.8}$$

for each $1 \leq p \leq \infty$. One can easily verify that $\mathbb{A}$ is invertible. Indeed, using the second inequality in (2.5), we find that the inverse of $\mathbb{A}$ is the operator $\mathbb{B}$ given by

$$(\mathbb{B}\mathbf{x})_n = A_n^{-1}x_{n+1}, \quad \mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_p, \quad n \in \mathbb{Z}.$$

We say that a bounded linear operator $\mathbb{A}$ on a Banach space Y is *hyperbolic* if its spectrum does not intersect the unit circle.

Theorem 2.1. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible bounded linear operators on X with a strong nonuniform exponential dichotomy. Then the operator $\mathbb{A}$ is hyperbolic on Y_p for each $1 \leq p \leq \infty$.*

Proof. Take $\lambda \in \mathbb{C}$ with $|\lambda| = 1$ and let $\bar{A}_m = \frac{1}{\lambda}A_m$ for $m \in \mathbb{Z}$. Moreover, let

$$\bar{\mathcal{A}}(m,n) = \begin{cases} \bar{A}_{m-1}\cdots\bar{A}_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ \bar{A}_m^{-1}\cdots\bar{A}_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

Since $|\lambda| = 1$, inequalities (2.6) and (2.7) hold when $\mathcal{A}$ is replaced by $\bar{\mathcal{A}}$. Hence, it follows from [2, Theorem 6.1] that the linear operator $T \colon Y_p \to Y_p$ defined by

$$(T\mathbf{x})_n = x_n - \bar{A}_{n-1}x_{n-1} = x_n - \frac{1}{\lambda}A_{n-1}x_{n-1}$$

for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_p$ and $n \in \mathbb{Z}$ is invertible. This readily implies that $\lambda\,\mathrm{Id} - \mathbb{A}$ is invertible and so λ does not belong to the spectrum of $\mathbb{A}$. □

It turns out that the converse of Theorem 2.1 also holds.

Theorem 2.2. *Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of invertible bounded linear operators on X and let $\|\cdot\|_n$, for $n \in \mathbb{Z}$, be a sequence of norms on X satisfying* (2.4) *and* (2.5) *for some constants $C > 0$ and $\varepsilon \geq 0$. If the operator $\mathbb{A}$ defined by* (2.8) *is hyperbolic on Y_p for some $1 \leq p \leq \infty$, then the sequence $(A_m)_{m\in\mathbb{Z}}$ has a strong nonuniform exponential dichotomy.*

Proof. Assume that $\mathbb{A}$ is hyperbolic on Y_p for some $1 \le p \le \infty$. In particular, $\mathrm{Id} - \mathbb{A}$ is invertible on Y_p. By [2, Theorem 6.1] there exists projections P_n, for $n \in \mathbb{Z}$, satisfying (2.1) and constants as in (2.2) such that (2.6) and (2.7) hold for $x \in X$ and $m \ge n$. Together with (2.4) this readily implies that the sequence $(A_m)_{m\in\mathbb{Z}}$ has a strong nonuniform exponential dichotomy. □

3. Nonautonomous Grobman-Hartman theorem

In this section we consider the nonlinear dynamics

$$x_{m+1} = A_m x_m + f_m(x_m), \tag{3.1}$$

where $(A_m)_{m\in\mathbb{Z}}$ is a sequence of invertible bounded linear operators with a strong nonuniform exponential dichotomy and $(f_m)_{m\in\mathbb{Z}}$ is a sequence of continuous functions $f_m \colon X \to X$ such that $f_m(0) = 0$ for $m \in \mathbb{Z}$. We assume that there exists $\delta > 0$ such that

$$\|f_m\|_\infty := \sup\{\|f_m(x)\| : x \in X\} \le \delta e^{-\varepsilon|m|}, \tag{3.2}$$

$$\|f_m(x) - f_m(y)\| \le \delta e^{-\varepsilon|m|}\|x - y\| \tag{3.3}$$

for $m \in \mathbb{Z}$ and $x, y \in X$ (with ε as in (2.2)). Let also $\|\cdot\|_n$, for $n \in \mathbb{Z}$, be the sequence of Lyapunov norms given by (2.3).

The following result is a nonautonomous Grobman-Hartman theorem.

Theorem 3.1. *If $\delta > 0$ is sufficiently small, then there exists a unique sequence of homeomorphisms $\tilde{h}_m \colon X \to X$, for $m \in \mathbb{Z}$, such that*

$$(A_m + f_m) \circ \tilde{h}_m = \tilde{h}_{m+1} \circ A_m \tag{3.4}$$

for $m \in \mathbb{Z}$ and

$$\sup_{m\in\mathbb{Z}} \sup_{v\in X} \|\tilde{h}_m(v) - v\|_m < +\infty. \tag{3.5}$$

Moreover, there exist $K, a > 0$ such that

$$\|\tilde{h}_m(v) - v - \tilde{h}_m(w) + w\|_m \le K\|v - w\|_m^a \tag{3.6}$$

for $m \in \mathbb{Z}$ and $v, w \in X$.

Proof. We first recall the autonomous version of the Grobman-Hartman theorem, including the Hölder continuity of the conjugacy (see [4, 7]).

Lemma 3.2. *Let $A \colon Y \to Y$ be a hyperbolic invertible bounded linear operator on a Banach space Y. Moreover, let $f \colon Y \to Y$ be a continuous map such that*

$$\|f(x)\| \le \delta \quad \text{and} \quad \|f(x) - f(y)\| \le \delta\|x - y\|$$

for all $x, y \in Y$ and some $\delta > 0$. If δ is sufficiently small then:

(1) *there exists a unique bounded continuous map $h \colon Y \to Y$ such that*

$$(A + f) \circ (\mathrm{Id} + h) = (\mathrm{Id} + h) \circ A;$$

(2) $\mathrm{Id} + h$ *is a homeomorphism;*
(3) h *and* $\overline{h} = (\mathrm{Id} + h)^{-1} - \mathrm{Id}$ *are Hölder continuous;*
(4) $\overline{h}$ *is the unique bounded continuous map such that*

$$A \circ (\mathrm{Id} + \overline{h}) = (\mathrm{Id} + \overline{h}) \circ (A + f).$$

Let $\mathbb{A}$ be an operator defined by (2.8) taking $p = \infty$. By Theorem 2.1, $\mathbb{A}$ is hyperbolic on Y_∞. We define a map $F\colon Y_\infty \to Y_\infty$ by

$$(F(\mathbf{x}))_m = f_{m-1}(x_{m-1})$$

for $m \in \mathbb{Z}$ and $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in Y_\infty$. By (2.4) and (3.2) we have

$$\begin{aligned}\|F(\mathbf{x})\|_\infty &= \sup_{m\in\mathbb{Z}} \|(F(\mathbf{x}))_m\|_m = \sup_{m\in\mathbb{Z}} \|f_{m-1}(x_{m-1})\|_m \\ &\le \sup_{m\in\mathbb{Z}} (Ce^{\varepsilon|m|}\delta e^{-\varepsilon|m-1|}) \le C\delta e^{\varepsilon}\end{aligned}$$

for $\mathbf{x} = (x_m)_{m\in\mathbb{Z}} \in Y_\infty$. Similarly, by (2.4) and (3.3) we have

$$\|F(\mathbf{x}) - F(\mathbf{y})\|_\infty \le C\delta e^{\varepsilon}\|\mathbf{x} - \mathbf{y}\|_\infty$$

for $\mathbf{x}$, $\mathbf{y} \in Y_\infty$. Hence, it follows from Lemma 3.2 that for δ sufficiently small, there exists a unique bounded continuous function H such that

$$(\mathbb{A} + F) \circ (\mathrm{Id} + H) = (\mathrm{Id} + H) \circ \mathbb{A}. \tag{3.7}$$

Moreover, H is Hölder continuous. Given $v \in X$ and $m \in \mathbb{Z}$, we define a sequence $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$ by

$$v_n = \begin{cases} v & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases} \tag{3.8}$$

Clearly, $\mathbf{v} \in Y_\infty$. Let

$$h_m(v) = (H(\mathbf{v}))_m$$

for $m \in \mathbb{Z}$ and $v \in X$. It follows from (3.7) that

$$((\mathbb{A} + F)((\mathrm{Id} + H)(\mathbf{v})))_{m+1} = ((\mathrm{Id} + H)(\mathbb{A}\mathbf{v}))_{m+1},$$

which readily implies that

$$(A_m + f_m)(v + h_m(v)) = (\mathrm{Id} + h_{m+1})(A_m v).$$

Hence, (3.4) holds taking $\tilde{h}_m = \mathrm{Id} + h_m$. Moreover,

$$\begin{aligned}\|\tilde{h}_m - \mathrm{Id}\|_{\infty,m} &:= \sup_{v\in X} \|\tilde{h}_m(v) - v\|_m = \sup_{v\in X} \|h_m(v)\|_m \\ &= \sup_{v\in X} \|(H(\mathbf{v}))_m\|_m\end{aligned}$$

and so

$$\sup_{m\in\mathbb{Z}} \|\tilde{h}_m - \mathrm{Id}\|_{\infty,m} \le \sup_{\mathbf{x}\in Y_\infty} \|H(\mathbf{x})\|_\infty < \infty$$

since H is bounded. Hence, (3.5) holds.

We also show that (3.6) holds. Since H is Hölder continuous, there exist $K, a > 0$ such that

$$\|H(\mathbf{x}) - H(\mathbf{y})\|_\infty \le K\|\mathbf{x} - \mathbf{y}\|_\infty^a \quad \text{for } \mathbf{x},\, \mathbf{y} \in Y_\infty. \tag{3.9}$$

Given v, $w \in X$, $m \in \mathbb{Z}$, we define $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$ and $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$ by

$$v_n = \begin{cases} v & \text{if } n = m, \\ 0 & \text{if } n \ne m \end{cases} \quad \text{and} \quad w_n = \begin{cases} w & \text{if } n = m, \\ 0 & \text{if } n \ne m. \end{cases}$$

Applying (3.9) with $\mathbf{x} = \mathbf{v}$ and $\mathbf{y} = \mathbf{w}$, we conclude that (3.6) holds.

Now we prove that $\tilde{h}_m$ is a homeomorphism for $m \in \mathbb{Z}$. Observe that by Lemma 3.2 the map $\mathrm{Id} + H$ is a homeomorphism and $G := (\mathrm{Id} + H)^{-1} - \mathrm{Id}$ is Hölder continuous on Y_∞ (we may assume that with the same constants K, a). Moreover,

$$\mathbb{A} \circ (\mathrm{Id} + G) = (\mathrm{Id} + G) \circ (\mathbb{A} + F). \tag{3.10}$$

Given $m \in \mathbb{Z}$ and $v \in X$, let $\mathbf{v}$ be defined as above. Moreover, let $g_m(v) = (G(\mathbf{v}))_m$ and $\tilde{g}_m = \mathrm{Id} + g_m$. It follows from (3.10) that

$$A_{m+1} \circ \tilde{g}_m = \tilde{g}_{m+1} \circ (A_m + f_m), \quad m \in \mathbb{Z}. \tag{3.11}$$

Moreover, in a similar manner that for $\tilde{h}_m$ we have

$$\sup_{m \in \mathbb{Z}} \|\tilde{g}_m - \mathrm{Id}\|_{\infty, m} < +\infty, \tag{3.12}$$

$$\|\tilde{g}_m(v) - v - \tilde{g}_m(w) + w\|_m \le K\|v - w\|_m^a \tag{3.13}$$

for $m \in \mathbb{Z}$ and $v, w \in X$. Now observe that it follows from (3.4) and (3.11) that

$$\tilde{g}_{m+1} \circ \tilde{h}_{m+1} \circ A_m = A_{m+1} \circ \tilde{g}_m \circ \tilde{h}_m, \quad m \in \mathbb{Z}. \tag{3.14}$$

We define a map $Z \colon Y_\infty \to Y_\infty$ by

$$(Z(\mathbf{x}))_m = \tilde{g}_m(\tilde{h}_m(x_m)), \quad \mathbf{x} = (x_m)_{m \in \mathbb{Z}} \in Y_\infty.$$

It follows from (3.14) that $Z \circ \mathbb{A} = \mathbb{A} \circ Z$. Moreover, using (3.5) and (3.12) we conclude that $\mathrm{Id} - Z$ is bounded. Finally, it follows from (3.6) and (3.13) that Z is continuous. Hence, the uniqueness in Lemma 3.2 implies that $Z = \mathrm{Id}$ and so

$$\tilde{g}_m \circ \tilde{h}_m = \mathrm{Id}, \quad \text{for } m \in \mathbb{Z}.$$

Similarly,

$$\tilde{h}_m \circ \tilde{g}_m = \mathrm{Id}, \quad \text{for } m \in \mathbb{Z},$$

and $\tilde{h}_m$ is a homeomorphism for each $m \in \mathbb{Z}$.

Finally, we establish the uniqueness of the sequence of maps $(\tilde{h}_m)_{m \in \mathbb{Z}}$. Let $(\tilde{h}_m^i)_{m \in \mathbb{Z}}$, for $i = 1, 2$, be sequences of continuous maps on X satisfying (3.4), (3.5) and (3.6). We define maps $\tilde{H}^i \colon Y_\infty \to Y_\infty$, for $i = 1, 2$, by

$$(\tilde{H}^i(\mathbf{x}))_m = \tilde{h}^i(x_m)$$

for $m \in \mathbb{Z}$ and $\mathbf{x} = (x_m)_{m \in \mathbb{Z}} \in Y_\infty$. Identity (3.4) implies that

$$(\mathbb{A} + F) \circ \tilde{H}^i = \tilde{H}^i \circ \mathbb{A} \quad \text{for } i = 1, 2.$$

Moreover, it follows from (3.5) that $\mathrm{Id} - \tilde{H}^i$ is bounded and from (3.6) that it is continuous (and so also is $\tilde{H}_i$). Hence, by the uniqueness in Lemma 3.2 we conclude that $\tilde{H}^1 = \tilde{H}^2$ and so $\tilde{h}_m^1 = \tilde{h}_m^2$ for each $m \in \mathbb{Z}$. This completes the proof of the theorem. □

4. Nonautonomous stable manifold theorem

In this section we consider again the nonlinear dynamics in (3.1), where the sequence $(A_m)_{m \in \mathbb{Z}}$ has a strong nonuniform exponential dichotomy and $(f_m)_{m \in \mathbb{Z}}$ is now a sequence of C^1 functions $f_m \colon X \to X$ such that $f_m(0) = 0$, $d_0 f_m = 0$ and

$$\|d_x f_{m-1} - d_y f_{m-1}\| \le B e^{-\varepsilon |m|} \|x - y\| \tag{4.1}$$

for $m \in \mathbb{Z}$ and $x, y \in X$ (for some $B > 0$ and with ε as in (2.2)). We shall write $E_m^s = \operatorname{Im} P_m$, $E_m^u = \operatorname{Im} Q_m$, $F_m = A_m + f_m$ and

$$\mathcal{F}(m, n) = F_{m-1} \circ \cdots \circ F_n \quad \text{for } m \ge n.$$

Moreover, for each $\rho > 0$ let

$$E^s_m(\rho) = \{v \in E^s_m : \|v\|_m < \rho\}.$$

The following theorem establishes the existence of local stable manifolds for the dynamics in (3.1).

Theorem 4.1. *If $\delta > 0$ is sufficiently small, then there exist $\rho > 0$ and a sequence $\varphi_m \colon E^s_m \to E^u_m$, for $m \in \mathbb{Z}$, of C^1 maps with $\varphi_m(0) = 0$ and $d_0\varphi_m = 0$ such that the graphs*

$$\mathcal{V}_m = \big\{(x, \varphi_m(x)) : x \in E^s_m(\rho)\big\}$$

satisfy $F_m(\mathcal{V}_m) \subset \mathcal{V}_{m+1}$ for $m \in \mathbb{Z}$. Moreover, there exist $\lambda \in (0, 1)$ and $C > 0$ such that

$$\|\mathcal{F}(m, n)(x, \varphi_n(x)) - \mathcal{F}(m, n)(y, \varphi_n(y))\| \le C\lambda^{m-n}e^{\varepsilon|n|}\|x - y\| \tag{4.2}$$

for $n \in \mathbb{Z}$, $x, y \in E^s_n(\rho)$ and $m \ge n$.

Proof. We first recall an autonomous version of the stable manifold theorem. For a proof see, for instance, [1, 8].

Given a hyperbolic operator $A \colon Y \to Y$, we denote the stable and unstable spaces, respectively, by E^s and E^u. Note that $Y = E^s \oplus E^u$. We shall always consider the norm

$$\|x\| = \max\big\{\|x^s\|, \|x^u\|\big\},$$

where $x = x^s + x^u$ with $x^s \in E^s$ and $x^u \in E^u$.

Lemma 4.2. *Let $A \colon Y \to Y$ be a hyperbolic invertible bounded linear operator on a Banach space Y and let $f \colon Y \to Y$ be a C^1 map with $f(0) = 0$ and $d_0f = 0$. Then there exist $\rho > 0$ and a C^1 map $\Phi \colon E^s \to E^u$ with $\Phi(0) = 0$ and $d_0\Phi = 0$ such that the graph*

$$\mathcal{W} = \{x + \Phi(x) : x \in E^s \cap B(0, \rho)\}$$

satisfies $F(\mathcal{W}) \subset \mathcal{W}$, where $F(v) = Av + f(v)$. Moreover,

$$\mathcal{W} = \big\{x \in Y : \|F^n(x)\| < \rho \text{ for } n \ge 0\big\}$$

and there exist $\lambda \in (0, 1)$ and $K > 0$ such that

$$\|F^n(x + \Phi(x)) - F^n(y + \Phi(y))\| \le K\lambda^n\|x - y\|$$

for $x, y \in E^s \cap B(0, \rho)$ and $n \ge 0$.

Now we define a map $F \colon Y_\infty \to Y_\infty$ by

$$(F(\mathbf{x}))_n = A_{n-1}x_{n-1} + f_{n-1}(x_{n-1})$$

where $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty$ and $n \in \mathbb{Z}$. Observe that by (4.1) we have

$$\|f_{n-1}(x)\| = \|f_{n-1}(x) - f_{n-1}(0)\| \le Be^{-\varepsilon|n|}\|x\|^2. \tag{4.3}$$

It follows from (2.4), (2.5) and (4.3) that the map F is well defined.

Lemma 4.3. *The map F is differentiable and*

$$d_{\mathbf{x}}F\xi = (A_{n-1}\xi_{n-1} + C_{n-1}\xi_{n-1})_{n\in\mathbb{Z}}$$

for each $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ and $\xi = (\xi_n)_{n\in\mathbb{Z}} \in Y_\infty$, where $C_{n-1} = d_{x_{n-1}}f_{n-1}$.

Proof. Given $\mathbf{x} \in Y_\infty$, we define an operator $L\colon Y_\infty \to Y_\infty$ by

$$L\xi = (A_{n-1}\xi_{n-1} + C_{n-1}\xi_{n-1})_{n\in\mathbb{Z}}.$$

It follows from (2.4), (2.5) and (4.1) that L is well defined. Moreover,

$$\begin{aligned}(F(\mathbf{x}+\mathbf{y}) - F(\mathbf{x}) - L\mathbf{y})_n &= f_{n-1}(x_{n-1}+y_{n-1}) - f_{n-1}(x_{n-1}) - C_{n-1}y_{n-1}\\ &= \int_0^1 d_{x_{n-1}+ty_{n-1}} f_{n-1}y_{n-1}\,dt - d_{x_{n-1}} f_{n-1}y_{n-1}\\ &= \int_0^1 \big(d_{x_{n-1}+ty_{n-1}} f_{n-1}y_{n-1} - d_{x_{n-1}} f_{n-1}y_{n-1}\big)\,dt.\end{aligned}$$

Using again (2.4) and (4.1) we obtain

$$\begin{aligned}&\|(F(\mathbf{x}+\mathbf{y}) - F(\mathbf{x}) - L\mathbf{y})_n\|_n\\ &\le \int_0^1 \|d_{x_{n-1}+ty_{n-1}} f_{n-1}y_{n-1} - d_{x_{n-1}} f_{n-1}y_{n-1}\|_n\,dt\\ &\le Ce^{\varepsilon|n|}\int_0^1 \|d_{x_{n-1}+ty_{n-1}} f_{n-1}y_{n-1} - d_{x_{n-1}} f_{n-1}y_{n-1}\|\,dt\\ &\le BC\|y_{n-1}\|_{n-1}^2.\end{aligned}$$

Hence,

$$\|F(\mathbf{x}+\mathbf{y}) - F(\mathbf{x}) - L\mathbf{y}\|_\infty \le BC\|\mathbf{y}\|_\infty^2,$$

which implies

$$\lim_{\mathbf{y}\to\mathbf{0}} \frac{\|F(\mathbf{x}+\mathbf{y}) - F(\mathbf{x}) - L\mathbf{y}\|_\infty}{\|\mathbf{y}\|_\infty} = 0.$$

This completes the proof. □

It follows from Lemma 4.3 and (4.1) that F is of class C^1. Note that $\mathbf{0} = (0)_{n\in\mathbb{Z}}$ is a hyperbolic fixed point of F. Indeed, by Lemma 4.3 and the assumption $d_0 f_n = 0$ we have $D_{\mathbf{0}}F = \mathbb{A}$, which by Theorem 2.1 is hyperbolic. Now let

$$\begin{aligned}Y_\infty^s &= \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty : x_n \in E_n^s \text{ for } n\in\mathbb{Z}\},\\ Y_\infty^u &= \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in Y_\infty : x_n \in E_n^u \text{ for } n\in\mathbb{Z}\}.\end{aligned}$$

Since $Y_\infty = Y_\infty^s \oplus Y_\infty^u$, we can write each $\mathbf{x} \in Y_\infty$ uniquely in the form

$$\mathbf{x} = \mathbf{x}^s + \mathbf{x}^u, \quad \mathbf{x}^s \in Y_\infty^s,\ \mathbf{x}^u \in Y_\infty^u.$$

Note that

$$\|\mathbf{x}\|_\infty = \max\{\|\mathbf{x}^s\|_\infty, \|\mathbf{x}^u\|_\infty\}.$$

By Lemma 4.2, there exists $\rho > 0$ such that the set

$$\mathcal{W} = \big\{\mathbf{x} \in Y_\infty : \|F^n(\mathbf{x})\|_\infty < \rho \text{ for } n \ge 0\big\}$$

is a C^1 manifold tangent to Y_∞^s and there exists a C^1 function $\Phi\colon Y_\infty^s \to Y_\infty^u$ such that $\Phi(\mathbf{0}) = \mathbf{0}$, $d_{\mathbf{0}}\Phi = 0$ and

$$\mathcal{W} = \{\mathbf{x} + \Phi(\mathbf{x}) : \mathbf{x} \in B^s(\mathbf{0},\rho)\},$$

where $B^s(\mathbf{0},\rho)$ denotes the ball in Y_∞^s of radius ρ centered at $\mathbf{0}$.

The next lemma is crucial for constructing the sequence of maps $(\varphi_m)_{m\in\mathbb{Z}}$ in the statement of the theorem.

Lemma 4.4. *Given $\mathbf{x}^1 = (x_m^1)_{m\in\mathbb{Z}}$, $\mathbf{x}^2 = (x_m^2)_{m\in\mathbb{Z}} \in B^s(\mathbf{0},\rho)$, if $x_k^1 = x_k^2$ for some $k \in \mathbb{Z}$, then $(\Phi(\mathbf{x}^1))_k = (\Phi(\mathbf{x}^2))_k$.*

Proof. We proceed by contradiction. Assume that $(\Phi(\mathbf{x}^1))_k \neq (\Phi(\mathbf{x}^2))_k$ and define $\mathbf{y} = (y_m)_{m\in\mathbb{Z}} \in Y^u_\infty$ by

$$y_m = \begin{cases} (\Phi(\mathbf{x}^2))_m & \text{if } m \neq k, \\ (\Phi(\mathbf{x}^1))_k & \text{if } m = k. \end{cases}$$

Then

$$(F^n(\mathbf{x}^2 + \mathbf{y}))_m = \begin{cases} \mathcal{F}(m, m-n)(x^2_{m-n} + (\Phi(\mathbf{x}^2))_{m-n}) & \text{if } m \neq n+k, \\ \mathcal{F}(m, m-n)(x^1_{m-n} + (\Phi(\mathbf{x}^1))_{m-n}) & \text{if } m = n+k, \end{cases}$$

for $n \geq 0$ and $m \in \mathbb{Z}$. Therefore,

$$\begin{aligned} &\sup_{m\in\mathbb{Z}} \|(F^n(\mathbf{x}^2 + \mathbf{y}))_m\|_m \\ &= \max\Big\{ \sup_{m\neq n+k} \|(F^n(\mathbf{x}^2 + \Phi(\mathbf{x}^2)))_m\|_m, \|(F^n(\mathbf{x}^1 + \Phi(\mathbf{x}^1)))_{n+k}\|_{n+k} \Big\} \end{aligned}$$

for $n \geq 0$. Hence,

$$\|F^n(\mathbf{x}^2 + \mathbf{y})\|_\infty < \rho \quad \text{for } n \geq 0$$

and so $\mathbf{y} = \Phi(\mathbf{x}^2)$. This contradiction shows that $(\Phi(\mathbf{x}^1))_k = (\Phi(\mathbf{x}^2))_k$. □

Now we construct the sequence of maps $(\varphi_m)_{m\in\mathbb{Z}}$. Given $v \in E^s_m(\rho)$, let $\mathbf{v} = (v_n)_{n\in\mathbb{Z}}$ be as in (3.8). Clearly, $\mathbf{v} \in B^s(\mathbf{0}, \rho)$ and we define

$$\varphi_m(v) = (\Phi(\mathbf{v}))_m \in E^u_m. \tag{4.4}$$

In view of Lemma 4.4 the maps φ_m are well defined. Moreover, since $\Phi(\mathbf{0}) = \mathbf{0}$ and $d_{\mathbf{0}}\Phi = 0$, we have $\varphi_m(0) = 0$ and $d_0\varphi_m = 0$. Finally, since Φ is of class C^1 one can easily verify that each map φ_m is also of class C^1.

Lemma 4.5. *For every $m \in \mathbb{Z}$ we have $F_m(\mathcal{V}_m) \subset \mathcal{V}_{m+1}$.*

Proof. Take $v + \varphi_m(v) \in \mathcal{V}_m$ and let $\mathbf{v}$ be as above. Then $\varphi_m(v) = (\Phi(\mathbf{v}))_m$ and $\mathbf{v} + \Phi(\mathbf{v}) \in \mathcal{W}$. Since $F(\mathcal{W}) \subset \mathcal{W}$, we conclude that

$$F(\mathbf{v} + \Phi(\mathbf{v})) = \mathbf{y} + \Phi(\mathbf{y})$$

for some $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in B^s(\mathbf{0}, \rho)$. Hence,

$$\begin{aligned} F_m(v + \varphi_m(v)) &= (F(\mathbf{v} + \Phi(\mathbf{v})))_{m+1} \\ &= (\mathbf{y} + \Phi(\mathbf{y}))_{m+1} \\ &= y_{m+1} + (\Phi(\mathbf{y}))_{m+1}. \end{aligned}$$

Since $\mathbf{y} \in B^s(\mathbf{0}, \rho)$, we have

$$\|y_{m+1}\|_{m+1} \leq \|\mathbf{y}\|_\infty < \rho$$

and so $y_{m+1} \in E^s_{m+1}(\rho)$. On the other hand, by (4.4) we have

$$(\Phi(\mathbf{y}))_{m+1} = \varphi_{m+1}(y_{m+1})$$

and thus,

$$F_m(v + \varphi_m(v)) = y_{m+1} + \varphi_{m+1}(y_{m+1}) \in \mathcal{V}_{m+1}.$$

This completes the proof. □

We proceed with the proof of the theorem. It follows from Lemma 4.2 that there exist $\lambda \in (0,1)$ and $K > 0$ such that

$$\|F^n(\mathbf{x} + \Phi(\mathbf{x})) - F^n(\mathbf{y} + \Phi(\mathbf{y}))\|_\infty \le K\lambda^n \|\mathbf{x} - \mathbf{y}\|_\infty, \tag{4.5}$$

for $\mathbf{x}, \mathbf{y} \in B^s(\mathbf{0}, \rho)$ and $n \ge 0$. Now take $n \in \mathbb{Z}$, $x, y \in E_n^s(\rho)$ and define $\mathbf{x} = (x_m)_{m\in\mathbb{Z}}$ and $\mathbf{y} = (y_m)_{m\in\mathbb{Z}}$ by

$$x_m = \begin{cases} x & \text{if } m = n, \\ 0 & \text{if } m \ne n \end{cases} \quad \text{and} \quad y_m = \begin{cases} y & \text{if } m = n, \\ 0 & \text{if } m \ne n. \end{cases}$$

Clearly, $\mathbf{x}, \mathbf{y} \in B^s(\mathbf{0}, \rho)$ and it follows from (4.5) that

$$\|\mathcal{F}(m,n)(x + \varphi_n(x)) - \mathcal{F}(m,n)(y + \varphi_n(y))\|_m \le K\lambda^{m-n}\|x - y\|_n$$

for $m \ge n$. Together with (2.4) this yields inequality (4.2). This concludes the proof. □

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Luis Barreira
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: barreira@math.tecnico.ulisboa.pt

Davor Dragičević
School of Mathematics and Statistics, University of New South Wales, Sydney NSW 2052, Australia
E-mail address: d.dragicevic@unsw.edu.au

Claudia Valls
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: cvalls@math.tecnico.ulisboa.pt