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TURING AND HOPF BIFURCATION OF GIERER-MEINHARDT ACTIVATOR-SUBSTRATE MODEL

RANCHAO WU, YAN SHAO, YUE ZHOU, LIPING CHEN

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Abstract. Gierer-Meinhardt model acts as one of prototypical reaction diffusion systems describing pattern formation phenomena in natural events. Bifurcation analysis, including theoretical and numerical analysis, is carried out on the Gierer-Meinhardt activator-substrate model. The effects of diffusion on the stability of equilibrium points and the bifurcated limit cycle from Hopf bifurcation are investigated. It shows that under some conditions, diffusion-driven instability, i.e, the Turing instability, about the equilibrium point will occur, which is stable without diffusion. While once the diffusive effects are present, the bifurcated limit cycle, which is the spatially homogeneous periodic solution and stable without the presence of diffusion, will become unstable. These diffusion-driven instabilities will lead to the occurrence of spatially nonhomogeneous solutions. Consequently, some pattern formations, like stripe and spike solutions, will appear. To understand the Turing and Hopf bifurcation in the system, we use dynamical techniques, such as stability theory, normal form and center manifold theory. To illustrate theoretical analysis, we carry out numerical simulations.

1. Introduction

Natural patterns are various in shape and form. The development processes of such patterns are complex, and also interesting to researchers. To understand the underlying mechanism for patterns of plants and animals, Turing [1] first proposed the coupled reaction-diffusion equations. It was showed that the stable process could evolve into an instability with diffusive effects. He showed that diffusion could destabilize spatially homogeneous states and cause nonhomogeneous spatial patterns, which accounted for biological patterns in plants and animals. Such instability is frequently called the Turing instability, also known as diffusion-driven instability. Gierer and Meinhardt [2] presented a prototypical model of coupled reaction diffusion equations, which described the interaction between two substances, the activator and the inhibitor, and was used to describe the Turing instability.

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The Gierer-Meinhardt model is expressed in the following form

$$\begin{aligned} \frac{\partial a}{\partial t} &= \rho_0\rho + c\rho\frac{a^r}{h^s} - \mu a + D_a\frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial h}{\partial t} &= c'\rho'\frac{a^T}{h^u} - vh + D_h\frac{\partial^2 h}{\partial x^2}. \end{aligned} \tag{1.1}$$

where $a(x,t)$ and $h(x,t)$ represent the population densities of the activator and the inhibitor at time $t > 0$ and spatial location x, respectively. D_a and D_h are the diffusion constants of the activator and the inhibitor, respectively; $\rho\rho_0$ is the source concentration for the activator; ρ' is the one for the inhibitor; the activator and the inhibitor are removed by the first order kinetics μa and vh, respectively, either by enzyme degradation, or leakage, or re-uptake by the source, or by any combination of such mechanisms; now the sources of activator and the inhibitor are assumed to be uniformly distributed, that is, ρ and ρ' are constants.

Several results about such model have been achieved. If $s \neq u$, it is said to have different sources. If $s = u$, then it is said to be the model with common sources. When $r = 2, s = 1, T = 2$ and $u = 0$, Ruan [3] investigated the instability of equilibrium points and the periodic solutions under diffusive effects, which were stable without diffusion. The perturbation method was employed to carry out the analysis there. In [4], they showed the analysis of the Turing instability for such model. By using the bifurcation technique, Liu et al. [5] obtained the results about the Hopf bifurcation, the steady state bifurcation and their interaction in this model. However, the model was subject to fixed Dirichlet boundary conditions. Recently, Song [6] further investigated the Turing-Hopf bifurcation and spatial resonance phenomena in this model. When $r = 2, s = 2, u = 0, T = 1$ and $u = 0$, Wang et al. [7] studied the Turing instability and the Hopf bifurcation.

Also, the Turing instability for the semi-discrete Gierer-Meinhardt model was considered in [8]. Bifurcation for the Gierer-Meinhardt model with saturation was analyzed in [9]. The influence of gene expression time delay on the patterns of Gierer-Meinhardt system was explored in [10]. The Turing bifurcation in models like Brusselator and Gierer-Meinhardt systems were analyzed in [11].

The existence, asymptotic behaviors of solutions and their stability in terms of diffusion effects have been extensively investigated, for example, [12, 13, 14, 15, 16] and references therein.

In view of the processes in morphogenesis, which was described in detail in [2], if it is assumed that the sources of distribution are activated by $a(x,t)$ and further by some substance of concentration $s(x,t)$, one could give the activator-substrate (depletion) model. Here, the substance of concentration $s(x,t)$ could be consumed by activation or some indirect effect of activation. In one dimension, the depletion model could be written as follows

$$\begin{aligned} \frac{\partial a}{\partial t} &= a^2 h - \mu a + D_a\frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial h}{\partial t} &= c_0 - c'\rho a^2 h - vh + D_h\frac{\partial^2 h}{\partial x^2}. \end{aligned} \tag{1.2}$$

This model was used to describe pigmentation patterns in sea shells [17, 18] and the ontogeny of ribbing on ammonoid shells [19]. By means of qualitative analysis, such as stability theory, normal form and bifurcation technique, we will investigate the

Turing instability of the system with Neumann boundary conditions. Furthermore, some patterns will be identified numerically.

By using the scaling transformation, let

$$t = \frac{\tau}{v}, \quad \bar{\mu} = \frac{\mu}{v}, \quad D_H = \frac{D_h}{v}, \quad D_A = \frac{D_a}{v},$$

then one has

$$\begin{aligned} \frac{\partial A}{\partial \tau} &= A^2 H - \bar{\mu} A + D_A \frac{\partial^2 A}{\partial x^2}, \\ \frac{\partial H}{\partial \tau} &= c_0 - A^2 H - H + D_H \frac{\partial^2 H}{\partial x^2}. \end{aligned}$$

For simplicity, we change parameters $A, H, \tau, \bar{\mu}, c_0, D_A, D_H$ into $a, h, t, \mu, c, D_a, D_h$, respectively. System (1.2) can be written as follows

$$\begin{aligned} \frac{\partial a}{\partial t} &= a^2 h - \mu a + D_a \frac{\partial^2 a}{\partial x^2}, \\ \frac{\partial h}{\partial t} &= c - a^2 h - h + D_h \frac{\partial^2 h}{\partial x^2}. \end{aligned} \tag{1.3}$$

where $\mu, c, D_a, D_h > 0, a, h \geq 0$. In the sequel, system (1.3) is assumed to be subjected to the Neumann boundary conditions

$$\frac{\partial a}{\partial x}(0,t) = \frac{\partial a}{\partial x}(\pi,t) = 0, \quad \frac{\partial h}{\partial x}(0,t) = \frac{\partial h}{\partial x}(\pi,t) = 0.$$

2. Analysis of system without diffusion

When the diffusive terms in system (1.3) are absent, it will reduce to the local system

$$\begin{aligned} \frac{da}{dt} &= a^2 h - \mu a, \\ \frac{dh}{dt} &= c - a^2 h - h. \end{aligned} \tag{2.1}$$

Let

$$f(a,h) = a^2 h - \mu a, \ g(a,h) = c - a^2 h - h.$$

Let $l = \frac{c}{\mu}$. Note that if $0 < l < 2$, system (2.1) has a unique equilibrium point $S(0, c)$; if $l = 2$, the system has equilibrium points S and $P^*(1, \mu)$; if $l > 2$, it has equilibrium points S,

$$P_0\Big(\frac{l + \sqrt{l^2 - 4}}{2}, \frac{l - \sqrt{l^2 - 4}}{2}\mu\Big), \quad P_1\Big(\frac{l - \sqrt{l^2 - 4}}{2}, \frac{l + \sqrt{l^2 - 4}}{2}\mu\Big).$$

In fact, when $l = 2$, P_0 and P_1 will coincide to be the point $P^*(1, \mu)$.

The Jacobian matrix of (2.1) evaluated at the equilibrium point S is

$$J(S) = \begin{pmatrix} -\mu & 0 \\ 0 & -1 \end{pmatrix},$$

so S is the asymptotically stable node.

The Jacobian matrix of (2.1) evaluated at the equilibrium point P_1 is

$$J(P_1) = \begin{pmatrix} \mu & \frac{l^2 - l\sqrt{l^2-4} - 2}{2} \\ -2\mu & -\frac{l^2 - l\sqrt{l^2-4}}{2} \end{pmatrix}.$$

Then $\mathrm{tr}(J) = \mu - \frac{l^2 - l\sqrt{l^2-4}}{2}$, $\det(J) = \frac{\mu}{2}\sqrt{l^2-4}(\sqrt{l^2-4} - l) < 0$, so P_1 is the unstable saddle.

From the above analysis about the existence and stability of equilibrium points, one has the following result.

Theorem 2.1. *The equilibrium point S is the asymptotically stable node. When $0 < l < 2$, system* (2.1) *only has the equilibrium point S. When $l > 2$, two other equilibrium points P_0 and P_1 will appear. The point P_1 is the unstable saddle.*

Remark 2.2. From the later analysis, note that when diffusion effect is taken into consideration, S will still be stable and P_1 will still be unstable. Thus, no Turing instability will occur at these points. Next the dynamical behaviors of P_0 without diffusion will be given.

The Jacobian matrix of (2.1) evaluated at the equilibrium point P_0 is

$$J(P_0) = \begin{pmatrix} \mu & \frac{l^2+l\sqrt{l^2-4}-2}{2} \\ -2\mu & -\frac{l^2+l\sqrt{l^2-4}}{2} \end{pmatrix}.$$

Note that the corresponding characteristic equation for $J(P_0)$ is

$$\lambda^2 - \mathrm{tr}(J(P_0))\lambda + Det J(P_0) = 0, \tag{2.2}$$

where $\mathrm{tr}(J(P_0)) = \mu - \frac{l(l+\sqrt{l^2-4})}{2}$ and $\det J(P_0) = \frac{\mu[l^2+l\sqrt{l^2-4}-4]}{2} > 0$.

Theorem 2.3. *The equilibrium point P_0 of system* (2.1) *is asymptotically stable if*

$$\mu < \frac{l(l+\sqrt{l^2-4})}{2} \tag{2.3}$$

and is unstable if

$$\mu > \frac{l(l+\sqrt{l^2-4})}{2}. \tag{2.4}$$

Proof. If (2.3) holds, then the eigenvalues are both negative or have negative real parts, so the equilibrium P_0 of (2.1) is stable; if (2.4) holds, then the eigenvalues are both positive or have positive real parts, so the equilibrium P_0 of (2.1) is unstable. □

From Theorem 2.3, we know that the Hopf bifurcation may occur at the point P_0 in system (2.1). Next the Hopf bifurcation at the point P_0 and its direction will be investigated. For simplicity, let $\mu_0 = \frac{l(l+\sqrt{l^2-4})}{2}$ and (a_0, h_0) denote the point P_0. In terms of the characteristic equation (2.2), the eigenvalues are

$$\lambda_{1,2} = \frac{\mathrm{tr}(J) \pm \sqrt{\mathrm{tr}(J)^2 - 4\det(J)}}{2}.$$

If $\mathrm{tr}(J)^2 < 4\det(J)$, then it has a pair of complex roots, with real parts $\frac{\mathrm{tr}(J)}{2}$. Note that $\frac{dRe\lambda_{1,2}}{d\mu}|_{\mu_0} = 1 > 0$, thus the Hopf bifurcation may occur in system (2.1) when $\mu = \mu_0$. The Hopf bifurcation curve is defined by $\mathrm{tr}(J) = 0$, i.e.,

$$\mu = \frac{l(l+\sqrt{l^2-4})}{2}.$$

As for the direction of Hopf bifurcation, it could be derived as the way in [20] as follows. Let $a \to a + a_0, h \to h + h_0$, so

$$\begin{pmatrix} \frac{da}{dt} \\ \frac{dh}{dt} \end{pmatrix} = \begin{pmatrix} f(a_0+a, h_0+h) \\ g(a_0+a, h_0+h) \end{pmatrix} = J(P_0)\begin{pmatrix} a \\ h \end{pmatrix} + \begin{pmatrix} f_2(a,h) \\ g_2(a,h) \end{pmatrix}, \tag{2.5}$$

where

$$f_2(a,h,\mu) = a^2 h_0 + 2a_0 a h + a^2 h,\ \ g_2(a,h,\mu) = -a^2 h_0 - 2a_0 a h - a^2 h.$$

When $\mu = \mu_0$, we verify that $\lambda_{1,2}(\mu_0) = \pm i\omega_0$, where $\omega_0^2 = \frac{\mu[l^2 + l\sqrt{l^2-4}-4]}{2} > 0$. We choose one of eigenvectors corresponding to the eigenvalue $i\omega_0$ of matrix $J(P_0)$ at $\mu = \mu_0$ to be $\xi = (i\omega_0 + \mu, -2\mu)^T$. Let

$$T = \begin{pmatrix} \omega_0 & \mu \\ 0 & -2\mu \end{pmatrix},$$

then

$$T^{-1} = \begin{pmatrix} \frac{1}{\omega_0} & \frac{1}{2\omega_0} \\ 0 & -\frac{1}{2\mu} \end{pmatrix}.$$

The transformation $\begin{pmatrix} a \\ h \end{pmatrix} = P\begin{pmatrix} u \\ v \end{pmatrix}$ changes (2.1) into

$$\begin{aligned} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} &= P^{-1}JP\begin{pmatrix} u \\ v \end{pmatrix} + P^{-1}\begin{pmatrix} f_2 P(u,v) \\ g_2 P(u,v) \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\omega_0 \\ \omega_0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_3(u,v) \\ g_3(u,v) \end{pmatrix}, \end{aligned}$$

where

$$\begin{gathered} \begin{pmatrix} f_3(u,v) \\ g_3(u,v) \end{pmatrix} = \begin{pmatrix} \frac{1}{2\omega_0} f_2(\omega_0 u + \mu v, -2\mu v) \\ -\frac{1}{2\mu} g_2(\omega_0 u + \mu v, -2\mu v) \end{pmatrix}, \\ \begin{aligned} f_2(\omega_0 u + \mu v, -2\mu v) &= \omega_0^2 u^2 h_0 + \mu^2 v^2 h_0 + 2\omega_0 \mu h u v - 4a_0\omega_0\mu u v \\ &\quad - 4a_0\mu^2 v^2 - 2\mu\omega_0^2 u^2 v - 2\mu^3 v^3 - 4\omega_0\mu^2 u v^2, \end{aligned} \\ g_2(\omega_0 u + \mu v, -2\mu v) = -f_2(\omega_0 u + \mu v, -2\mu v). \end{gathered}$$

Then

$$\begin{gathered} f_{2uu} = 2h_0\omega_0^2 - 4\mu\omega_0^2 v, \quad f_{2uuv} = -4\mu\omega_0^2, \quad f_{2uuu} = 0, \\ f_{2uv} = 2\omega_0\mu h_0 - 4a_0\mu\omega_0, \quad f_{2uvv} = -8\mu^2\omega_0, \quad f_{2vv} = 2\mu^2 h_0 - 8a_0\mu^2, \\ g_{2uu} = -2h_0\omega_0^2 + 4\mu\omega_0^2 v, \quad g_{2vvv} = 12\mu^3, \quad g_{2uuu} = 0, \\ g_{2uv} = -2\omega_0\mu h_0 + 4a_0\mu\omega_0, \quad g_{2uvv} = 8\mu^2\omega_0, \quad g_{2vv} = -2\mu^2 h_0 + 8a_0\mu^2. \end{gathered}$$

So the stability of Hopf bifurcation in system (2.1) at $P_0(a_0, h_0)$ is determined by the sign of the following quantity [20]

$$\begin{aligned} \sigma &= \frac{1}{16}(f_{3uuu} + g_{3uuv} + f_{3uvv} + g_{3vvv}) \\ &\quad + \frac{1}{16\omega_0}[f_{3uv}(f_{3uu} + f_{3vv}) - g_{3uv}(g_{3uu} + g_{3vv}) - f_{3uu}g_{3uu} + f_{3vv}g_{3vv}], \end{aligned}$$

where all the partial derivatives are evaluated at the bifurcation point $(u, v, \mu) = (0, 0, \mu_0)$.

We can find that

$$f_{3uu} = h\omega_0, \quad f_{3uuu} = 0, \quad f_{3uv} = \mu h - 2a\mu, \quad f_{3uvv} = -4\mu^2,$$
$$f_{3vv} = \frac{\mu^2 h - 4a\mu^2}{\omega_0}, \quad g_{3uu} = \frac{h\omega_0^2}{\mu}, \quad g_{3uuv} = -2\omega_0^2,$$
$$g_{3uv} = \omega_0 h - 2a\omega_0, \quad g_{3vv} = \mu h - 4a\mu, \quad g_{3vvv} = -6\mu^2,$$

then

$$\sigma = \frac{-\omega_0^4(2\mu + h^2) - \mu^3[10\omega_0^2 - 2(h-2a)(-l - \sqrt{l^2-4}) - \mu(h-4a)^2]}{16\omega_0^2\mu} < 0.$$

From the above analysis, one has the the following Hopf bifurcation result at the point P_0.

Theorem 2.4. *When* $\operatorname{tr}(J)^2 < 4\det(J)$, *system* (2.1) *undergoes a supercritical Hopf bifurcation at* $\mu = \mu_0$ *and the bifurcated limit cycle is stable as* $\mu > \mu_0$.

For an illustration of the Hopf bifurcation, see Figures 1 and 2.

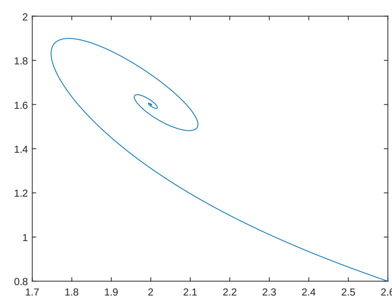

FIGURE 1. Equilibrium point P_0 is a stable focus.

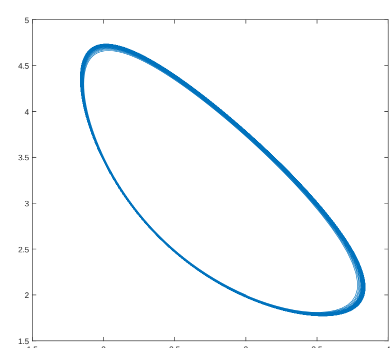

FIGURE 2. A stable limit cycle is bifurcated from Hopf bifurcation.

When $l = 2$, the equilibrium points

$$P_0\Big(\frac{l+\sqrt{l^2-4}}{2}, \frac{l-\sqrt{l^2-4}}{2}\mu\Big), \quad P_1\Big(\frac{l-\sqrt{l^2-4}}{2}, \frac{l+\sqrt{l^2-4}}{2}\mu\Big),$$

with $(l > 2)$, coincide to be the point $P^*(1,\mu)$. The Jacobian matrix evaluated at P^* is

$$J(P^*) = \begin{pmatrix} \mu & 1 \\ -2\mu & -2 \end{pmatrix}.$$

The corresponding eigenvalues are $\lambda_1 = 0$, $\lambda_2 = \mu - 2$ and $\mathrm{tr}(J(P^*)) = \mu - 2$, $\det(J(P^*)) = 0$. So it is nonhyperbolic. When $\mu \neq 2$, its stability could be analyzed in the following by employing center manifold reduction theory in [20].

Let $a \to a+1$, $h \to h+\mu$, system (2.1) becomes

$$\begin{pmatrix} \frac{da}{dt} \\ \frac{dh}{dt} \end{pmatrix} = J(P^*)\begin{pmatrix} a \\ h \end{pmatrix} + \begin{pmatrix} f_2(a,h) \\ g_2(a,h) \end{pmatrix}, \tag{2.6}$$

where

$$\begin{gathered} f_2(a,h,\mu) = a^2\mu + 2ah + a^2h, \\ g_2(a,h,\mu) = -a^2\mu - 2ah - a^2h. \end{gathered}$$

The Jacobian matrix at P^* can be diagonalized as

$$T^{-1}J(P^*)T = \begin{pmatrix} 0 & 0 \\ 0 & \mu-2 \end{pmatrix},$$

where

$$T = \begin{pmatrix} 1 & 1 \\ -\mu & -2 \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \frac{-2}{\mu-2} & \frac{-1}{\mu-2} \\ \frac{\mu}{\mu-2} & \frac{1}{\mu-2} \end{pmatrix}.$$

Consequently, the system will be changed into

$$\begin{aligned} \begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} &= \begin{pmatrix} 0 & 0 \\ 0 & \mu-2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \\ &\quad + \begin{pmatrix} -\frac{1}{\mu-2}(u+v)[(u+v)\mu - (\mu u + 2v)(u+v+2)] \\ \frac{\mu-1}{\mu-2}(u+v)[(u+v)\mu - (\mu u + 2v)(u+v+2)] \end{pmatrix}. \end{aligned} \tag{2.7}$$

Its local center manifold at the origin can be represented as

$$W^c(0) = \{(u,v) \in R^2 \mid v = \gamma(u), \mid u \mid < \delta, \gamma(0) = D\gamma(0) = 0\},$$

for $\delta > 0$ sufficiently small. Assume that $\gamma(x)$ takes the form

$$\gamma(u) = u^2 + bu^3 + O(u^4). \tag{2.8}$$

Substituting it into system (2.7) and equating coefficients on each power of x to zero, we have

$$a = \frac{\mu(\mu-1)}{(\mu-2)^2}, \quad b = \frac{\mu^2(\mu+2)(\mu-1)}{(\mu-2)^4},$$

thus

$$\gamma(u) = \frac{\mu(\mu-1)}{(\mu-2)^2}u^2 + \frac{\mu^2(\mu+2)(\mu-1)}{(\mu-2)^4}u^3 + O(x^4).$$

As a result, system (2.7) restricted to the local center manifold is

$$\dot{u} = \frac{\mu}{\mu-2}u^2 + O(u^3),$$

from which we know that the origin is unstable, that is, the point P^* is unstable.

3. Turing instability induced by diffusion

When the diffusive effects are considered, it is desirable to know how the diffusive terms affect the stability of fixed points and the bifurcated limit cycle. If the stable equilibrium points and stable limit cycles become unstable under such effects, then it is often known as the Turing instability, namely, the diffusion-driven instability. In this section, instability induced by diffusive effects on the equilibrium points and the bifurcated limit cycle will be investigated.

3.1. **Turing instability of the equilibrium points.** Let $u = a - a_0, v = h - h_0$, then the linearized system of (1.3) at (a_0, h_0) is

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 2a_0h_0 - \mu + D_a \frac{\partial^2}{\partial x^2} & a_0^2 \\ -2a_0h_0 & -a_0^2 - 1 + D_h \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} =: \begin{pmatrix} u \\ v \end{pmatrix}, \tag{3.1}$$

with the Neumann boundary condition

$$u_x(0,t) = v_x(0,t) = u_x(\pi,t) = v_x(\pi,t) = 0. \tag{3.2}$$

System (3.1)-(3.2) has the solution (u, v) formally described as

$$\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{\lambda t} \cos kx. \tag{3.3}$$

Substituting this into (3.1), we have

$$\begin{aligned} &\sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ b_k \end{pmatrix} \lambda e^{\lambda t} \cos kx \\ &= L(P_0) \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \sum_{k=0}^{\infty} \begin{pmatrix} \mu + D_a \frac{\partial^2}{\partial x^2} & \frac{l^2 + l\sqrt{l^2-4} - 2}{2} \\ -2\mu & -\frac{l^2 + l\sqrt{l^2-4}}{2} + D_h \frac{\partial^2}{\partial x^2} \end{pmatrix} \begin{pmatrix} a_k \\ b_k \end{pmatrix} e^{\lambda t} \cos kx. \end{aligned}$$

Comparing the equal powers of k, we have

$$(\lambda I - J_k(\mu)) \begin{pmatrix} a_k \\ b_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k = 0, 1, 2, \dots, \tag{3.4}$$

where

$$J_k(\mu) = \begin{pmatrix} \mu - k^2 D_a & \frac{l^2 + l\sqrt{l^2-4} - 2}{2} \\ -2\mu & -\frac{l^2 + l\sqrt{l^2-4}}{2} - k^2 D_h \end{pmatrix}.$$

So system (3.4) has a nonzero solution $(a_k, b_k)^T$ if and only if

$$\det(\lambda I - J_k(\mu)) = 0.$$

Which is the characteristic equation of the original system (1.3) at P_0. Rewrite it as the equation

$$\lambda^2 - \operatorname{tr}(k)\lambda + \det(k) = 0, \tag{3.5}$$

where

$$\operatorname{tr}(k) = \mu - \frac{l^2 + l\sqrt{l^2-4}}{2} - k^2(D_a + D_h) = \operatorname{tr}(J(P_0)) - k^2(D_a + D_h),$$

$$
\begin{aligned}
\det(k) &= \frac{l^2 + l\sqrt{l^2-4} - 4}{2}\mu - \mu k^2 D_h + \frac{l^2 + l\sqrt{l^2-4}}{2} k^2 D_a + k^4 D_a D_h \\
&= k^2 D_h [k^2 D_a - \mu D_h] + \frac{l^2 + l\sqrt{l^2-4}}{2} k^2 D_a + \det(J(P_0)).
\end{aligned}
$$

Under condition (2.3), one has $\operatorname{tr}(k) < 0$ for all $k = 0, 1, 2, \dots$ and $\det(0) = \det(J(P_0)) > 0$. Let

$$
r_m = \min_{1 \le k \le m} \frac{\det(J) + \frac{l^2 + l\sqrt{l^2-4}}{2} D_a k^2}{(\mu - D_a k^2) k^2}.
$$

If $\frac{\mu}{D_a} \le 1$ or $m^2 < \frac{\mu}{D_a} \le (m+1)^2$, and $r < r_m$, then $\det(k) \ge \det(0) > 0$, so (a_0, h_0) is linearly asymptotically stable for system (1.3); if $m^2 < \frac{\mu}{D_a} \le (m+1)^2$ and $D_h > r_m$, then there exists at least one negative in $\det(1), \dots, \det(m)$, and so (a_0, h_0) is unstable for system (1.3). Hence, the above analysis can be summarized as follows.

Theorem 3.1. *Assume that condition* (2.3) *holds, let*

$$
r_m = \min_{1 \le k \le m} \frac{\det(J) + \frac{l^2 + l\sqrt{l^2-4}}{2} D_a k^2}{(\mu - D_a k^2) k^2},
$$

then (a_0, h_0) *is a stable equilibrium for* (1.3) *if either*

$$
\frac{\mu}{D_a} \le 1, \tag{3.6}
$$

or

$$
m^2 < \frac{\mu}{D_a} \le (m+1)^2 \quad \text{and} \quad D_h < r_m \tag{3.7}
$$

hold. Also (a_0, h_0) *is an unstable equilibrium for* (1.3) *if*

$$
m^2 < \frac{\mu}{D_a} \le (m+1)^2 \quad \text{and} \quad D_h > r_m. \tag{3.8}
$$

Remark 3.2. (1) At the point S, the characteristic equation (3.5) changes into

$$
\lambda^2 - \operatorname{tr}(k)\lambda + \det(k) = 0, \tag{3.9}
$$

where

$$
\begin{aligned}
\operatorname{tr}(k) &= -\mu - 1 - k^2(D_a + D_h) = \operatorname{tr}(J(s)) - k^2(D_a + D_h), \\
\det(k) &= \mu + k^2(D_a + \mu D_h) + k^4 D_a D_h \\
&= \det(J(S)) + k^2(D_a + \mu D_h) + k^4 D_a D_h.
\end{aligned}
$$

Note that in this case $\operatorname{tr}(k) < 0$ and $\det(k) > 0$, for all $k = 0, 1, 2, \dots$, so the point S is still stable under diffusive effects.

(2) At the point P_1, the characteristic equation (3.5) changes into

$$
\lambda^2 - \operatorname{tr}(k)\lambda + \det(k) = 0, \tag{3.10}
$$

where

$$
\begin{aligned}
\operatorname{tr}(k) &= \mu - \frac{l^2 - l\sqrt{l^2-4}}{2} - k^2(D_a + D_h) \\
&= \operatorname{tr}(J(P_1)) - k^2(D_a + D_h),
\end{aligned}
$$

$$\begin{aligned}\det(k) &= \frac{l^2 - l\sqrt{l^2-4} - 4}{2}\mu - \mu k^2 D_h + \frac{l^2 - l\sqrt{l^2-4}}{2}k^2 D_a + k^4 D_a D_h \\ &= \det(J(P_1)) - \mu k^2 D_h + \frac{l^2 - l\sqrt{l^2-4}}{2}k^2 D_a + k^4 D_a D_h.\end{aligned}$$

Note that $\det(0) = \det(J(P_1)) < 0$, so the point P_1 is still unstable under diffusive effects.

(3) At the point $P^*(1,\mu)$, the characteristic equation (3.5) changes into

$$\lambda^2 - \mathrm{tr}(k)\lambda + \det(k) = 0, \tag{3.11}$$

where

$$\begin{gathered}\mathrm{tr}(k) = \mu - 2 - k^2(D_a + D_h) = \mathrm{tr}(J(P^*)) - k^2(D_a + D_h), \\ \det(k) = -\mu k^2 D_h + 2k^2 D_a + k^4 D_a D_h.\end{gathered}$$

Note that $\mathrm{tr}(0) = \mu - 2, \det(0) = \det(J(P^*)) = 0$, from the previous analysis, it is unstable ($\mu \neq 2$). So the point P^* is still unstable under diffusive effects, from analysis of the Turing and Hopf interaction in [21].

3.2. **Turing instability of the bifurcated limit cycle.** To analyze the effects induced by diffusion on the stability of the bifurcated limit cycle, we apply the center manifold reduction and normal form technique to system (1.3). From the first nonvanishing coefficient of Poincaré normal form, combined with the eigenvalues of linearized system at the point P_0 and the bifurcation value μ_0, the stability of the bifurcated limit cycle could be identified. To this end, some necessary transformation procedures and analysis about the eigenvalues will be carried out.

Let $\mu = \mu_0$ and take the transformation $u = a - a_0,\ v = h - h_0,\ U = (u,v)^T$, so system (1.3) can be rewritten as

$$\begin{gathered}U_t = \Big[J(\mu_0) + \begin{pmatrix} D_a\partial_{xx} & 0 \\ 0 & D_h\partial_{xx}\end{pmatrix}\Big]U + F(U,\mu_0), \\ U_x(0,t) = U_x(\pi,t) = (0,0)^T,\end{gathered} \tag{3.12}$$

where

$$F(U,\mu_0) = (f_2(u,v,\mu_0),(g_2(u,v,\mu_0))^T,$$

f_2 and g_2 are defined in (2.5). As in [22], $F(U,\mu_0)$ can be rewritten into

$$F(U,\mu_0) = \frac{1}{2}Q(U,U) + \frac{1}{6}C(U,U,U) + O(|U|^4)$$

and

$$Q(U,V) = \begin{pmatrix} Q_1(U,V) \\ Q_2(U,V)\end{pmatrix}, \quad C(U,V,W) = \begin{pmatrix} C_1(U,V,W) \\ C_2(U,V,W)\end{pmatrix},$$

so

$$\begin{aligned}Q_1(U,V) &= f_{2uu}u_1v_1 + f_{2uv}u_1v_2 + f_{2vu}u_2v_1 + f_{2vv}u_2v_2 \\ &= 2(h_0u_1v_1 + a_0u_1v_2 + au_2v_1), \\ Q_2(U,V) &= g_{2uu}u_1v_1 + g_{2uv}u_1v_2 + g_{2vu}u_2v_1 + g_{2vv}u_2v_2 \\ &= -Q_1(U,V),\end{aligned}$$

$$\begin{aligned}C_1(U,V,W) = &\ f_{2uuu}u_1v_1w_1 + f_{2uuv}u_1v_1w_2 + f_{2uvu}u_1v_2w_1 + f_{2uvv}u_1v_2w_2 \\ &+ f_{2vuu}u_2v_1w_1 + f_{2vuv}u_2v_1w_2 + f_{2vvu}u_2v_2w_1 + f_{2vvv}u_2v_2w_2 \\ = &\ 2(u_1v_1w_2 + u_1v_1w_1 + u_2v_1w_1),\end{aligned}$$

$$\begin{aligned}C_2(U,V,W) &= g_{2uuu}u_1v_1w_1 + g_{2uuv}u_1v_1w_2 + g_{2uvu}u_1v_2w_1 + g_{2uvv}u_1v_2w_2 \\ &\quad + g_{2vuu}u_2v_1w_1 + g_{2vuv}u_2v_1w_2 + g_{2vvu}u_2v_2w_1 + g_{2vvv}u_2v_2w_2 \\ &= -C_1(U,V,W),\end{aligned}$$

for any $U=(u_1,u_2)^T$, $V=(v_1,v_2)^T$, $W=(w_1,w_2)^T$, and $U,\ V,\ W \in H^2([0,\pi]) \times H^2([0,\pi])$. For $\mu=\mu_0$, the linear operator $L=L(\mu_0)$ is defined by

$$LU = \Big[J(\mu_0) + \begin{pmatrix} D_a\partial_{xx} & 0 \\ 0 & D_h\partial_{xx}\end{pmatrix}\Big]U$$

and let L^* be the adjoint operator of L, then

$$L^*U = \Big[J^*(\mu_0) + \begin{pmatrix} D_a\partial_{xx} & 0 \\ 0 & D_h\partial_{xx}\end{pmatrix}\Big]U,$$

with

$$\begin{aligned}J(\mu_0) &= \begin{pmatrix} \frac{l^2+l\sqrt{l^2-4}}{2} & \frac{l^2+l\sqrt{l^2-4}-2}{2} \\ -(l^2+l\sqrt{l^2-4}) & -\frac{l^2+l\sqrt{l^2-4}}{2}\end{pmatrix}, \\ J^*(\mu_0) &= \begin{pmatrix} \frac{l^2+l\sqrt{l^2-4}}{2} & -(l^2+l\sqrt{l^2-4}) \\ \frac{l^2+l\sqrt{l^2-4}-2}{2} & -\frac{l^2+l\sqrt{l^2-4}}{2}\end{pmatrix}.\end{aligned}$$

Clearly, $\langle L^*U,V\rangle = \langle U,LV\rangle$ for any $U,V \in H^2([0,\pi]) \times H^2([0,\pi])$ and the inner product in $H^2([0,\pi]) \times H^2([0,\pi])$ is defined as $\langle U,V\rangle = \frac{1}{\pi} \times \int_0^\pi \overline{U}^T V dx$ for any $U,V \in H^2([0,\pi]) \times H^2([0,\pi])$. The linearized system of (3.12) at the equilibrium $(0,0)$ is

$$U_t = LU \tag{3.13}$$

with the Neumann boundary condition

$$U_x(0,t) = U_x(0,t) = (0,0)^T. \tag{3.14}$$

System (3.13) with boundary condition (3.14) has a solution that can be formally represented as

$$U = \sum_{k=0}^{\infty} \begin{pmatrix} a_k \\ h_k \end{pmatrix} e^{\lambda t}\cos kx, \tag{3.15}$$

where a_k and h_k are complex numbers, k is the wave number $k=0,1,2\dots$, and $\lambda \in \mathbf{C}$ is the temporal spectrum. Substituting (3.15) into (3.12) and collecting the like terms about k, one has

$$(\lambda I - L_k)\begin{pmatrix} a_k \\ h_k \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad k=0,1,2,\dots, \tag{3.16}$$

where

$$L_k = \begin{pmatrix} \frac{l^2+l\sqrt{l^2-4}}{2} - D_ak^2 & \frac{l^2+l\sqrt{l^2-4}-2}{2} \\ -(l^2+l\sqrt{l^2-4}) & -\frac{l^2+l\sqrt{l^2-4}}{2} - D_hk^2\end{pmatrix}.$$

For some k, Equation (3.16) has a nonzero solution $(a_k,h_k)^T$ if and only if the dispersion relation is satisfied, $\det(\lambda I - L_k)=0$. From such dispersion relation, the characteristic equation follows immediately

$$\lambda^2 - \operatorname{tr}(L_k) + \det(k) = 0, \quad k=0,1,2,\dots, \tag{3.17}$$

where

$$\operatorname{tr}(L_k) = -(D_a+D_h)k^2,$$

$$\det(L_k) = D_h k^2 (D_a k^2 - \frac{l^2 + l\sqrt{l^2-4}}{2}) + D_a k^2 \frac{l^2 + l\sqrt{l^2-4}}{2} + \det(J).$$

Note that when $\mu = \mu_0$, one has $\mathrm{tr}(L_0) = 0$, $\det(L_0) = \det(J) = \mu_0(\mu_0 - 2) > 0$, $\mathrm{tr}(L_k) < 0$ for $k = 1, 2, \dots$. Then for $k = 0$, L has eigenvalues with zero real parts, i.e., a pair of purely imaginary eigenvalues. Signs of the remaining eigenvalues of L could be judged as follows.

If $D_a \geq \frac{l^2+l\sqrt{l^2-4}}{2}$, then $\det(L_k) \geq \det(L_0) > 0$ for $k = 1, 2, \dots$. Moreover, if $m^2 < \frac{l^2+l\sqrt{l^2-4}}{2Da} \leq (m+1)^2$, $m \in \mathbf{N}^+$ and $D_h < \bar{r}$, then $\det(L_k) > 0$, $k = 1, 2, \dots$, where

$$\bar{r} = \min_{1 \leq k \leq m} \frac{D_a k^2 \frac{l^2+l\sqrt{l^2-4}}{2} + \det(J)}{(\frac{l^2+l\sqrt{l^2-4}}{2} - D_a k^2)k^2}.$$

Consequently, the remaining eigenvalues of L all have negative real parts. If $m^2 < \frac{l^2+l\sqrt{l^2-4}}{2Da} \leq (m+1)^2$, $m \in \mathbf{N}^+$ and $D_h > \bar{r}$, then there must exist at least one of $\det(L_1)$, $\det(L_2)$, $\dots$, $\det(L_m)$ to be negative. Then some eigenvalues of L will have positive real parts.

Next the center manifold reduction and normal form technique are applied to system (1.3). Let $Lq = i\omega_0 q$ and $L^* q^* = -i\omega_0 q^*$, then one has

$$q = (i\omega_0 + \frac{l^2 + l\sqrt{l^2-4}}{2}, -(l^2 + l\sqrt{l^2-4}))^T, \quad q^* = \frac{1}{2\omega_0}(i, -\frac{\omega_0}{l^2 + l\sqrt{l^2-4}} + \frac{i}{2})^T,$$

respectively. Note that $\langle q^*, q\rangle = 1$ and $\langle q^*, \bar{q}\rangle = 0$.

According to [22], for each $U \in Dom(L)$, the pair (z, w) could be associated, where $U = zq + \bar{z}\bar{q} + w$, $z = \langle q^*, U\rangle$ and $w = (w_1, w_2)^T$. Then

$$u = (i\omega_0 + \frac{l^2 + l\sqrt{l^2-4}}{2})z + (-i\omega_0 + \frac{l^2 + l\sqrt{l^2-4}}{2})\bar{z} + w_1,$$
$$v = -(l^2 + l\sqrt{l^2-4})(z + \bar{z}) + w_2.$$

System (3.12) in (z, w) coordinates is changed to

$$\begin{aligned} \frac{dz}{dt} &= i\omega_0 z + \langle q^*, \widetilde{f}\rangle, \\ \frac{dw}{dt} &= Lw + H(z, \bar{z}, w), \end{aligned} \tag{3.18}$$

where

$$\widetilde{f} = F(zq + \bar{z}\bar{q} + w, \mu_0), \quad H(z, \bar{z}, w) = \widetilde{f} - \langle q^*, \widetilde{f}\rangle q - \langle \bar{q}^*, \widetilde{f}\rangle \bar{q},$$

$$\begin{aligned} \widetilde{f} &= \frac{1}{2}Q(zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w) \\ &\quad + \frac{1}{6}C(zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w, zq + \bar{z}\bar{q} + w) + O(|zq + \bar{z}\bar{q} + w|^4) \\ &= \frac{1}{2}Q(q, q)z^2 + Q(q, \bar{q})z\bar{z} + \frac{1}{2}Q(\bar{q}, \bar{q})\bar{z}^2 + O(|z|^3, |z| \cdot |w|, |w|^2), \end{aligned}$$

$$\begin{aligned} \langle q^*, \widetilde{f}\rangle &= \frac{1}{2}\langle q^*, Q(q, q)\rangle z^2 + \langle q^*, Q(q, \bar{q})\rangle z\bar{z} + \frac{1}{2}\langle q^*, Q(\bar{q}, \bar{q})\rangle z^2 \\ &\quad + O(|z|^3, |z| \cdot |w|, |w|^2), \\ \langle \bar{q}^*, \widetilde{f}\rangle &= \frac{1}{2}\langle \bar{q}^*, Q(q, q)\rangle z^2 + \langle \bar{q}^*, Q(q, \bar{q})\rangle z\bar{z} + \frac{1}{2}\langle \bar{q}^*, Q(\bar{q}, \bar{q})\rangle z^2 \\ &\quad + O(|z|^3, |z| \cdot |w|, |w|^2). \end{aligned}$$

Hence

$$H(z,\bar z,w)=\frac{1}{2}z^2H_{20}+z\bar zH_{11}+\frac{1}{2}\bar z^2H_{02}+O(|z|^3,|z|\cdot|w|,|w|^2),$$

where

$$\begin{gathered}H_{20}=Q(q,q)-\langle q^*,Q(q,q)\rangle q-\langle\bar q^*,Q(q,q)\rangle\bar q,\\ H_{11}=Q(q,\bar q)-\langle q^*,Q(q,\bar q)\rangle q-\langle\bar q^*,Q(q,\bar q)\rangle\bar q,\\ H_{02}=Q(\bar q,\bar q)-\langle q^*,Q(\bar q,\bar q)\rangle q-\langle\bar q^*,Q(\bar q,\bar q)\rangle\bar q.\end{gathered}$$

Moreover, $H_{20}=H_{11}=H_{02}=(0,0)^T$, so $H(z,\bar z,w)=O(|z|^3,|z|\cdot|w|,|w|^2)$. Hence, system (3.12) possesses a center manifold. It can be described as

$$w=\frac{1}{2}z^2w_{20}+z\bar zw_{11}+\frac{1}{2}\bar z^2w_{02}+O(|z|^3).$$

In view of $Lw+H(z,\bar z,w)=\frac{dw}{dt}=\frac{\partial w}{\partial z}\frac{dz}{dt}+\frac{\partial w}{\partial\bar z}\frac{d\bar z}{dt}$, one has

$$\begin{gathered}w_{20}=[2i\omega_0-L]^{-1}H_{20}=(0,0)^T,\\ w_{11}=-L^{-1}H_{11}=(0,0)^T,\\ w_{02}=[-2i\omega_0-L]^{-1}H_{02}=(0,0)^T.\end{gathered}$$

Thus, the reaction-diffusion system (3.12) restricted to the center manifold is

$$\frac{dz}{dt}=i\omega_0z+\langle q^*,\widetilde f\rangle=i\omega_0z+\sum_{2\le i+j\le 3}\frac{g_{ij}}{i!j!}z^i\bar z^j+O(|z|^4), \tag{3.19}$$

where

$$\begin{gathered}g_{20}=\langle q^*,Q(q,q)\rangle,\quad g_{11}=\langle q^*,Q(q,\bar q)\rangle,\quad g_{02}=\langle q^*,Q(\bar q,\bar q)\rangle,\\ g_{21}=2\langle q^*,Q(w_{11},q)\rangle+\langle q^*,Q(w_{20},\bar q)\rangle+\langle q^*,C(q,q,\bar q)\rangle=\langle q^*,C(q,q,\bar q)\rangle.\end{gathered}$$

The dynamics of (3.12) is determined by that of (3.19). Furthermore, its Poincaré normal form could be given in the form

$$\frac{dz}{dt}=(\alpha(\mu)+i\omega(\mu))z+z\sum_{j=1}^{M}\delta_j(\mu)(z\bar z)^j, \tag{3.20}$$

where z is a complex variable, $M\ge 1$, and $\delta_j(\mu)$ are complex-valued coefficients. Then one has

$$\delta_1(\mu)=\frac{g_{20}g_{11}\left[3\alpha(\mu)+i\omega(\mu)\right]}{2\left[\alpha^2(\mu)+\omega^2(\mu)\right]}+\frac{|g_{11}|^2}{\alpha(\mu)+i\omega(\mu)}+\frac{|g_{02}|^2}{2\left[\alpha(\mu)+3i\omega(\mu)\right]}+\frac{g_{21}}{2}.$$

Note that $\mathrm{Re}(\delta_1(\mu_0))=\mathrm{Re}[\frac{g_{20}g_{11}}{2\omega_0}i+\frac{g_{21}}{2}]$, since $\alpha(\mu_0)=0$ and $\omega(\mu_0)=\omega_0>0$. From

$$\begin{gathered}g_{20}=\frac{1}{\omega_0}\left[(i\omega_0+\mu)^2-4a\mu(i\omega_0+\mu)\right]\Big(\frac{\omega_0-3i\mu}{2\mu}\Big),\\ g_{11}=\frac{1}{\omega_0}\left[(\mu^2+\omega_0^2)h-4a\mu^2\right]\Big(\frac{\omega_0-3i\mu}{2\mu}\Big),\\ g_{21}=\frac{1}{\omega_0}(3\mu i-\omega_0)(3\mu^2+\omega_0^2+2i\omega_0\mu),\end{gathered}$$

one has

$$\mathrm{Re}(\delta_1(\mu_0))=\frac{1}{4\omega_0^2\mu_0}\big\{(h_0\mu_0^2+h_0\omega_0^2-4a_0\mu_0^2)[3(h_0\mu_0^2-h_0\omega_0^2-4a_0\mu_0^2)$$

$$- (\omega_0^2 - 9\mu_0^2)(h_0 - 2a_0)] - 2\omega_0^2\mu_0(\omega_0^2 + 9\mu_0^2)\},$$

from which one can obtain that the bifurcated limit cycle is locally stable in the center manifold if $\mathrm{Re}(\delta_1(\mu_0)) < 0$, otherwise, it is unstable in the center manifold. In fact, when $\mathrm{Re}(\delta_1(\mu_0)) < 0$, the corresponding Floquet exponent is negative, otherwise the corresponding Floquet exponent is positive. However, the stability of such homogeneous limit cycle for system (1.3) may be different for system (2.1). If P_0 is unstable for system (1.3) but stable for system (2.1), i.e., some of eigenvalues of system (1.3) at the equilibrium point P_0 have positive real parts, then the limit cycle bifurcated from Hopf bifurcation will also be unstable for system (1.3). That means if some eigenvalue of L has positive real part, then the bifurcated limit cycle is unstable. When eigenvalues of L have negative real parts except a pair of imaginary roots, the limit cycle will be stable when $\mathrm{Re}(\delta_1(\mu_0)) < 0$, and unstable when $\mathrm{Re}(\delta_1(\mu_0)) > 0$. After tedious calculation, it shows that

$$\mathrm{Re}(\delta_1(\mu_0)) = \frac{l^2 + l\sqrt{l^2-4}}{4\sqrt{l^2-4}}\Big(5l^3 - 14l + 5l^2\sqrt{l^2-4} + 4\sqrt{l^2-4}\Big) > 0.$$

That implies the limit cycle from the Hopf bifurcation is unstable for system (1.3) under diffusive effects, otherwise, it is stable for system (2.1). Now the above discussions are summarized as in the following theorem.

Theorem 3.3. *Assume* (2.4) *holds, so the spatially homogeneous periodic solution bifurcated from the equilibrium $P_0(a_0, b_0)$ is stable for system* (2.1). *However, the spatially homogenous periodic solution is always unstable for system* (1.3) *with diffusive terms.*

4. Numerical simulations

In this section, numerical simulations about the Turing and Hopf bifurcation will be illustrated. From Theorem 3.1, we know that under conditions (2.3) and either (3.6) or (3.7), the point (a_0, h_0) will be still stable, so no Turing instability will be induced. Meanwhile, under (2.3) and (3.8), the point (a_0, h_0) will become unstable from stable under diffusive effects, so the Turing instability will occur. Consequently, some patterns will form in the original system. For further understanding above theoretical analysis, the corresponding numerical results are presented.

Now we take parameters as $\mu = 1, c = 2.2, D_a = 1.5, D_h = 10$, so (2.3) and (3.6) are satisfied. The point P_0 is still stable under diffusive effects, see Figure 3. Initial states are $a(0,t) = 1.7179 + 0.8\cos(x)$, $h(0,t) = 0.5821 + 0.8\cos(x)$.

If the parameters are taken as $\mu = 1, c = 2.3, D_a = 0.5, D_h = 3$, so (2.3) and (3.7) are satisfied. The point P_0 is still stable under diffusive effects, see Figure 4. We take $a(0,t) = 1.7179 + 0.5\cos(x)$, $h(0,t) = 0.5821 + 0.5\cos(x)$.

However, when $\mu = 1$, $c = 2.3$, $D_a = 0.5$, $D_h = 10$, so (2.3) and (3.8) are satisfied, the point P_0 become unstable under diffusive effects, see Figure 5. Initial values are taken as $a(0,t) = 1.7179 + 0.5\cos(x)$, $h(0,t) = 0.5821 + 0.5\cos(x)$, Then consequently it is found that the patterns appear, see Figure 6.

From the numerical results, note that the Turing instability of the equilibrium point occur under the diffusive effects. As for such effects on the bifurcated limit cycle, we take $\mu = 7.86$, $c = 23.58$, $D_a = 7.86$, $D_h = 5$. Then $\mu > \mu_0 \approx 7.8541$ and $D_a > \mu_0$, so (2.4) is satisfied and the eigenvalues of L have negative real parts. In this case, the limit cycle from the Hopf bifurcation is stable for system (2.1), but the spatially homogeneous periodic solution is unstable for system (1.3) since

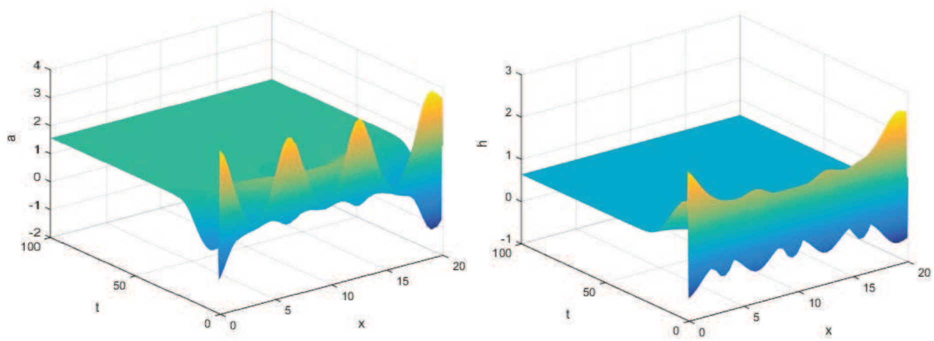

Figure 3. The point P_0 is stable.

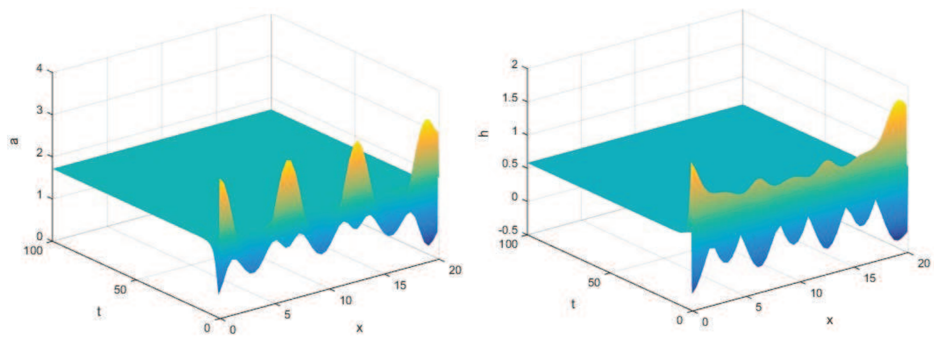

Figure 4. The point P_0 is stable.

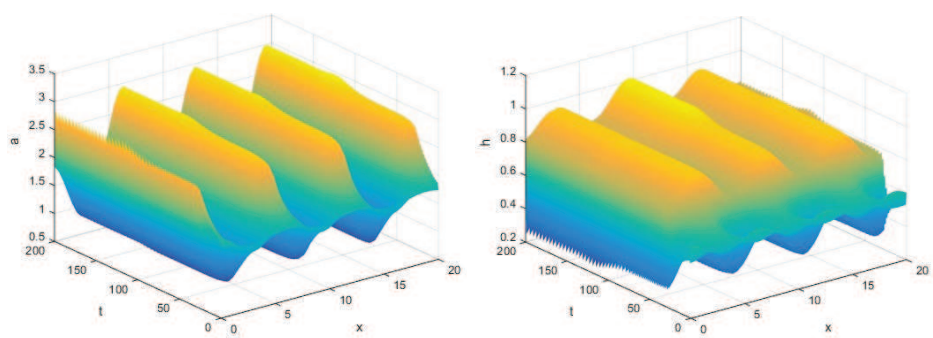

Figure 5. The point P_0 is unstable with diffusion.

$\mathrm{Re}(\delta_1(\mu_0)) > 0$, see Figure 7. Initial values are taken as $a(0,t) = 2.5 + 0.5\cos(x)$, $h(0,t) = 3 + 0.5\cos(x)$, close to the limit cycle.

The similar phenomenon occurs when parameters are taken as $\mu = 7.86$, $c = 23.58$, $D_a = 7$, $D_h = 2$, then $\mu > \mu_0 \approx 7.8541$, $D_a < \mu_0$ and $D_h < \bar{r}$, so (2.4) is satisfied and the eigenvalues of L have negative real parts. In this case, the limit cycle from the Hopf bifurcation is stable for system (2.1), but the spatially homogeneous periodic solution is unstable for system (1.3), see Figure 8. Initial

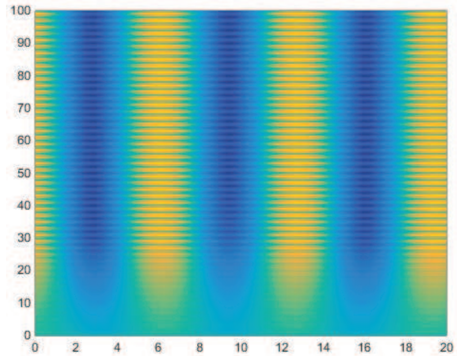

Figure 6. Patterns appear in system (1.3).

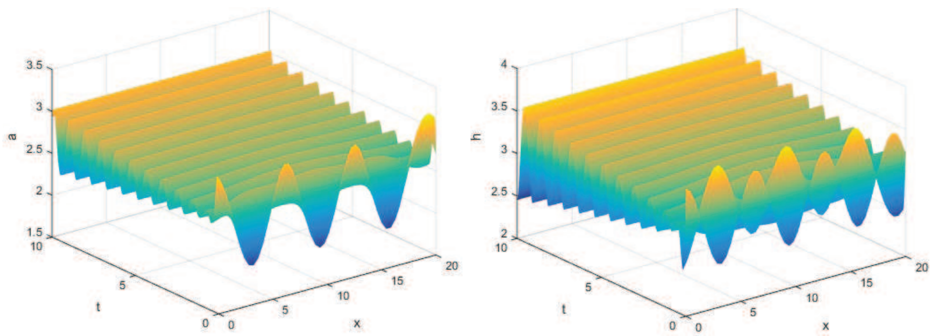

Figure 7. Turing instability of bifurcated periodic solution occurs.

values are taken as $a(0,t) = 2 + 0.02\,\mathrm{rand}(1)$, $h(0,t) = 3 + 0.02\,\mathrm{rand}(1)$, close to the limit cycle.

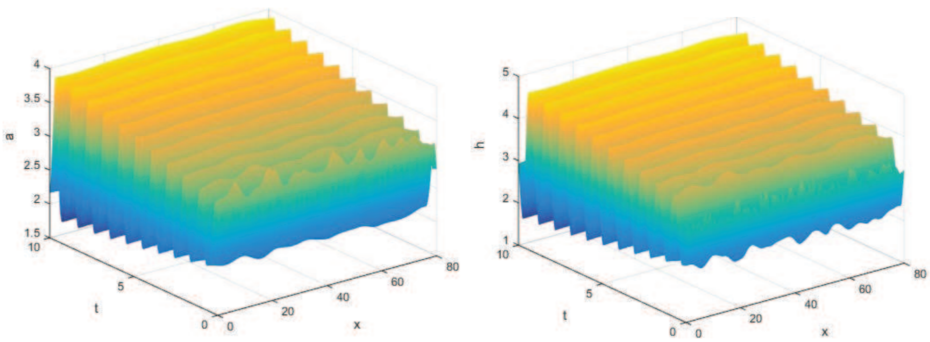

Figure 8. Turing instability of bifurcated periodic solution occurs.

Further, the Turing instability still occur when parameters are taken as $\mu = 7.86$, $c = 23.58$, $D_a = 2$, $D_h = 10.6$, then $\mu > \mu_0 \approx 7.8541$, $D_a < \mu_0$ and $D_h > \bar{r}$, so (2.4) is satisfied and some eigenvalues of L have positive real parts. The limit

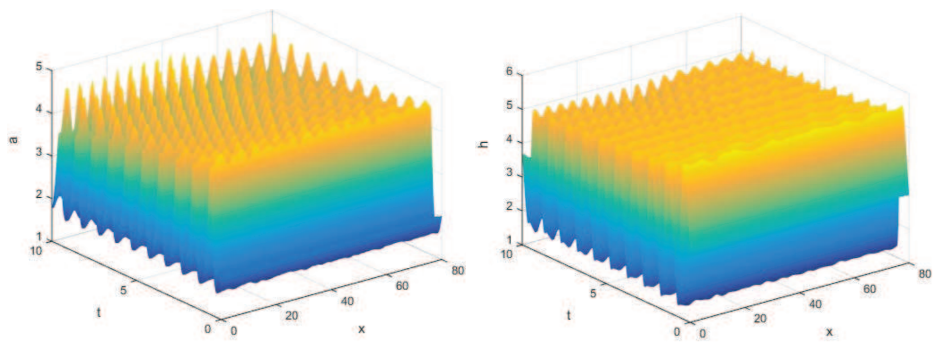

FIGURE 9. Turing instability of bifurcated periodic solution occurs under diffusion.

cycle from the Hopf bifurcation is still stable for system (2.1), but it is unstable for system (1.3), see Figure 9. Initial values are taken as $a(0,t) = 2.6 + 0.5\,\mathrm{rand}(1)$, $h(0,t) = 3.1 + 0.5\,\mathrm{rand}(1)$, close to the limit cycle.

From the numerical simulations in Figures 7–9, it shows that some patterns like stripe and spike solutions appear in system (1.3). Figure 10 corresponds to the cases in Figures 7 and 8. Figure 11 is the pattern in Figure 9.

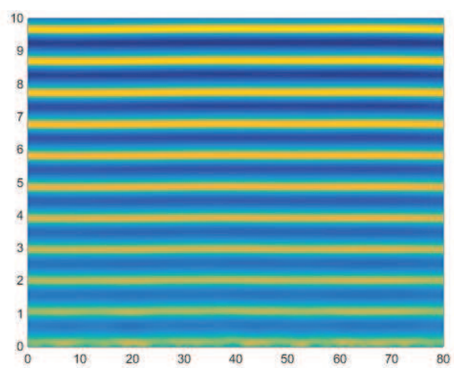

FIGURE 10. Patterns appearing in system (1.3).

Conclusions. To understand the dynamical behavior under diffusive effects, we consider the Gierer-Meinhardt depletion model here. The model is commonly used to explain the underlying complex mechanism for pattern formation in nature, describing the interaction of two sources in processes such as biological and chemical ones. If the equilibrium points and periodic solutions become unstable under diffusion terms, it is said to be the Turing instability, namely, the diffusion-driven instability. It is frequently noted that on such occasions some patterns will form in the system. By dynamical techniques, stability and the Hopf bifurcation of fixed points are analyzed in detail. Afterwards, it shows that the Turing instability will occur under some conditions, with the diffusive effects on stable fixed points and the stable bifurcated limit cycle. That implies the spatially nonhomogeneous solutions

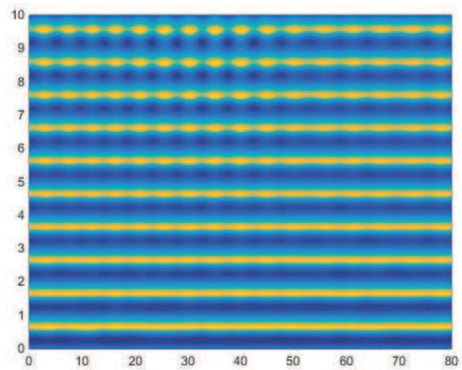

Figure 11. Patterns appearing in system (1.3).

will appear in the system, which will cause the formation of patterns. Numerical simulations verify the effectiveness of theoretical analysis. The other complex and interesting dynamical behaviors related to such model will be further investigated.

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Ranchao Wu
School of Mathematics, Anhui University, Hefei 230601, China
E-mail address: `rcwu@ahu.edu.cn`

Yan Shao
School of Mathematics, Anhui University, Hefei 230601, China
E-mail address: `yanshao28@126.com`

Yue Zhou
School of Mathematics, Anhui University, Hefei 230601, China
E-mail address: `zhouyue1016@126.com`

Liping Chen
School of Electrical Engineering and Automation, Hefei University of Technology, Hefei 230009,China
E-mail address: `lip_chenhut@126.com`