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WELL-POSEDNESS AND EXPONENTIAL DECAY OF SOLUTIONS FOR A TRANSMISSION PROBLEM WITH DISTRIBUTED DELAY

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ABSTRACT. In this article, we consider a transmission problem in a bounded domain with a distributed delay in the first equation. Using a semigroup theorem, we prove the existence and uniqueness of global solution under suitable assumptions on the weight of damping and the weight of distributed delay. Also we establish the exponential stability of the solution by introducing a suitable Lyapunov functional.

1. INTRODUCTION

In this article, we study the transmission problem with a distributed delay,

$$\begin{gathered} u_{tt}(x,t) - au_{xx}(x,t) + \mu_1 u_t(x,t) + \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s)ds = 0, \quad x \in \Omega,\ t > 0, \\ v_{tt}(x,t) - bv_{xx}(x,t) = 0, \quad x \in (L_1, L_2),\ t \geq 0, \end{gathered} \tag{1.1}$$

under the boundary and the transmission conditions

$$\begin{gathered} u(0,t) = u(L_3,t) = 0, \\ u(L_i,t) = v(L_i,t), \quad i = 1,2, \\ au_x(L_i,t) = bv_x(L_i,t), \quad i = 1,2 \end{gathered} \tag{1.2}$$

and the initial conditions

$$\begin{gathered} u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad x \in (L_1, L_2), \\ u_t(x,-t) = f_0(x,-t), \quad x \in \Omega, \quad t \in (0, \tau_2) \end{gathered} \tag{1.3}$$

where $0 < L_1 < L_2 < L_3$, $\Omega = (0, L_1) \cup (L_2, L_3)$, a, b, μ_1 are positive constants, and the initial data $(u_0, u_1, v_0, v_1, f_0)$ belongs to suitable space. Moreover, $\mu_2 : [\tau_1, \tau_2] \to \mathbb{R}$ is a bounded function, where τ_1 and τ_2 are two real number satisfying $0 \leq \tau_1 < \tau_2$.

It is known that transmission problems happen frequently in applications where the domain is occupied by two or several materials, whose elastic properties are different, joined together over the whole of a surface. From the mathematical point of view, a transmission problem for wave propagation consists on a hyperbolic

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equation for which the corresponding elliptic operator has discontinuous coefficients, see [2, 6].

In absence of delay ($\mu_2(s) = 0$), the system (1.1)-(1.3) has been investigated in [2] by Bastaos and Raposo; for $\Omega = [0, L_1]$, they showed that the well-posedness and exponential stability of the total energy. Rivera and Oquendo [17] studied the transmission problem of viscoelastic waves and established that the dissipation produced by the viscoelastic part is strong enough to produce the exponential stability, no matter small its size is. Interested readers are referred to [12, 13, 14, 16], for more results concerning other types of transmission problems.

Introducing the delay term makes the problem different from those considered in the literatures. Delay effect arises in many applications depending not only on the present state but also on some past occurrences. It may turn a well-behaved system into a wild one. The presence of delay may be a source of instability. For example, it was shown in [5, 4, 8, 20, 21, 26] that an arbitrarily small delay may destabilize a system that is uniformly asymptotically stable in the absence of delay unless additional control terms have been used. Here we mention the some interesting results on the relation between the delay term and source term [11, 10, 7, 23].

Nicaise and Pignotti [21] considered the wave equation with liner frictional damping and internal distributed delay

$$u_{tt} - \Delta u + \mu_1 u_t + a(x) \int_{\tau_1}^{\tau_2} \mu_2(s) u_t(t-s) ds = 0$$

in $\Omega \times (0, \infty)$, with initial and mixed Dirichlet-Neumann boundary conditions and a is a suitable function. They obtained exponential decay of the solution under the assumption that

$$\|a\|_\infty \int_{\tau_1}^{\tau_2} \mu_2(s) ds < \mu_1.$$

The authors also obtained the same result when the distributed delay acted on the part of the boundary. Mustafa and Kafini [19] considered a thermoelastic system with internal distributed delay, they obtained exponential stability under suitable condition; for the boundary distributed delay, similar result was obtained by [18]. Here we also mention the work on Timoshenko system with second sound and internal distributed delay in [1] by Apalara, and wave equation with strong distributed delay [15] by Messsaoudi et al.

The effect of the delay term $u_t(x, t-\tau)$ in the transmission system has been investigated by Benseghir [3]. Recently, the well-posedness and the decay of solution for a transmission problem in a bounded domain with a viscoelastic term and a delay term $u_t(x, t-\tau)$ have been studied in [9, 25].

In this work we consider the transmission system (1.1)-(1.3), and prove the well-posedness and the exponential stability. Our work extends the stability results in [2, 3] to the transmission system with distributed delay.

The plan of this paper is as follows. In section 2, we present some notations and assumptions needed for our work, and then establish the well-posedness of our problem by virtue of the semigroup methods. In section 3, we state and prove the stability result by introducing a suitable Lyapunov function.

2. Well-posedness of the problem

Throughout this paper, c and c_i are used to denote the generic positive constant. From now on, we shall omit x and t in all functions of x and t if there is no ambiguity.

As in [21], we introduce the new variable

$$z(x,\rho,t,s)=u_t(x,t-\rho s),\quad x\in\Omega,\ \rho\in(0,1),\ t>0,\ s\in(\tau_1,\tau_2).$$

Then the above variable z satisfies

$$sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0,\quad x\in\Omega,\ \rho\in(0,1),\ t>0,\ s\in(\tau_1,\tau_2). \tag{2.1}$$

Consequently, system (1.1) is equivalent to

$$\begin{gathered}
u_{tt}(x,t)-au_{xx}(x,t)+\mu_1u_t(x,t)+\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)ds=0,\\
x\in\Omega,\ t>0,\\
v_{tt}(x,t)-bv_{xx}(x,t)=0,\quad x\in(L_1,L_2),\ t\ge 0,\\
sz_t(x,\rho,t,s)+z_\rho(x,\rho,t,s)=0,\quad x\in\Omega,\ \rho\in(0,1),\ t>0,\ s\in(\tau_1,\tau_2).
\end{gathered} \tag{2.2}$$

Defining $U=(u,v,\varphi,\psi,z)^T$, we formally get that U satisfies

$$\begin{gathered}
U'=\mathcal{A}U,\\
U(0)=U_0=(u_0,v_0,u_1,v_1,f_0),
\end{gathered} \tag{2.3}$$

where the operator $\mathcal{A}$ is defined as

$$\mathcal{A}\begin{pmatrix}u\\ v\\ \varphi\\ \psi\\ z\end{pmatrix}=\begin{pmatrix}\varphi\\ \psi\\ au_{xx}-\mu_1\varphi-\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)ds\\ bv_{xx}\\ -\frac{1}{s}z_\rho(x,\rho,t,s)\end{pmatrix}.$$

Introducing the space

$$\begin{aligned}
X_*=\big\{(u,v)=H^1(\Omega)\cap H^1(L_1,L_2):u(0,t)=u(L_3,t)=0,\\
u(L_i,t)=v(L_i,t),\ au_x(L_i,t)=bv_x(L_i,t),\ i=1,2\big\},
\end{aligned}$$

we define the energy space as

$$\mathcal{H}=X_*\times L^2(\Omega)\times L^2(L_1,L_2)\times L^2\big(\Omega\times(0,1)\times(\tau_1,\tau_2)\big)$$

equipped with the inner product

$$\begin{aligned}
\langle U,\tilde U\rangle_{\mathcal{H}}&=\int_\Omega(\varphi\tilde\varphi+au_x\tilde u_x)dx+\int_{L_1}^{L_2}(\psi\tilde\psi+bv_x\tilde v_x)dx\\
&\quad+\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2}s|\mu_2(s)|z(x,\rho,s)\tilde z(x,\rho,s)\,ds\,d\rho\,dx.
\end{aligned}$$

The domain of $\mathcal{A}$ is

$$\begin{aligned}
D(\mathcal{A})=\big\{&(u,v,\varphi,\psi,z)^T\in\mathcal{H}:(u,v)\in(H^2(\Omega)\times H^2(L_1,L_2))\cap X_*,\\
&\varphi\in H^1(\Omega),\ \psi\in H^1(L_1,L_2),z(x,0,s)=\varphi,\\
&z,z_\rho\in L^2\big(\Omega\times(0,1)\times(\tau_1,\tau_2)\big)\big\}.
\end{aligned}$$

Clearly, $D(\mathcal{A})$ is dense in $\mathcal{H}$.

Concerning the weight of the distributed delay, we assume that

$$\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \leq \mu_1. \tag{2.4}$$

The well-posedness of the system (2.2), (1.2)and (1.3) is ensured by the following theorem.

Theorem 2.1. *Under the assumption* (2.4)*, for any* $U_0 \in \mathcal{H}$*, there exists a unique weak solution* $U \in C(\mathbb{R}^+, \mathcal{H})$ *of problem* (2.3)*. Moreover, if* $U_0 \in D(\mathcal{A})$*, then* $U \in C(\mathbb{R}^+, D(\mathcal{A})) \cap C(\mathbb{R}^+, \mathcal{H})$*.*

Proof. We use the semigroup approach and the Hille-Yosida theorem to prove the well-posedness of the problem. First, we prove that the operator $\mathcal{A}$ is dissipative. Indeed, for $U = (u, v, \varphi, \psi, z) \in D(\mathcal{A})$, where $\varphi(L_i) = \psi(L_i), i = 1, 2$, we have

$$\begin{aligned} &\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \\ &= \int_\Omega \big(au_{xx} - \mu_1\varphi - \int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, t, s) ds\big)\varphi \, dx + a\int_\Omega u_x\varphi_x dx \\ &\quad + \int_{L_1}^{L_2} bv_{xx}\psi \, dx + \int_{L_1}^{L_2} bv_x\psi_x \, dx + \int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)| zz_\rho d\rho \, ds \, dx. \end{aligned} \tag{2.5}$$

For the last term of the right hand side of (2.5), we have

$$\begin{aligned} &\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)| zz_\rho d\rho \, ds \, dx \\ &= \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2}\int_0^1 |\mu_2(s)| \frac{d}{d\rho} |z(x, \rho, t, s)|^2 d\rho \, ds \, dx \\ &\quad + \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) dsx - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_\Omega z^2(x, 0, s) dx. \end{aligned} \tag{2.6}$$

Integrating by parts in (2.5), and noticing the fact $z(x, 0, t, s) = \varphi(x, t)$, from (2.6), we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= [au_x\varphi]_{\partial\Omega} + [bv_x\psi]_{L_1}^{L_2} - \Big(\mu_1 - \frac{1}{2}\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big)\int_\Omega \varphi^2 dx \\ &\quad + \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) \, ds \, dx - \int_\Omega\int_{\tau_1}^{\tau_2} \mu_2(s) z(x, 1, s)\varphi \, ds \, dx. \end{aligned}$$

Using Young's inequality, and the equality $\varphi(L_i) = \psi(L_i)$, $i = 1, 2$, from (1.2) and (2.6) we have

$$\begin{aligned} \langle \mathcal{A}U, U\rangle_{\mathcal{H}} &\leq -(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|) \int_\Omega \varphi^2 dx - \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) \, ds \, dx \\ &\quad + \frac{1}{2}\int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x, 1, s) \, ds \, dx \\ &\leq -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|\Big)\int_\Omega \varphi^2 dx \leq 0, \end{aligned}$$

by (2.4). Hence, the operator $\mathcal{A}$ is dissipative.

Next, we prove the operator $\mathcal{A}$ is maximal. It is sufficient to show that the operator $\lambda I-\mathcal{A}$ is surjective for a fixed $\lambda > 0$. Indeed, given $\mathcal{F} = (f_1, f_2, f_3, f_4, f_5) \in \mathcal{H}$, we prove that there exists $U = (u, v, \varphi, \psi, z) \in D(\mathcal{A})$ satisfying

$$(\lambda I - \mathcal{A})U = \mathcal{F}, \tag{2.7}$$

that is

$$\begin{gathered} \lambda u - \varphi = f_1, \\ \lambda v - \psi = f_2, \\ \lambda\varphi - au_{xx} + \mu_1\varphi + \int_{\tau_1}^{\tau_2} |\mu_2(s)| z(x, 1, t, s) ds = f_3, \\ \lambda\psi - bv_{xx} = f_4, \\ \lambda sz + z_\rho = sf_5. \end{gathered} \tag{2.8}$$

Suppose we have obtained (u, v) with the suitable regularity, then

$$\begin{gathered} \varphi = \lambda u - f_1, \\ \psi = \lambda v - f_2, \end{gathered} \tag{2.9}$$

so we have $\varphi \in H^1(\Omega)$ and $\psi \in H^1(L_1, L_2)$. Moreover, using the approach as in Nicaise and Pignotti [20], we obtain that the last equation in (2.8) with $z(x, 0, s)$ has a unique solution

$$z(x, \rho, s) = \varphi(x)e^{-\lambda\rho s} + se^{\lambda\rho s} \int_0^\rho e^{\lambda\sigma s} f_5(x, \sigma, s) d\sigma.$$

It follows from (2.9) that

$$z(x, \rho, s) = \lambda ue^{-\lambda\rho s} - f_1e^{-\lambda\rho s} + se^{\lambda\rho s} \int_0^\rho e^{\lambda\sigma s} f_5(x, \sigma, s) d\sigma, \tag{2.10}$$

in particular, $z(x, 1, s) = \lambda ue^{-\lambda s} + z_0(x, s)$ with $z_0 \in L^2(\Omega \times (\tau_1, \tau_2))$ defined by

$$z_0(x, s) = -f_1e^{-\lambda s} + se^{\lambda s} \int_0^1 e^{\lambda\sigma s} f_5(x, \sigma, s) d\sigma.$$

By (2.8) and (2.9), the functions (u, v) satisfy the equations

$$\begin{gathered} \tilde{k}u - au_{xx} = \tilde{f}, \\ \lambda^2 v - bv_{xx} = f_2 + \lambda f_4, \end{gathered} \tag{2.11}$$

where

$$\begin{gathered} \tilde{k} = \lambda^2 + \lambda\mu_1 + \int_{\tau_1}^{\tau_2} \lambda|\mu_2(s)|e^{-\lambda s} ds > 0, \\ \tilde{f} = f_3 + (\lambda + \lambda\mu_1)f_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| z_0(x, s) ds \in L^2(\Omega), \end{gathered}$$

which can be reformulated as

$$\begin{gathered} \int_\Omega (\tilde{k}u - au_{xx})w_1 dx = \int_\Omega \tilde{f} w_1 dx, \\ \int_{L_1}^{L_2} (\lambda^2 v - bv_{xx})w_2 dx = \int_{L_1}^{L_2} (f_2 + \lambda f_4)w_2 dx, \end{gathered} \tag{2.12}$$

for any $(w_1, w_2) \in X_*$.

Integrating by parts in (2.12), we obtain that the variational formulation corresponding to (2.11) takes the form

$$\Phi\big((u,v),(w_1,w_2)\big)=l(w_1,w_2), \tag{2.13}$$

where the bilinear form $\Phi:(X_*,X_*)\to\mathbb{R}$ and the linear form $l:X_*\to\mathbb{R}$ are defined by

$$\begin{aligned}\Phi\big((u,v),(w_1,w_2)\big)&=\int_\Omega \tilde{k}uw_1dx+\int_\Omega au_xw_{1x}-[au_xw_1]_\Omega+\int_{L_1}^{L_2}\lambda^2vw_2dx\\&\quad+\int_{L_1}^{L_2}v_xw_{2x}dx-[bv_xw_2]_{L_1}^{L_2},\end{aligned}$$

and

$$l(w_1,w_2)=\int_\Omega \tilde{f}w_1dx+\int_{L_1}^{L_2}(f_2+\lambda f_4)w_2dx.$$

By the properties of the space X_*, it is easy to see that Φ is continuous and coercive, and l is continuous. Applying the Lax-Milgram theorem, we deduce that problem l (2.13) admits a unique solution $(u,v)\in X_*$ for all $(w_1,w_2)\in X_*$. It follows from (2.11) that $(u,v)\in\big((H^2(\Omega)\times H^2(L_1,L_2))\big)\cap X_*$. Thus, the operator $\lambda I-\mathcal{A}$ is surjective for any $\lambda>0$. Hence the Hille-Yosida theorem guarantees the existence of a unique solution to the problem (2.7). This completes the proof. $\square$

3. Exponential stability

In this section, we state and prove the stability result for the energy of the system (1.1)-(1.3). For the regular solution of the system (1.1)-(1.3), we define the energy as (see [3])

$$E_1(t)=\frac{1}{2}\int_\Omega u_t^2(x,t)dx+\frac{a}{2}\int_\Omega u_x^2(x,t)dx, \tag{3.1}$$

$$E_2(t)=\frac{1}{2}\int_{L_1}^{L_2} v_t^2(x,t)dx+\frac{b}{2}\int_{L_1}^{L_2} v_x^2(x,t)dx. \tag{3.2}$$

And the total energy is defined as

$$E(t)=E_1(t)+E_2(t)+\frac{1}{2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)|z^2(x,\rho,t,s)\,ds\,d\rho\,dx. \tag{3.3}$$

For the energy decay result, we assume a restriction on the weight of the distribute delay and the damping as

$$\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds<\mu_1. \tag{3.4}$$

The stability result reads as follows.

Theorem 3.1. *Let (u,v,z) be the solution of the system (2.2), (1.2) and (1.3). Assume (3.4) and*

$$\frac{a}{b}<\frac{L_1+L_3-L_2}{2(L_2-L_1)},\quad L_3>3(L_2-L_1). \tag{3.5}$$

Then there exist two positive constants K and κ, such that

$$E(t)\le Ke^{-\kappa t},\quad \forall t\ge 0. \tag{3.6}$$

The proof will be established through the following Lemmas.

Lemma 3.2. *Let assumption* (3.4) *holds. Then the energy functional defined by* (3.3)*, satisfies the estimate*

$$E'(t) \le -\Big(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds\Big) \int_\Omega u_t^2(x,t)dx \le 0. \tag{3.7}$$

Proof. By differentiating (3.1), using the first equation in (2.2), and integrating by parts, we obtain

$$E_1'(t) = [au_x u_t]_\Omega - \mu_1 \int_\Omega u_t^2(x,t)dx - \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) u_t(x,t)\, ds\, dx.$$

Similarly,

$$E_2'(t) = [bv_x v_t]_{L_1}^{L_2}.$$

Noticing that $z(x,0,t,s) = u_t(x,t)$, from (2.2), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx \\
&= -\frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx + \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| u_t^2(x,t)\, ds\, dx.
\end{aligned}$$

Meanwhile, using Young's inequality, we have

$$\begin{aligned}
&- \int_\Omega \int_{\tau_1}^{\tau_2} \mu_2(s) z(x,1,t,s) u_t(x,t)\, ds\, dx \\
&\le \frac{1}{2} \int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \int_\Omega u_t^2(x,t) dx + \frac{1}{2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx.
\end{aligned}$$

Combining the above equalities and using (3.4), we show that (3.7) holds, where we also use the fact $[au_x u_t]_{\partial\Omega} = [bv_x v_t]_{L_1}^{L_2}$ from (1.2). □

As in [15], we define the functional

$$I(t) = \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s e^{-\rho s} |\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx,$$

then we have the following estimate.

Lemma 3.3. *The functional $I(t)$ satisfies the estimate*

$$\begin{aligned}
I'(t) \le& -e^{-\tau_2} \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\, ds\, dx \\
&+ \Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)| ds \Big) \int_\Omega u_t^2(x,t) dx \\
&- e^{-\tau_2} \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\, ds\, d\rho\, dx.
\end{aligned} \tag{3.8}$$

Proof. By differentiating $I(t)$ and using the third equation in (2.2), we obtain

$$\begin{aligned}
I'(t) &= - \int_\Omega \int_{\tau_1}^{\tau_2} \int_0^1 e^{-\rho s} |\mu_2(s)| \frac{d}{d\rho} z^2(x,\rho,t,s) d\rho\, ds\, dx \\
&= - \int_\Omega \int_{\tau_1}^{\tau_2} |\mu_2(s)| \int_0^1 \frac{d}{d\rho} (e^{-\rho s} z^2(x,\rho,t,s)) d\rho\, ds\, dx \\
&\quad - \int_\Omega \int_0^1 \int_{\tau_1}^{\tau_2} s|\mu_2(s)| e^{-\rho s} z^2(x,\rho,t,s)\, ds\, d\rho\, dx.
\end{aligned}$$

Hence

$$\begin{aligned} I'(t) = &-\int_\Omega \int_{\tau_1}^{\tau_2} e^{-s}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx + \Big(\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds\Big)\int_\Omega u_t^2(x,t)dx \\ &- \int_\Omega \int_{\tau_1}^{\tau_2} s|\mu_2(s)|\int_0^1 e^{-\rho s} z^2(x,rho,t,s)\,d\rho\,ds\,dx. \end{aligned}$$

Recalling $e^{-s} \le e^{-\rho s} \le 1$, for all $\rho \in [0,1]$, and $-e^{-s} \le -e^{-\tau_2}$, for all $s \in [\tau_1, \tau_2]$, we obtain (3.8). □

Now we define the functional

$$\mathcal{D}(t) = \int_\Omega u u_t dx + \frac{\mu_1}{2}\int_\Omega u^2 dx + \int_{L_1}^{L_2} v v_t dx.$$

Then we have the following estimate.

Lemma 3.4. *The functional $\mathcal{D}(t)$ satisfies*

$$\begin{aligned} \mathcal{D}'(t) \le &-(a - \varepsilon_0 C_0^2)\int_\Omega u_x^2 dx - b\int_{L_1}^{L_2} v_x^2 dx + \int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx \\ &+ \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds \int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx. \end{aligned} \tag{3.9}$$

Proof. Taking the derivative of $\mathcal{D}(t)$ with respect to t, using (2.2), we obtain

$$\begin{aligned} \mathcal{D}'(t) = &\int_\Omega u_t^2 dx + \int_{L_1}^{L_2} v_t^2 dx - a\int_\Omega u_x^2 dx - \int_{L_1}^{L_2} v_x^2 dx \\ &- \int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)u(x,t)\,ds\,dx + [a u_x u]_{\partial\Omega} + [b v_x v]_{L_1}^{L_2}. \end{aligned} \tag{3.10}$$

It follows from the boundary condition (1.2) that

$$[a u_x u]_{\partial\Omega} + [b v_x v]_{L_1}^{L_2} = 0.$$

Using the boundary condition (1.2), we obtain

$$\begin{gathered} u^2(x,t) = \big(\int_0^x u_x(x,t)dx\big)^2 \le L_1\int_0^{L_1} u_x^2(x,t)dx, \quad x \in [0, L_1], \\ u^2(x,t) \le (L_3 - L_2)\int_{L_2}^{L_3} u_x^2(x,t)dx, \;\; x \in [L_2, L_3], \end{gathered}$$

which imply the following Poincaré's inequality

$$\int_\Omega u^2(x,t)dx \le C_0^2\int_\Omega u_x^2(x,t)dx, \quad x \in \Omega, \tag{3.11}$$

where $C_0 = \max\{L_1, L_3 - L_2\}$ is the Poincaré's constant. Using Young's inequality and (3.11), we have

$$\begin{aligned} &-\int_\Omega\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)u(x,t)\,ds\,dx \\ &\le \varepsilon_0 C_0^2\int_\Omega u_x^2(x,t)dx + \frac{1}{4\varepsilon_0}\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx, \end{aligned}$$

for any $\varepsilon_0 > 0$. Inserting the above estimates in (3.10), then (3.9) is fulfilled. □

Inspired by [13], we introduce the functional

$$q(x)=\begin{cases} x-\frac{L_1}{2}, & x\in[0,L_1],\\ x-\frac{L_2+L_3}{2}, & x\in[L_2,L_3],\\ \frac{L_1}{2}+\frac{L_2-L_3-L_1}{2(L_2-L_1)}(x-L_1), & x\in[L_1,L_2]. \end{cases} \tag{3.12}$$

We define the two functionals

$$\mathcal{F}_1(t)=-\int_\Omega q(x)u_xu_t\,dx,\quad \mathcal{F}_2(t)=-\int_{L_1}^{L_2} q(x)v_xv_t\,dx.$$

Then, we have the following estimates.

Lemma 3.5. *For any $\varepsilon_1>0$, the functionals $\mathcal{F}(t)$ and $\mathcal{F}_2(t)$ satisfy*

$$\begin{aligned} \mathcal{F}_1'(t)&\le C(\varepsilon_1)\int_\Omega u_t^2dx+(\frac{a}{2}+\varepsilon_1)\int_\Omega u_x^2dx\\ &\quad+C(\varepsilon_1)\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx\\ &\quad-\frac{a}{4}[(L_3-L_2)u_x^2(L_2,t)+L_1u_x^2(L_1,t)]-\frac{1}{4}[L_1u_t^2(L_1,t)+(L_3-L_2)u_t^2(L_2,t)], \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} \mathcal{F}_2'(t)&=-\frac{L_1+L_3-L_2}{4(L_2-L_1)}\Big(\int_{L_1}^{L_2}v_t^2dx+\int_{L_1}^{L_2}bv_x^2dx\Big)+\frac{L_1}{4}v_t^2(L_1,t)\\ &\quad+\frac{L_3-L_2}{4}v_t^2(L_2,t)+\frac{b}{4}[(L_3-L_2)v_x^2(L_2,t)+L_1v_x^2(L_1,t)]. \end{aligned} \tag{3.14}$$

Proof. Taking the derivative of $\mathcal{F}_1(t)$ with respect to t and using (2.2), we have

$$\begin{aligned} \mathcal{F}_1'(t)&=-\int_\Omega q(x)u_xu_{tt}-\int_\Omega q(x)u_{xt}u_t\,dx\\ &=-\int_\Omega q(x)u_x\Big(au_{xx}-\mu_1u_t-\int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)ds\Big)\,dx\\ &\quad-\int_\Omega q(x)u_{xt}u_t\,dx. \end{aligned} \tag{3.15}$$

Integrating by parts, we have

$$\begin{aligned} \int_\Omega q(x)u_{xt}u_tdx&=-\frac{1}{2}\int_\Omega q'(x)u_t^2dx+\frac{1}{2}[q(x)u_t^2]_{\partial\Omega},\\ \int_\Omega q(x)au_xu_{xx}dx&=-\frac{1}{2}\int_\Omega aq'(x)u_x^2dx+\frac{1}{2}[aq(x)u_x^2]_{\partial\Omega}. \end{aligned}$$

Inserting the above two equalities into (3.15), and noticing (3.12) and Young's inequality, we obtain

$$
\begin{aligned}
\mathcal{F}_1'(t) &= \frac{1}{2}\int_\Omega u_t^2 dx + \frac{1}{2}\int_\Omega u_x^2 dx - \frac{1}{2}[aq(x)u_x^2]_{\partial\Omega} \\
&\quad - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} + \int_\Omega q(x)u_x\big(\mu_1 u_t + \int_{\tau_1}^{\tau_2}\mu_2(s)z(x,1,t,s)ds\big)dx \\
&\le C(\varepsilon_1)\int_\Omega u_t^2 dx + (\frac{a}{2}+\varepsilon_1)\int_\Omega u_x^2 dx - \frac{1}{2}[aq(x)u_x^2]_{\partial\Omega} - \frac{1}{2}[q(x)u_t^2]_{\partial\Omega} \\
&\quad + C(\varepsilon_1)\int_{\tau_1}^{\tau_2}|\mu_2(s)|ds\int_\Omega\int_{\tau_1}^{\tau_2}|\mu_2(s)|z^2(x,1,t,s)\,ds\,dx,
\end{aligned}
\tag{3.16}
$$

for any $\varepsilon_1 > 0$. On the other hand, by the boundary conditions (1.2), we have

$$
\begin{gathered}
\frac{1}{2}[q(x)u_t^2]_{\partial\Omega} = \frac{1}{4}[L_1 u_t^2(L_1,t) + (L_3 - L_2)u_t^2(L_2,t)] \ge 0, \\
\frac{1}{2}[aq(x)u_x^2]_{\partial\Omega} = \frac{a}{4}[(L_3 - L_2)u_x^2(L_2,t) + L_1 u_x^2(L_1,t)].
\end{gathered}
$$

Inserting the above two equalities into (3.16), then (3.16) gives (3.13).

By the same method, taking the derivative of $\mathcal{F}_2(t)$ with respect to t, we have

$$
\begin{aligned}
\mathcal{F}_2'(t) &= -\int_{L_1}^{L_2} q(x)v_{xt}v_t dx - \int_{L_1}^{L_2} q(x)v_x v_{tt} dx \\
&= \frac{1}{2}\int_{L_1}^{L_2} q'(x)v_t^2 dx - \frac{1}{2}[q(x)v_t^2]_{L_1}^{L_2} + \frac{1}{2}\int_{L_1}^{L_2} bq'(x)v_x^2 dx - \frac{1}{2}[bq(x)v_x^2]_{L_1}^{L_2} \\
&= -\frac{L_1 + L_3 - L_2}{4(L_2 - L_1)}\big(\int_{L_1}^{L_2} v_t^2 dx + \int_{L_1}^{L_2} bv_x^2 dx\big) + \frac{L_1}{4}v_t^2(L_1,t) \\
&\quad + \frac{L_3 - L_2}{4}v_t^2(L_2,t) + \frac{b}{4}[(L_3 - L_2)v_x^2(L_2,t) + L_1 v_x^2(L_1,t)].
\end{aligned}
$$

Hence, the proof is complete. □

Proof of Theorem 3.1. We define the Lyapunov functional

$$
L(t) = N_1 E(t) + N_2 I(t) + \gamma_1\mathcal{F}_1(t) + \gamma_2\mathcal{F}_2(t) + \gamma_3\mathcal{D}(t), \tag{3.17}
$$

where $N_1, N_2, \gamma_1, \gamma_2, \gamma_3$ are positive constants that will be chosen later.

It follows from the boundary conditions (1.2) that

$$
a^2 u_x^2(L_i,t) = bv_x^2(L_i,t), \quad i = 1,2. \tag{3.18}
$$

Taking the derivative of (3.17) with respective to t, using the above lemmas and (3.18), we have

$$
\begin{aligned}
&L'(t)\\
&\le -\Big\{N_1(\mu_1 - \int_{\tau_1}^{\tau_2} |\mu_2(s)|ds) - N_2\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds - \gamma_1 C(\varepsilon_1) - \gamma_3\Big\}\int_\Omega u_t^2 dx\\
&\quad - \Big\{N_2 e^{-\tau_2} - \gamma_1 C(\varepsilon_1)\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds - \gamma_3 \frac{\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds}{4\varepsilon_0}\Big\}\\
&\quad \times \int_\Omega\int_{\tau_1}^{\tau_2} |\mu_2(s)| z^2(x,1,t,s)\,ds\,dx\\
&\quad - \big\{(a-\varepsilon_0 C_0^2)\gamma_3 - (\frac{a}{2}+\varepsilon_1)\gamma_1\big\}\int_\Omega u_x^2 dx\\
&\quad - \big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 + \gamma_3\big\}\int_{L_1}^{L_2} b v_x^2 dx\\
&- \big\{\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 - \gamma_3\big\}\int_{L_1}^{L_2} v_t^2 dx\\
&\quad - N_2 e^{-\tau_2}\int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx\\
&\quad - \big\{\gamma_1-\gamma_2\big\}\big(\frac{L_1}{4}u_t^2(L_1,t) + \frac{L_3-L_2}{4}u_t^2(L_2,t)\big)\\
&\quad - \big\{\gamma_1 - \frac{a}{b}\gamma_2\big\}\big(\frac{a}{4}[L_1 u_x^2(L_1,t) + (L_3-L_2)u_x^2(L_2,t)]\big).
\end{aligned}
\tag{3.19}
$$

At this point we will choose all the constants, carefully, such that all the coefficients in (3.19) will be negative. In fact, it follows from the assumption (3.5) that we can always choose γ_1,γ_2 and γ_3 such that

$$
\frac{L_1+L_3-L_2}{4(L_2-L_1)}\gamma_2 - \gamma_3 > 0, \quad \gamma_1 > \frac{a}{b}\gamma_2, \quad \gamma_1 > \gamma_2, \quad \gamma_3 > \frac{\gamma_1}{2}.
$$

Once the above constants $\gamma_1,\gamma_2,\gamma_3$ are fixed, we may choose ε_0 and ε_1 sufficiently small such that

$$
\gamma_3\varepsilon_0 C_0^2 + \gamma_1\varepsilon_1 < a(\gamma_3 - \frac{\gamma_1}{2}).
$$

Then we can take N_2 sufficiently large such that

$$
N_2 e^{-\tau_2} - \gamma_1 C(\varepsilon_1)\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds - \gamma_3\frac{\int_{\tau_1}^{\tau_2} |\mu_2(s)|ds}{4\varepsilon_0} > 0.
$$

Finally, noticing the assumption (3.4), we can always choose N_1 sufficiently large such that the first coefficient in (3.19) is negative.

Thus, we obtain that there exists a positive constant α such that (3.19) yields

$$
\begin{aligned}
L'(t) \le -\alpha\Big(&\int_\Omega u_t^2 dx + \int_\Omega a u_{xx}^2 dx + \int_{L_1}^{L_2} v_t^2 dx\\
&+ \int_{L_1}^{L_2} b v_{xx}^2 dx + \int_\Omega\int_0^1\int_{\tau_1}^{\tau_2} s|\mu_2(s)| z^2(x,\rho,t,s)\,ds\,d\rho\,dx\Big),
\end{aligned}
$$

recalling (3.3), which implies

$$L'(t) \le -\frac{\alpha}{2}E(t), \quad \forall \ge 0. \tag{3.20}$$

On the hand, it is not hard to see that $L(t) \sim E(t)$, i.e. there exist two positive constants β_1 and β_2 such that

$$\beta_1 E(t) \le L(t) \le \beta_2 E(t), \quad t \ge 0. \tag{3.21}$$

Combining (3.20) and (3.21), we obtain that

$$L'(t) \le -\kappa L(t), \quad t \ge 0$$

for the positive constant $\kappa = \alpha/\beta_2$. Integration over $(0, t)$ gives

$$L(t) \le L(0)e^{-\kappa t}, \quad t \ge 0,$$

recall (3.21) again, then (3.6) holds. Hence, the proof is complete. □

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