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SIGN-CHANGING SOLUTIONS FOR NON-LOCAL ELLIPTIC EQUATIONS

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Abstract. This article concerns the existence of sign-changing solutions for equations driven by a non-local integrodifferential operator with homogeneous Dirichlet boundary conditions,

$$\begin{gathered} -\mathcal{L}_K u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \mathbb{R}^n \setminus \Omega, \end{gathered}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a bounded, smooth domain and the nonlinear term f satisfies suitable growth assumptions. By using Brouwer's degree theory and Deformation Lemma and arguing as in [2], we prove that there exists a least energy sign-changing solution. Our results generalize and improve some results obtained in [27].

1. Introduction

In recent years, fractional and non-local operators of elliptic type have been widely investigated. This type of operators arises in several areas such as anomalous diffusion, the thin obstacle problem, optimization, finance, phase transitions, stratified materials, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, quasi-geostrophic flows, multiple scattering and materials science. One can see [8, 9, 12, 14, 20, 21, 22, 23, 24, 25, 26, 28] and their references. Many publications [3, 4, 5, 6, 7, 10, 11, 13, 15, 17, 18, 19] are devoted to the study of the existence of sign-changing solutions of classical elliptic boundary value problems such as

$$\begin{gathered} -\Delta u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.1}$$

where $f \in C(\bar{\Omega} \times \mathbb{R}, \mathbb{R})$, $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$. There have been several methods developed in studying sign-changing solutions of nonlinear elliptic equations, such as the invariant sets of descending flow method developed by Liu and Sun [4, 15, 19], and the minimax method which is established by Berestycki and Lions in the classical paper [7]. Teng, Wang and Wang [27] established the existence of a least energy sign-changing solution for nonlinear

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problems involving the fractional Laplacian by using a constrained minimization method.

Ambrosio and Isernia [2] studied the fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = K(x)f(u) \quad \text{in } \mathbb{R}^n, \tag{1.2}$$

where $0 < s < 1$, $n > 2s$, $(-\Delta)^s$ is the fractional Laplacian operator, which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy.$$

By using a minimization argument and a quantitative deformation Lemma, Ambrosio and Isernia proved the existence of sign-changing solutions for (1.2).

Motivated by the above works, in this article, we study the non-local elliptic problems

$$\begin{gathered} -\mathcal{L}_K u = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \mathbb{R}^n \setminus \Omega, \end{gathered} \tag{1.3}$$

where f satisfies the following assumptions

(A1) $f \in C^1(\bar{\Omega} \times \mathbb{R}^n)$, $\lim_{\tau\to 0} \frac{f(x,\tau)}{\tau} = 0$, uniformly in $x \in \bar{\Omega}$;

(A2) $|f(x,\tau)| \le C(1+|\tau|^{p-1})$ for some $C > 0$ and $p \in (2, 2_s^*)$, where $2_s^* = \frac{2n}{n-2s}$;

(A3) There exists a constant $\mu > 2$ such that

$$0 < \mu F(x,\tau) \le \tau f(x,\tau), \quad \forall x \in \bar{\Omega},\ \tau \in \mathbb{R}\setminus\{0\},$$

where $F(x,\tau) = \int_0^\tau f(x,t)dt$;

(A4) For every $x \in \bar{\Omega}$ the function $\tau \mapsto \frac{f(x,\tau)}{|\tau|}$ is strictly increasing for all $|\tau| > 0$.

The non-local integrodifferential operator $\mathcal{L}_K$ is defined as follows

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y)+u(x-y)-2u(x))K(y)dy, \quad x \in \mathbb{R}^n,$$

where $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a function with the properties

(A5) $\gamma K \in L^1(\mathbb{R}^n)$, where $\gamma(x) = \min\{|x|^2, 1\}$;

(A6) there exists $\lambda > 0$ such that $K(x) \ge \lambda |x|^{-(n+2s)}$ for any $x \in \mathbb{R}^n \setminus \{0\}$.

A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case $\mathcal{L}_K$ is the fractional Laplace operator $-(-\Delta)^s$.

To prove the existence of sign-changing solutions for (1.3), we argue exactly as in [2], where the authors deal with fractional Schrödinger equations.

We remark that the Dirichlet datum is given in $\mathbb{R}^n \setminus \Omega$ and not simply on $\partial\Omega$, consistent with the non-local character of the operator $\mathcal{L}_K$.

As is explained in [7, Remark 9.2], the minimax method of Berestycki and Lions strongly depends on a kind of nodal structure associated with equation (1.1), which is unknown for problem (1.3). The variational methods used in [4, 17] heavily rely on the following decomposition, for $u \in H_0^1(\Omega)$,

$$\begin{gathered} \langle \Phi'(u), u^+\rangle = \langle \Phi'(u^+), u^+\rangle, \quad \langle \Phi'(u), u^-\rangle = \langle \Phi'(u^-), u^-\rangle, \\ \Phi(u) = \Phi(u^+) + \Phi(u^-), \end{gathered} \tag{1.4}$$

where $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$ and Φ is the energy functional of (1.1) given by

$$\Phi(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega F(x,u)dx.$$

But for problem (1.3) and $u \in X_0$ (the space X_0 was introduced in [20, 21], and will also be defined in Section 2), we have

$$I(u) = I(u^+) + I(u^-) - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)\,dx\,dy,$$

$$\langle I'(u), u^+\rangle = \langle I'(u^+), u^+\rangle - \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)\,dx\,dy,$$

where I is the energy functional of (1.3) given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^{2n}} |u(x) - u(y)|^2 K(x-y)\,dx\,dy - \int_{\Omega} F(x,u)dx.$$

Clearly, the functional I does no longer satisfy the decomposition (1.4). In this article, motivated by [4, 5, 17], we try to get sign-changing solutions for (1.3) by seeking the minimizer of the energy functional I over the following constraint:

$$\mathcal{M} := \{u \in \mathcal{N} : u^\pm \not\equiv 0, \langle I'(u), u^+\rangle = \langle I'(u), u^-\rangle = 0\},$$

where the set $\mathcal{N}$ defined as

$$\mathcal{N} := \{u \in X_0 \setminus \{0\} : \langle I'(u), u\rangle = 0\}.$$

Since $\mathcal{L}_K$ is nonlocal, we need some technical analysis to show that $\mathcal{M} \neq \emptyset$. Since we look for sign-changing solution to (1.3), it is natural to seek functions $w \in \mathcal{M}$ such that

$$I(w) = \inf_{v \in \mathcal{M}} I(v).$$

As in [1, 3], we are able to prove the existence of a minimizer of I on $\mathcal{M}$ and that it is a weak solution to (1.3) by using a suitable deformation argument. Now, we are ready to state the main results of this paper.

Theorem 1.1. *Suppose that* (A1)–(A4) *hold. Then problem* (1.3) *admits a least energy sign-changing solution.*

For equation (1.1), we can follow the argument of [4] to show that the least energy sign-changing solution has exactly two nodal domains. But in this framework, because there is a nonlocal term $\mathcal{L}_K u$, we can not get the same result.

The rest of this article is organized as follows. In Section 2, we prove some lemmas, which are crucial to investigate our main result. The proof of Theorem 1.1 is given in Section 3.

2. Preliminaries

Recall that the space X introduced by Servadei and Valdinoci [20, 21, 22, 23, 24, 25] is defined as a linear space of Lebesgue measurable functions from $\mathbb{R}^n$ to $\mathbb{R}$, such that, any function u restricted in X belongs to $L^2(\Omega)$ and the map $(x,y) \mapsto (u(x) - u(y))\sqrt{K(x-y)}$ is in $L^2(\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega), dx\,dy)$, where $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. Moreover,

$$X_0 = \{u \in X : u = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\}.$$

The function space X is equipped with the norm

$$\|u\|_X = \Big(\|u\|^2_{L^2(\Omega)} + \int_Q |u(x) - u(y)|^2 K(x-y)\,dx\,dy\Big)^{1/2}, \tag{2.1}$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. For any $u \in X_0$, the space X_0 is endowed with the norm

$$\|u\|_{X_0} = \Big(\int_Q |u(x) - u(y)|^2 K(x-y)\, dx\, dy \Big)^{1/2},$$

which is equivalent to the usual one defined in (2.1). In the following, we also denote the norm $\|\cdot\|_{X_0}$ as $\|\cdot\|$.

For the reader's convenience, we review the main embedding results for the space X_0.

Lemma 2.1 ([20, 21, 22, 23, 24, 25])**.** *The embedding $X_0 \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [2, 2_s^*]$, and compact for any $r \in [2, 2_s^*)$.*

Now, we collect some preliminary lemmas which will be used in the last section to prove our main result.

Lemma 2.2. *Let $\{u_j\}$ be a sequence such that $u_j \rightharpoonup u$ in X_0, then, up to a subsequence,*

(i)
$$\lim_{j\to\infty} \int_\Omega f(x,u_j)u_j dx = \int_\Omega f(x,u)u dx,$$

(ii)
$$\lim_{j\to\infty} \int_\Omega F(x,u_j) dx = \int_\Omega F(x,u) dx,$$

(iii)
$$\begin{aligned} &\liminf_{j\to\infty} \int_{\mathbb{R}^{2n}} (-u_j^-(x)u_j^+(y) - u_j^-(y)u_j^+(x))K(x-y)\, dx\, dy \\ &\geq \int_{\mathbb{R}^{2n}} (-u^-(x)u^+(y) - u^-(y)u^+(x))K(x-y)\, dx\, dy. \end{aligned} \tag{2.2}$$

Proof. (i) By the compact embedding $X_0 \hookrightarrow L^p(\Omega)(2 \leq p < 2_s^*)$, taking if necessary a subsequence, we have $u_j \to u$ in $L^p(\Omega)$ and $u_j(x) \to u(x)$ a.e. on $\mathbb{R}^n$. By a standard discussion, there exists a function $g \in L^p(\Omega)$ such that

$$|u(x)|, |u_j(x)| \leq g(x).$$

By (A2) and $u \in L^p(\Omega)$, we have

$$|f(x,u)|^{\frac{p}{p-1}} \leq C^{\frac{p}{p-1}}(1 + |u|^{p-1})^{\frac{p}{p-1}} \leq C^{\frac{p}{p-1}} 2^{\frac{p}{p-1}}(1 + |u|^p) \in L^1(\Omega),$$

it follows that $f(\cdot,u) \in L^{\frac{p}{p-1}}(\Omega)$. Since

$$|f(x,u_j) - f(x,u)|^{\frac{p}{p-1}} \leq 2^{\frac{p}{p-1}} C^{\frac{p}{p-1}}(1 + |g|^{p-1})^{\frac{p}{p-1}} \in L^1(\Omega),$$

it follows from the Lebesgue dominated convergence theorem that

$$\int_\Omega |f(x,u_j) - f(x,u)|^{\frac{p}{p-1}} dx \to 0, \quad \text{as } j \to \infty.$$

By the Hölder inequality, we have

$$\begin{aligned} \int_\Omega (f(x,u_j) - f(x,u))u_j dx &\leq \Big(\int_\Omega |f(x,u_j) - f(x,u)|^{\frac{p}{p-1}} dx \Big)^{\frac{p-1}{p}} \Big(\int_\Omega |u_j|^p dx \Big)^{1/p} \\ &\to 0, \quad \text{as } j \to \infty. \end{aligned}$$

Thus

$$\lim_{j\to\infty} \int_\Omega f(x,u_j)u_j\, dx = \lim_{n\to\infty} \int_\Omega f(x,u)u_j\, dx = \int_\Omega f(x,u)u\, dx.$$

(ii) By the mean value theorem, there exists $\lambda \in [0,1]$ such that

$$
\begin{aligned}
&\Big|\int_\Omega (F(x,u_j)-F(x,u))dx\Big| \\
&= \Big|\int_\Omega f(x,u+\lambda(u_j-u))(u_j-u)dx\Big| \\
&\le \int_\Omega C(1+|u+\lambda(u_j-u)|^{p-1})|u_j-u|dx \\
&\le C\int_\Omega |u_j-u|dx + 2^pC\int_\Omega |u|^{p-1}|u_j-u|dx + 2^pC\int_\Omega |u_j-u|^p dx \\
&\le C\|u_j-u\|_1 + 2^pC\|u\|_p^{p-1}\|u_i-u\|_p + 2^pC\|u_j-u\|_p.
\end{aligned}
$$

Thus

$$
\lim_{j\to\infty}\int_\Omega (F(x,u_j)-F(x,u))dx = 0.
$$

(iii) By using $u_j(x)\to u(x)$ a.e. on $\mathbb{R}^n$, Fatou's Lemma and

$$
(-u_j^-(x)u_j^+(y) - u_j^-(y)u_j^+(x))K(x-y) \ge 0,
$$

we have that (2.2) holds. □

Lemma 2.3. (i) *For all $u\in\mathcal{N}$ such that $\|u\|\to+\infty$, we have $I(u)\to+\infty$;*
(ii) *There exists $\rho>0$ such that $\|u\|\ge\rho$ for all $u\in\mathcal{N}$ and $\|w^\pm\|\ge\rho$ for all $w\in\mathcal{M}$.*

Proof. (i) By the definition of $\mathcal{N}$ and assumption (A3), we have

$$
\begin{aligned}
I(u) &= I(u) - \frac{1}{\mu}\langle I'(u),u\rangle \\
&= (\frac12 - \frac1\mu)\|u\|^2 - \int_\Omega (F(x,u) - \frac1\mu f(x,u)u)dx \\
&\ge (\frac12 - \frac1\mu)\|u\|^2.
\end{aligned}
$$

Thus, $I(u)\to+\infty$ as $\|u\|\to+\infty$.

(ii) By assumptions (A1) and (A2), we have that for any $\varepsilon>0$, there exists a positive constant C_ε such that

$$
|f(t)t| \le \varepsilon|t|^2 + C_\varepsilon|t|^p, \quad \text{for all } t\in\mathbb{R}, \tag{2.3}
$$

where $2<p<2_s^*$. Since $u\in\mathcal{N}$ we have $\langle I'(u),u\rangle = 0$, that is

$$
\|u\| = \int_\Omega f(x,u)udx.
$$

By (2.3), $X_0 \hookrightarrow L^2(\Omega)$ and $X_0\hookrightarrow L^p(\Omega)$, we have

$$
\|u\|^2 \le \varepsilon\int_\Omega |u|^2dx + C_\varepsilon\int_\Omega |u|^pdx \le \varepsilon\gamma_2\|u\|^2 + C_\varepsilon\gamma_p\|u\|^p, \tag{2.4}
$$

where γ_2 and γ_p are the embedding constants. We can choose ε small enough in order to find $\rho>0$ such that $\|u\|\ge\rho$. Now, for $w\in\mathcal{M}$, we have that $\langle I'(w),w^\pm\rangle = 0$, so

$$
\langle I'(w^\pm),w^\pm\rangle = \int_{\mathbb{R}^{2n}} (u^-(x)u^+(y) + u^-(y)u^+(x))K(x-y)\,dx\,dy \le 0,
$$

which gives

$$\|w^{\pm}\|^2 \le \int_\Omega w^{\pm} f(x, w^{\pm}) dx.$$

Then we can proceed as in the proof of (i). $\square$

Lemma 2.4. *Let $\{w_j\} \subset \mathcal{M}$ such that $w_j \rightharpoonup w$ in X_0. Then $w^{\pm} \neq 0$.*

Proof. Firstly we observe that by Lemma 2.3 there exists $\rho > 0$ such that

$$\|w_j^{\pm}\| \ge \rho \quad \text{for all } j \in \mathbb{N}. \tag{2.5}$$

Since $w_j \in \mathcal{M}$, we have $\langle I'(w_j), w_j^{\pm} \rangle = 0$; that is,

$$\begin{aligned} &\|w_j^{\pm}\|^2 - \int_{\mathbb{R}^{2n}} (w_j^-(x) w_j^+(y) + w_j^-(y) w_j^+(x)) K(x-y)\, dx\, dy \\ &= \int_\Omega f(x, w_j^{\pm}) w_j^{\pm} dx. \end{aligned} \tag{2.6}$$

At this point, recalling that

$$- \int_{\mathbb{R}^{2n}} (w_j^-(x) w_j^+(y) + w_j^-(y) w_j^+(x)) K(x-y)\, dx\, dy \ge 0,$$

by (2.5) and (2.6) we deduce that

$$\rho^2 \le \|w_j^{\pm}\|^2 \le \int_\Omega f(x, w_j^{\pm}) w_j^{\pm} dx. \tag{2.7}$$

Now, by the fact that $w_j \rightharpoonup w$ in X_0 and the compactly embedding $X_0 \hookrightarrow L^q(\Omega) (q \in [2, 2_s^*))$, we know that $w_j \to w$ in $L^q(\Omega)$. Moreover, by using that $|t^{\pm} - s^{\pm}| \le |t - s|$ for all $t, s \in \mathbb{R}$, we can deduce that $w_j^{\pm} \to w^{\pm}$ in $L^q(\Omega)$, and for all $x \in \Omega$, we also have $w_j^{\pm} \to w^{\pm}$ *a.e.* in Ω. Similarly to Lemma 2.2, it is easy to see that

$$\int_\Omega f(x, w_j^{\pm}) w_j^{\pm} dx \to \int_\Omega f(x, w^{\pm}) w^{\pm} dx. \tag{2.8}$$

Putting together (2.7) and (2.8) we have

$$0 < \rho^2 \le \int_\Omega f(x, w^{\pm}) w^{\pm} dx$$

showing that $w^{\pm} \neq 0$. $\square$

Lemma 2.5. *If $v \in X_0 : v^{\pm} \neq 0$, then there exist $s, t > 0$ such that*

$$\langle I'(tv^+ + sv^-), v^+ \rangle = 0 \quad \text{and} \quad \langle I'(tv^+ + sv^-), v^- \rangle = 0.$$

As a consequence $tv^+ + sv^- \in \mathcal{M}$.

Proof. Let $W : (0, +\infty) \times (0, +\infty) \to \mathbb{R}^2$ be a continuous vector field given by

$$W(t, s) = (\langle I'(tv^+ + sv^-), tv^+ \rangle, \langle I'(tv^+ + sv^-), sv^- \rangle)$$

for every $t, s \in (0,+\infty) \times (0,+\infty)$. By using (2.3), $X_0 \hookrightarrow L^2(\Omega)$ and $X_0 \hookrightarrow L^p(\Omega)$, we have

$$\begin{aligned} &\langle I'(tv^+ + sv^-), tv^+\rangle \\ &= t^2\|v^+\|^2 - st\int_{\mathbb{R}^{2n}} (v^-(x)v^+(y) + v^-(y)v^+(x))K(x-y)\,dx\,dy \\ &\quad - \int_\Omega tv^+ f(x,tv^+)dx \\ &\geq t^2\|v^+\|^2 - \int_\Omega tv^+ f(x,tv^+)dx \geq t^2\|v^+\|^2 - \varepsilon t^2\|v^+\|_2^2 - C_\varepsilon t^p\|v^+\|_p^p \\ &\geq (1-\varepsilon\gamma_2)t^2\|v^+\|^2 - C_\varepsilon\gamma_p t^p\|v^+\|_p^p. \end{aligned} \tag{2.9}$$

We choose $\varepsilon = \frac{1}{2\gamma_2}$, then there exists $r > 0$ small enough such that $\langle I'(rv^+ + sv^-), rv^+\rangle > 0$ for all $s > 0$, and similarly there exists $\bar r > 0$ small enough such that $\langle I'(tv^+ + \bar r v^-), \bar r v^-\rangle > 0$ for all $t > 0$. By assumption (A3) there exists a positive constant C_1 such that

$$F(x,t) \geq C_1 t^\mu, \text{ for every } t \text{ sufficiently large, uniformly in } x \in \bar\Omega. \tag{2.10}$$

Hence, taking into account that $\mu > 2$, we obtain

$$\begin{aligned} &\langle I'(tv^+ + sv^-), tv^+\rangle \\ &= t^2\|v^+\|^2 - st\int_{\mathbb{R}^{2n}} (v^-(x)v^+(y) + v^-(y)v^+(x))K(x-y)\,dx\,dy \\ &\quad - \int_\Omega tv^+ f(x,tv^+)dx \\ &\leq t^2\|v^+\|^2 - st\int_{\mathbb{R}^{2n}} (v^-(x)v^+(y) + v^-(y)v^+(x))K(x-y)\,dx\,dy \\ &\quad - \mu\int_\Omega F(x,tv^+)dx \\ &\leq t^2\|v^+\|^2 - st\int_{\mathbb{R}^{2n}} (v^-(x)v^+(y) + v^-(y)v^+(x))K(x-y)\,dx\,dy \\ &\quad - t^\mu C_1\mu\int_\Omega |v^+|^\mu dx \to -\infty \quad \text{as } t \to +\infty. \end{aligned}$$

Then, there exists $R > 0$ sufficiently large such that $\langle I'(Rv^+ + sv^-), Rv^+\rangle < 0$ for all $s > 0$ and similarly we can find $\bar R > 0$ such that $\langle I'(tv^+ + \bar R v^-), \bar R v^-\rangle < 0$ for all $t > 0$. As a consequence, we have proved the existence of suitable $0 < r < R$ such that, for all $t, s \in [r, R]$ it holds

$$\begin{gathered} \langle I'(rv^+ + sv^-), rv^+\rangle > 0, \quad \langle I'(tv^+ + \bar r v^-), \bar r v^-\rangle > 0, \\ \langle I'(Rv^+ + sv^-), Rv^+\rangle < 0, \quad \langle I'(tv^+ + \bar R v^-), \bar R v^-\rangle < 0. \end{gathered} \tag{2.11}$$

By applying Miranda's theorem [16] we have the conclution. $\square$

Definition 2.6. For each $v \in X_0$ with $v^\pm \neq 0$, let us consider the function $h^v : [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ given by

$$h^v(t,s) = I(tv^+ + sv^-)$$

and its gradient $\Phi^v : [0,+\infty) \times [0,+\infty) \to \mathbb{R}^2$ defined by

$$\begin{aligned}\Phi^v(t,s) = (\Phi_1^v(t,s), \Phi_2^v(t,s)) &= \Big(\frac{\partial h^v}{\partial t}(t,s), \frac{\partial h^v}{\partial s}(t,s)\Big) \\ &= (\langle I'(tv^+ + sv^-), v^+\rangle, \langle I'(tv^+ + sv^-), v^-\rangle).\end{aligned} \tag{2.12}$$

Furthermore, we consider the Jacobian matrix of Φ^v:

$$(\Phi^v)'(t,s) = \begin{pmatrix} \frac{\partial \Phi_1^v}{\partial t}(t,s) & \frac{\partial \Phi_1^v}{\partial s}(t,s) \\ \frac{\partial \Phi_2^v}{\partial t}(t,s) & \frac{\partial \Phi_1 2^v}{\partial s}(t,s) \end{pmatrix}.$$

In the following we aim to prove that, if $w \in \mathcal{M}$, the function h^w has a critical point and in particular a global minimum in $(t,s) = (1,1)$.

Lemma 2.7. *If $w \in \mathcal{M}$, then*

(i) *$h^w(t,s) < h^w(1,1) = I(w)$, for all $t,s \ge 0$ such that $(t,s) \neq (1,1)$;*
(ii) *$\det(\Phi^w)'(1,1) > 0$.*

Proof. (i) Since $w \in \mathcal{M}$, then $\langle I'(w), w^\pm\rangle = 0$, that is

$$\begin{aligned}\|w^+\|^2 - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy = \int_\Omega f(x,w^+)w^+dx,\\ \|w^-\|^2 - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy = \int_\Omega f(x,w^-)w^-dx.\end{aligned}$$

From this and by the definition of Φ^w, it follows that $(1,1)$ is a critical point of h^w. By (A3), there exists constants $C_1 > 0$ and $C_2 > 0$ such that

$$F(x,\tau) \ge C_1|\tau|^\mu - C_2, \quad \forall x \in \bar{\Omega}.$$

It follows that

$$\begin{aligned}h^w(t,s) &= I(tw^+ + sw^-)\\ &\le \frac{1}{2}\|tw^+ + sw^-\|^2 - C_1\int_\Omega |tw^+ + sw^-|^\mu dx + |\Omega|C_2\\ &\le \frac{t^2+s^2}{2}\Big[\|w^+\|^2 + \|w^-\|^2\\ &\quad - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big]\\ &\quad - C_1t^\mu\int_\Omega |w^+|^\mu dx - C_1s^\mu\int_\Omega |w^-|^\mu dx + |\Omega|C_2\\ &\le \frac{t^2+s^2}{2}\Big[\|w^+\|^2 + \|w^-\|^2\\ &\quad - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big]\\ &\quad - C_1t^\mu\int_\Omega |w^+|^\mu dx + |\Omega|C_2.\end{aligned}$$

Let us suppose that $|t| \ge |s| > 0$, thus, by using $t^2 + s^2 \le 2t^2$ we see that

$$\begin{aligned}\frac{h^w(t,s)}{t^2+s^2} &\le \frac{1}{2}\Big[\|w^+\|^2 + \|w^-\|^2 - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big]\\ &\quad - C_1\frac{t^{\mu-2}}{2}\int_\Omega |w^+|^\mu dx + \frac{|\Omega|C_2}{t^2+s^2}.\end{aligned}$$

Taking into account that $\mu > 2$, we infer that

$$\lim_{|(t,s)|\to+\infty} h^w(t,s) = -\infty.$$

By using the continuity of h^w we deduce the existence of $(\bar t,\bar s)\in[0,+\infty)\times[0,+\infty)$ that is a global maximum point of h^w.

Now we prove that $\bar t,\bar s > 0$. Suppose by contradiction that $\bar s = 0$. Then $\langle I'(\bar t w^+),\bar t w^+\rangle = 0$, that is

$$\|w^+\|^2 = \int_\Omega (w^+)^2 \frac{f(x,\bar t w^+)}{\bar t w^+}dx. \tag{2.13}$$

Since

$$\begin{aligned}
\langle I'(w^+),w^+\rangle &= \langle I'(w),w^+\rangle + \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\\
&= \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy < 0,
\end{aligned}$$

we obtain

$$\|w^+\|^2 < \int_\Omega (w^+)^2 \frac{f(x,w^+)}{w^+}dx. \tag{2.14}$$

Then, combining (2.13) and (2.14) we obtain

$$0 < \int_\Omega (w^+)^2\Big[\frac{f(x,w^+)}{w^+} - \frac{f(x,\bar t w^+)}{\bar t w^+}\Big]dx.$$

So, by (A4), we deduce that $\bar t < 1$. It follows from the condition (A4) that for every $x\in\bar\Omega$

$$\begin{gathered}
\tau\mapsto \frac12\tau f(x,\tau) - F(x,\tau) \text{ is strictly increasing for all } \tau>0,\\
\tau\mapsto \frac12\tau f(x,\tau) - F(x,\tau) \text{ is strictly decreasing for all } \tau<0.
\end{gathered} \tag{2.15}$$

Thus

$$\begin{aligned}
&h^w(\bar t,0) = I(\bar t w^+)\\
&= I(\bar t w^+) - \frac12\langle I'(\bar t w^+),\bar t w^+\rangle\\
&= \int_\Omega \big[\frac12 \bar t w^+ f(x,\bar t w^+) - F(x,\bar t w^+)\big]dx\\
&\le \int_\Omega \big[\frac12 \bar t w^+ f(x,\bar t w^+) - F(x,\bar t w^+)\big]dx + \int_\Omega \big[\frac12 \bar t w^- f(x,\bar t w^-) - F(x,\bar t w^-)\big]dx\\
&< \int_\Omega \big[\frac12 w^+ f(x,w^+) - F(x,w^+)\big]dx + \int_\Omega \big[\frac12 w^- f(x,w^-) - F(x,w^-)\big]dx\\
&= I(w) - \frac12\langle I'(w),w\rangle\\
&= I(w) = h^w(1,1).
\end{aligned}$$

So $h^w(\bar t,0) < h^w(1,1)$, and this gives a contradiction because $(\bar t,0)$ is a global maximum point. Similarly we can prove that $\bar t > 0$.

Now we show that $\bar s,\bar t \le 1$. Since $(h^w)'(\bar t,\bar s) = 0$, we obtain

$$\bar t^2\|w^+\|^2 - \bar s\bar t \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy$$

$$
\begin{aligned}
&= \int_\Omega \bar t w^+ f(x,\bar t w^+)dx,\\
&\bar s^2\|w^-\|^2 - \bar s\bar t\int_{\mathbb{R}^{2n}}(w^-(x)w^+(y)+w^-(y)w^+(x))K(x-y)\,dx\,dy\\
&= \int_\Omega \bar s w^- f(x,\bar s w^-)dx.
\end{aligned}
$$

Assume that $\bar t \ge \bar s$. Since

$$
\int_{\mathbb{R}^{2n}}(w^-(x)w^+(y)+w^-(y)w^+(x))K(x-y)\,dx\,dy \le 0,
$$

we have

$$
\begin{aligned}
&\bar t^2\|w^+\|^2 - \bar t^2\int_{\mathbb{R}^{2n}}(w^-(x)w^+(y)+w^-(y)w^+(x))K(x-y)\,dx\,dy\\
&\ge \int_\Omega \bar t w^+ f(x,\bar t w^+)dx.
\end{aligned}
\tag{2.16}
$$

Since $\langle I'(w), w^+\rangle = 0 (w\in\mathcal{M})$, we deduce that

$$
\|w^+\|^2 - \int_{\mathbb{R}^{2n}}(w^-(x)w^+(y)+w^-(y)w^+(x))K(x-y)\,dx\,dy = \int_\Omega w^+ f(x,w^+)dx
$$

which together with (2.16) gives

$$
\int_\Omega \Big[\frac{f(x,\bar t w^+)}{\bar t w^+} - \frac{f(x,w^+)}{w^+}\Big]dx \le 0.
$$

By (A4) we can infer that $\bar t \le 1$.

Now we aim to prove that h^w does not assume a global maximum in $[0,1]\times[0,1]\setminus\{(1,1)\}$, namely $h^w(\bar t,\bar s) < h^w(1,1)$ for every $(\bar t,\bar s)\in[0,1]\times[0,1]\setminus\{(1,1)\}$. By the definition of h^w and (2.15) we have

$$
\begin{aligned}
h^w(\bar t,\bar s) &= I(\bar t w^+ + \bar s w^-) - \frac12\langle I'(\bar t w^+ + \bar s w^-), \bar t w^+\rangle - \frac12\langle I'(\bar t w^+ + \bar s w^-), \bar s w^-\rangle\\
&= \int_{\Omega\cap\{w\ge 0\}}\big[\frac12 \bar t w^+ f(x,\bar t w^+) - F(x,\bar t w^+)\big]dx\\
&\quad + \int_{\Omega\cap\{w\le 0\}}\big[\frac12 \bar s w^- f(x,\bar s w^-) - F(x,\bar s w^-)\big]dx\\
&< \int_{\Omega\cap\{w\ge 0\}}\Big[\frac12 w^+ f(x,w^+) - F(x,w^+)\Big]\,dx\\
&\quad + \int_{\Omega\cap\{w\le 0\}}\big[\frac12 w^- f(x,w^-) - F(x,w^-)\big]dx\\
&= \int_\Omega\big[\frac12 w f(x,w) - F(x,w)\big]dx = h^w(1,1).
\end{aligned}
$$

(ii) Firstly, let us observe that

$$\begin{aligned} \frac{\partial \Phi_1^w}{\partial t}(t,s) &= \|w^+\|^2 - \int_\Omega f'(x,tw^+)(w^+)^2dx \\ \frac{\partial \Phi_2^w}{\partial s}(t,s) &= \|w^-\|^2 - \int_\Omega f'(x,tw^-)(w^-)^2dx \\ \frac{\partial \Phi_1^w}{\partial s}(t,s) = \frac{\partial \Phi_2^w}{\partial t}(t,s) & \\ &= -\int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy. \end{aligned} \tag{2.17}$$

By (A4), it is easy to see that for every $x \in \bar{\Omega}$,

$$\tau^2 f'(x,\tau) - \tau f(x,\tau) > 0 \quad \text{for all } \tau \neq 0. \tag{2.18}$$

Then, by using the fact that $w \in \mathcal{M}$, (2.17) and (2.18), we have

$$\begin{aligned} &\det(\Phi^w)'(1,1) \\ &= \Big[\|w^+\|^2 - \int_\Omega f'(x,w^+)(w^+)^2dx\Big]\Big[\|w^-\|^2 - \int_\Omega f'(x,w^-)(w^-)^2dx\Big] \\ &\quad - \Big[\int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big]^2 \\ &= \Big[\int_\Omega ((w^+)^2 f'(x,w^+) - w^+ f(x,w^+))dx \\ &\quad - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big] \\ &\quad \times \Big[\int_\Omega ((w^-)^2 f'(x,w^-) - w^- f(x,w^-))dx \\ &\quad - \int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big] \\ &\quad - \Big[\int_{\mathbb{R}^{2n}} (w^-(x)w^+(y) + w^-(y)w^+(x))K(x-y)\,dx\,dy\Big]^2 > 0. \end{aligned}$$

□

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1 by following two steps.

Step 1. We show that there exists $w \in \mathcal{M}$ such that $I(w) = \inf_{v\in\mathcal{M}} I(v)$. By Lemma 2.3, there exists a minimizing sequence $\{w_j\} \subset \mathcal{M}$, bounded in X_0, such that

$$I(w_j) \to \inf_{v\in\mathcal{M}} I(v) =: c_0 > 0. \tag{3.1}$$

By Lemma 2.1, up to a subsequence, we have

$$\begin{gathered} w_j^\pm \rightharpoonup w^\pm \quad \text{in } X_0, \\ w_j^\pm \to w^\pm \quad \text{in } L^p(\Omega), \\ w_j^\pm \to w^\pm \quad \text{a.e. in } \mathbb{R}^n. \end{gathered}$$

From Lemma 2.4 we deduce that $w^{\pm} \neq 0$, so $w = w^+ + w^-$ is sign-changing. By Lemma 2.5, there exist $s, t > 0$ such that

$$\langle I'(tw^+ + sw^-), w^+\rangle = 0, \quad \langle I'(tw^+ + sw^-), w^-\rangle = 0 \tag{3.2}$$

and $tw^+ + sw^- \in \mathcal{M}$. Now, we prove that $s, t \leq 1$. Since $w_j \in \mathcal{M}$, we have $\langle I'(w_j), w_j^{\pm}\rangle = 0$ or equivalently

$$\begin{aligned}
&\|w_j^+\|^2 - \int_{\mathbb{R}^{2n}} (w_j^-(x)w_j^+(y) + w_j^-(y)w_j^+(x))K(x-y)\,dx\,dy \\
&= \int_\Omega w_j^+ f(x, w_j^+)dx, \\
&\|w_j^-\|^2 - \int_{\mathbb{R}^{2n}} (w_j^-(x)w_j^+(y) + w_j^-(y)w_j^+(x))K(x-y)\,dx\,dy \\
&= \int_\Omega w_j^- f(x, w_j^-)dx.
\end{aligned} \tag{3.3}$$

The weak lower semicontinuity of the norm $\|\cdot\|$ in X_0 yields

$$\|w^{\pm}\|^2 \leq \liminf_{n\to\infty} \|w_j^{\pm}\|^2. \tag{3.4}$$

By using (3.3), (3.4) and Lemma 2.2, we obtain

$$\langle I'(w), w^+\rangle \leq 0 \quad \text{and} \quad \langle I'(w), w^-\rangle \leq 0. \tag{3.5}$$

Then, we combine (3.2) with (3.5), and arguing similarly as in the proof of Lemma 2.7(i) we obtain that $s, t \leq 1$.

Next, we show that $I(tw^+ + sw^-) = c_0$ and $t = s = 1$. By using $tw^+ + sw^- \in \mathcal{M}, w_j \in \mathcal{M}$, (2.15), (3.1), $s, t \in (0, 1]$ and Lemma 2.2 we can see

$$\begin{aligned}
c_0 &\leq I(tw^+ + sw^-) = I(tw^+ + sw^-) - \frac{1}{2}\langle I'(tw^+ + sw^-), tw^+ + sw^-\rangle \\
&= \int_\Omega \Big[\frac{1}{2} f(x, tw^+ + sw^-)(tw^+ + sw^-) - F(x, tw^+ + sw^-)\Big]dx \\
&= \int_{\Omega\cap\{w\geq 0\}} \Big[\frac{1}{2} tw^+ f(x, tw^+) - F(x, tw^+)\Big]dx \\
&\quad + \int_{\Omega\cap\{w\leq 0\}} \Big[\frac{1}{2} sw^- f(x, sw^-) - F(x, sw^-)\Big]dx \\
&\leq \int_\Omega \Big[\frac{1}{2} w^+ f(x, w^+) - F(x, w^+)\Big]dx \\
&\quad + \int_\Omega \Big[\frac{1}{2} w^- f(x, w^-) - F(x, w^-)\Big]\,dx \\
&= \lim_{j\to\infty} \Big[\int_\Omega \Big(\frac{1}{2} w_j^+ f(x, w_j^+) - F(x, w_j^+)\Big)dx + \int_\Omega \Big(\frac{1}{2} w_j^- f(x, w_j^-) - F(x, w_j^-)\Big)dx\Big] \\
&= \lim_{j\to\infty} \int_\Omega \Big(\frac{1}{2} w_j f(x, w_j) - F(x, w_j)\Big)dx = \lim_{j\to\infty} \Big[I(w_j) - \frac{1}{2}\langle I'(w_j), w_j\rangle\Big] \\
&= \lim_{j\to\infty} I(w_j) = c_0.
\end{aligned}$$

Thus, we have proved that there exist $t, s \in (0, 1]$ such that $tw^+ + sw^- \in \mathcal{M}$ and $I(tw^+ + sw^-) = c_0$. Let us observe that by the above calculation we can infer that $t = s = 1$, so $w = w^+ + w^- \in \mathcal{M}$ and $I(w^+ + w^-) = c_0$.

Step 2. Now, we prove that $I'(w) = 0$. We argue by contradiction. Then, we can find a positive constant $\beta > 0$ and $v_0 \in X_0$ with $\|v_0\| = 1$, such that $\langle I'(w), v_0\rangle = 2\beta > 0$. By the continuity of I', we can choose a radius R so that $\langle I'(v), v_0\rangle = \beta > 0$ for every $v \in B_R(w) \subset X_0$ with $v^{\pm} \neq 0$.

Let $D := (a, b) \times (a, b) \subset \mathbb{R}^2$ with $0 < a < 1 < b$ such that

(i) $(1, 1) \in D$ and $\Phi^w(t, s) = (0, 0)$ in $\bar{D}$ if, and only if, $(t, s) = (1, 1)$,

(ii) $c_0 \notin h^w(\partial D)$,

(iii) $\{tw^+ + sw^- : (t, s) \in \bar{D}\} \subset B_R(w)$,

where h^w and Φ^w are defined by Definition 2.6, and satisfy Lemma 2.7. Then we can choose a radius $0 < r < R$ such that

$$B = \bar{B}_r(w) \subset B_R(w) \quad \text{and} \quad B \cap \{tw^+ + sw^- : (t, s) \in \partial D\} = \emptyset. \tag{3.6}$$

Now, let us define a continuous mapping $\rho : X_0 \to [0, +\infty)$ such that $\rho(u) := \operatorname{dist}(u, B^c)$ for all $u \in X_0$, and we consider a bounded Lipschitz vector field $\mathbb{V} : X_0 \to X_0$ given by $\mathbb{V}(u) = -\rho(u)v_0$. For every $u \in X_0$, denoting by $\eta(\tau) = \eta(\tau, u)$, we consider the following Cauchy problem

$$\begin{gathered} \eta'(\tau) = \mathbb{V}(\eta(\tau)) \quad \text{for all } \tau > 0, \\ \eta(0) = u. \end{gathered}$$

We observe that there exist a continuous deformation $\eta(\tau, u)$ and $\tau_0 > 0$ such that for all $\tau \in [0, \tau_0]$ the following properties hold:

(a) $\eta(\tau, u) = u$ for all $u \notin B$,

(b) $\tau \to I(\eta(\tau, u))$ is decreasing for all $\eta(\tau, u) \in B$,

(c) $I(\eta(\tau, w)) \leq I(w) - \frac{r\beta}{2}\tau$.

(a) follows from the definition of ρ. Regarding (b), we observe that $\langle I'(\eta(\tau)), v_0\rangle = \beta > 0$ for $\eta(\tau) \in B \subset B_R(w)$, and, by the definition of ρ, we have $\rho(\eta(\tau)) > 0$. Then

$$\frac{d}{d\tau}(I(\eta(\tau))) = \langle I'(\eta(\tau)), \eta'(\tau)\rangle = -\rho(\eta(\tau))\langle I'(\eta(\tau)), v_0\rangle = -\rho(\eta(\tau))\beta < 0,$$

for all $\eta(\tau) \in B$, that is $I(\eta(\tau, u))$ is decreasing with respect to τ. Now we prove (c). Being $\tau_0 > 0$ such that $\eta(\tau, u) \in B$ for every $\tau \in [0, \tau_0]$, we can assume that

$$\|\eta(\tau, w) - w\| \leq \frac{r}{2}, \quad \text{for any } \tau \in [0, \tau_0].$$

Since $\rho(\eta(\tau, w)) = \operatorname{dist}(\eta(\tau, w), B^c) \geq \frac{r}{2}$, we deduce that

$$\frac{d}{d\tau} I(\eta(\tau, w)) \leq -\rho(\eta(\tau, w))\beta \leq -\frac{r\beta}{2}$$

and, integrating on $[0, \tau_0]$, we obtain

$$I(\eta(\tau_0, w)) - I(w) \leq -\frac{r\beta}{2}\tau_0.$$

Now, we consider a suitable deformed path $\bar{\eta}_0 : \bar{D} \to X_0$ defined by

$$\bar{\eta}_0(t, s) := \eta(\tau_0, tw^+ + sw^-), \quad \text{for all } (t, s) \in \bar{D}.$$

We note that

$$\max_{(t,s)\in\bar{D}} I(\bar{\eta}_0(t, s)) < c_0.$$

Indeed, by (b) and the fact that $\eta(0, u) = u$, we have

$$I(\bar{\eta}_0(t, s)) = I(\eta(\tau_0, tw^+ + sw^-)) \leq I(\eta(0, tw^+ + sw^-))$$

$$
\begin{aligned}
&= I(tw^+ + sw^-) = h^w(t,s) \\
&< c_0, \quad \forall (t,s) \in \bar{D} \setminus \{(1,1)\},
\end{aligned}
$$

and for $(t,s) = (1,1)$, by (c) we have

$$
\begin{aligned}
I(\bar{\eta}_0(1,1)) &= I(\eta(\tau_0, w^+ + w^-)) = I(\eta(\tau_0, w)) \\
&=\leq I(w) - \frac{r\beta}{2}\tau_0 < I(w) = c_0.
\end{aligned}
$$

Then, $\bar{\eta}_0(t,s) \cap \mathcal{M} = \emptyset$; that is,

$$
\bar{\eta}_0(t,s) \notin \mathcal{M} \quad \text{for all } (t,s) \in \bar{D}. \tag{3.7}
$$

On the other hand, defining $\Psi_{\tau_0} : D \to \mathbb{R}^2$ such that

$$
\Psi_{\tau_0} := \Big(\frac{1}{t} \langle I'(\bar{\eta}_0(t,s)), (\bar{\eta}_0(t,s))^+ \rangle, \frac{1}{s} \langle I'(\bar{\eta}_0(t,s)), (\bar{\eta}_0(t,s))^- \rangle \Big).
$$

We see that, for all $(t,s) \in \partial D$, by (3.6) and (a) for $\tau = \tau_0$, it holds

$$
\Psi_{\tau_0}(t,s) = (\langle I'(tw^+ + sw^-), w^+ \rangle, \langle I'(tw^+ + sw^-), w^- \rangle) = \Phi^w(t,s).
$$

Then, by using Brouwer's topological degree, we have

$$
\deg(\Psi_{\tau_0}, D, (0,0)) = \deg(\Phi^w, D, (0,0)) = \operatorname{sgn}(\det(\Phi^w)'(1,1)) = 1,
$$

so we deduce that Ψ_{τ_0} has a zero $(\bar{t}, \bar{s}) \in D$, that is

$$
\langle I'(\bar{\eta}_0(\bar{t}, \bar{s})), (\bar{\eta}_0(\bar{t}, \bar{s}))^{\pm} \rangle = 0.
$$

Therefore, there exists $(\bar{t}, \bar{s}) \in D$ such that $\bar{\eta}_0(\bar{t}, \bar{s}) \in \mathcal{M}$ and this contradicts (3.7).

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