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MULTIPLICITY RESULTS OF FRACTIONAL-LAPLACE SYSTEM WITH SIGN-CHANGING AND SINGULAR NONLINEARITY

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Abstract. In this article, we study the following fractional-Laplacian system with singular nonlinearity

$$\begin{gathered}(-\Delta)^s u = \lambda f(x)u^{-q} + \frac{\alpha}{\alpha+\beta} b(x)u^{\alpha-1}w^{\beta} \quad \text{in } \Omega\\ (-\Delta)^s w = \mu g(x)w^{-q} + \frac{\beta}{\alpha+\beta} b(x)u^{\alpha}w^{\beta-1} \text{ in } \Omega\\ u, w > 0 \text{ in } \Omega, \quad u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, $n > 2s$, $s \in (0,1)$, $0 < q < 1$, $\alpha > 1$, $\beta > 1$ satisfy $2 < \alpha+\beta < 2_s^* - 1$ with $2_s^* = \frac{2n}{n-2s}$, the pair of parameters $(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$. The weight functions $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < f,\ g \in L^{\frac{\alpha+\beta}{\alpha+\beta-1+q}}(\Omega)$, and $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b(x) \in L^{\infty}(\Omega)$. Using variational methods, we show existence and multiplicity of positive solutions with respect to the pair of parameters (λ,μ).

1. Introduction

Let $\Omega \subset \mathbb{R}^n$ is a bounded domain with smooth boundary, $n > 2s$ and $s \in (0,1)$. We consider the following fractional system with singular nonlinearity

$$\begin{gathered}(-\Delta)^s u = \lambda f(x)u^{-q} + \frac{\alpha}{\alpha+\beta} b(x)u^{\alpha-1}w^{\beta} \quad \text{in } \Omega\\ (-\Delta)^s w = \mu g(x)w^{-q} + \frac{\beta}{\alpha+\beta} b(x)u^{\alpha}w^{\beta-1} \quad \text{in } \Omega\\ u, w > 0 \text{ in } \Omega, \quad u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega.\end{gathered} \tag{1.1}$$

Here, $(-\Delta)^s$ is the fractional Laplacian operator defined as

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}dy \quad \text{for all } x \in \mathbb{R}^n.$$

We assume the following assumptions on f and g:

(A1) $f, g : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ such that $0 < f, g \in L^{q^*}(\Omega)$, where $q^* = \frac{\alpha+\beta}{\alpha+\beta-1+q}$.

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(A2) $b : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is a sign-changing function such that $b^+ = \max\{f, 0\} \not\equiv 0$ and $b(x) \in L^\infty(\Omega)$.

Also the pair of parameters $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}$, $0 < q < 1$ and $\alpha > 1$, $\beta > 1$ satisfy $2 < \alpha + \beta < 2_s^* - 1$, with $2_s^* = \frac{2n}{n-2s}$, known as fractional critical Sobolev exponent.

In this work, we prove the existence of multiple positive solutions for a system of fractional operator with singular and sign changing nonlinearity by studying the nature of Nehari manifold with respect to the pair of parameter (λ, μ). These same result can be easily extended to the following $p-$fractional Laplacian system with singular and sign-changing nonlinearity

$$
\begin{gathered}
(-\Delta)_p^s u = \lambda f(x) u^{-q} + \frac{\alpha}{\alpha+\beta} b(x) u^{\alpha-1} w^{\beta} \quad \text{in } \Omega \\
(-\Delta)_p^s w = \mu g(x) w^{-q} + \frac{\beta}{\alpha+\beta} b(x) u^{\alpha} w^{\beta-1} \quad \text{in } \Omega \\
u, w > 0 \text{ in } \Omega, \quad u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega.
\end{gathered} \tag{1.2}
$$

where $0 < q < 1 \leq p - 1 < \alpha + \beta < p_s^* - 1$ and $(-\Delta)_p^s$ is a $p-$fractional operator which is defined as

$$
(-\Delta)_p^s u(x) = 2 \lim_{\epsilon \to 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|x-y|^{n+ps}} dy.
$$

This definition is consistent, up to a normalization constant depending on n and s, with the fractional Laplacian $(-\Delta)^s$, for the case $p = 2$.

For $u = v$, $\alpha = \beta$, $\alpha + \beta = r$, $\lambda = \mu$ and $f = g$, problem (1.2) reduces to the p-fractional Laplace equation with singular nonlinearities

$$
\begin{gathered}
(-\Delta)_p^s = f(x) w^{-q} + \lambda b(x) w^r \quad \text{in } \Omega, \\
w > 0 \text{ in } \Omega, \quad w = 0 \text{ in } \mathbb{R}^n \setminus \Omega.
\end{gathered} \tag{1.3}
$$

In [19], the author studied the existence and multiplicity of positive solutions to problem (1.3) with sign-changing and singular nonlinearity. In the scalar case, the problems involving the fractional operator with singular nonlinearity have been studied by many authors, see [29, 4, 14] and references therein. Also, in [20], the author used the Caffarelli and Silvestre [5] approach to obtain the multiplicity results for singular and sign-changing nonlinearity.

The fractional power of Laplacian is the infinitesimal generator of Lévy stable diffusion process and arise in anomalous diffusions in plasma, population dynamics, geophysical fluid dynamics, flames propagation, chemical reactions in liquids and American options in finance. For more details, one can see [3, 15] and reference therein. Recently the fractional elliptic problem have been investigated by many authors for polynomial type nonlinearities see [28, 4, 30] and reference therein. Moreover, by Nehari manifold and fibering maps, the author obtained the existence of multiple solutions for p-fractional equations [21, 22] and reference therein.

For $s = 1$, the paper by Crandall, Robinowitz and Tartar [7] is the starting point on semilinear problem with singular nonlinearity. There is a large body of literature on singular nonlinearity see [1, 2, 7, 8, 9, 10, 12, 13, 23, 24, 25, 26, 16, 17, 18] and reference therein. In [6], Chen showed the existence and multiplicity of the problem

$$
-\Delta w - \frac{\lambda}{|x|^2} w = \frac{f(x)}{w^q} + \mu g(x) w^p \quad \text{in } \Omega \setminus \{0\}
$$

$$w > 0 \text{ in } \Omega \setminus \{0\}, \quad w = 0 \text{ in } \partial\Omega,$$

where $0 \in \Omega$ is a bounded smooth domain of $\mathbb{R}^n$ with smooth boundary, $0 < \lambda < \frac{(n-2)^2}{4}$, $0 < q < 1 < p < \frac{n+2}{n-2}$, $f(x) > 0$ and g is sign-changing continuous function.

The natural space to look for solutions of the problem (1.2) is the product space $W_0^{s,p}(\Omega) \times W_0^{s,p}(\Omega)$. To study (1.1), it is important to encode the 'boundary condition' $u = v = 0$ in $\mathbb{R}^n \setminus \Omega$ in the weak formulation. Servadei and Valdinoci [28] introduced the new function spaces to study the variational functionals related to the fractional Laplacian by observing the interaction between Ω and $\mathbb{R}^n \setminus \Omega$.

To the best of our knowledge, there is no work related the fractional Laplacian system with singular and sign-changing nonlinearity. In this work, we studied the multiplicity results for the system of fractional Laplacian equation with singular nonlinearity and sign-changing weight function with respect to the pair of parameter (λ, μ). This work is motivated by the work of Chen and Chen in [6]. But one can not directly extend all the results for fractional Laplacian, due to the non-local behavior of the operator and the bounded support of the test function is not preserved. Also due to the singularity of the problem, the associated functional is not differentiable in the sense of Gâteaux. The results obtained here are somehow expected but we show how the results arise out of nature of the Nehari manifold.

This article is organized as follows: Section 2 is devoted to some preliminaries and notations. We also state our main results. In section 3, we study the decomposition of Nehari manifold and the associated energy functional is bounded below and coercive. Section 3 contains the existence of a nontrivial solutions in $\mathcal{N}^+_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$.

We will use the following notation throughout this paper: $\|f\|_{q^*}$, $\|g\|_{q^*}$ denote the norm in $L^{\frac{\alpha+\beta}{\alpha+\beta-1+q}}(\Omega)$.

2. Preliminaries

In this section we give some definitions and functional settings. At the end of this section, we state our main results. For this we define $H^s(\Omega)$, the usual fractional Sobolev space

$$H^s(\Omega) := \big\{w \in L^2(\Omega) : \frac{(w(x) - w(y))}{|x-y|^{\frac{n}{2}+s}} \in L^2(\Omega \times \Omega)\big\}$$

endowed with the norm

$$\|w\|_{H^s(\Omega)} = \|w\|_{L^2(\Omega)} + \Big(\int_{\Omega\times\Omega} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}. \tag{2.1}$$

for details on fractional Sobolev spaces, we refer the reader to [27].

Because of the non-localness of the operator, we define the linear space

$$X_0 = \Big\{w : \mathbb{R}^n \to \mathbb{R} : w \text{ is measurable}, w|_\Omega \in L^p(\Omega) \\ \frac{w(x)-w(y)}{|x-y|^{\frac{n+2s}{2}}} \in L^2(Q),\ w = 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega\Big\}$$

where $Q = \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$ and $\mathcal{C}\Omega := \mathbb{R}^n \setminus \Omega$. The space X_0 was firstly introduced by Servadei and Valdinoci [28]. The space X_0 endowed with the norm

$$\|w\| = \Big(\int_Q \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2} \tag{2.2}$$

is a Hilbert space. We note that, the norms in (2.1) and (2.2) are not same because $\Omega\times\Omega$ is strictly contained in Q. Let $Y = X_0 \times X_0$ be the cartesian product of two reflexive Banach spaces, which is also reflexive Banach space with the norm

$$\begin{aligned}\|(u,w)\| &= (\|u\|_{X_0}^2 + \|w\|_{X_0}^2)^{1/2}\\ &= \Big(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \int_Q \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}.\end{aligned}$$

Now we define the space

$$C_Y := \{(u,w) : u,w \in C_c^\infty(\mathbb{R}^n) : u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}.$$

Then C_Y is a dense in the space Y.

Denote

$$S := \inf_{u\in X_0} \Big\{ \frac{\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 |x-y|^{-(n+2s)}\,dx\,dy}{(\int_{\mathbb{R}} |u|^{\alpha+\beta} dx)^{\frac{2}{\alpha+\beta}}} \Big\},$$

$$K_{\lambda,\mu} := \lambda \int_\Omega f(x)(u_+)^{1-q} dx + \mu \int_\Omega g(x)(w_+)^{1-q} dx.$$

Definition 2.1. A weak solution of problem (1.1) is a function $(u,w) \in Y$, with u, $w > 0$ in Ω such that for every $(\phi,\psi) \in Y$,

$$\begin{aligned}&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{(n+2s)}}\,dx\,dy + \int_Q \frac{(w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{(n+2s)}}\,dx\,dy\\ &= \lambda \int_\Omega f(x)(u^{-q}\phi)(x)dx + \mu \int_\Omega g(x)(w^{-q}\psi)(x)dx\\ &\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u^{\alpha-1}w^\beta\phi)(x)dx + \frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u^\alpha w^{\beta-1}\psi)(x)dx.\end{aligned}$$

To show the existence of positive solution of (1.1), we consider the problem

$$\begin{gathered}(-\Delta)^s u = \lambda f(x) u_+^{-q} + \frac{\alpha}{\alpha+\beta} b(x) u_+^{\alpha-1} w_+^\beta \quad \text{in } \Omega\\ (-\Delta)^s w = \mu g(x) w_+^{-q} + \frac{\beta}{\alpha+\beta} b(x) u_+^{\alpha} w_+^{\beta-1} \quad \text{in } \Omega\\ u,w > 0 \text{ in } \Omega, \quad u = w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,\end{gathered}$$

where $w_+ := \max\{w, 0\}$ denotes the positive part of w. Then the function $(u,w) \in Y$ with $u,w > 0$ in $\Omega\times\Omega$ is a weak solution of the problem (2) if for every $(\phi,\psi) \in Y$, we have

$$\begin{aligned}&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{(n+2s)}}\,dx\,dy + \int_Q \frac{(w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{(n+2s)}}\,dx\,dy\\ &= \lambda \int_\Omega f(x)(u_+^{-q}\phi)(x)dx + \mu \int_\Omega g(x)(w_+^{-q}\psi)(x)dx\\ &\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_+^{\alpha-1}w_+^\beta\phi)(x)dx + \frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_+^\alpha w_+^{\beta-1}\psi)(x)dx.\end{aligned}$$

We note that if $(u,w) > 0$ is a solution of (2) then one can easily see that (u,w) is also a solution (1.1). To find the solution of (2), we will use variational approach. So we define the associated functional $J_{\lambda,\mu} : Y \to [-\infty,\infty)$ as

$$J_{\lambda,\mu}(u,w) = \frac{1}{2}\|(u,w)\|^2 - \frac{1}{1-q}\int_\Omega \Big(\lambda f(x) u_+^{1-q} + \mu g(x) w_+^{1-q}\Big)\,dx$$

$$- \frac{1}{\alpha+\beta}\int_\Omega b(x)u_+^\alpha w_+^\beta dx.$$

Here $J_{\lambda,\mu}$ is not bounded below on Y but is bounded below on appropriate subset $\mathcal{N}_{\lambda,\mu}$ of Y. Therefore in order to obtain the existence results, we introduce the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} = \big\{(u,w)\in Y : \langle J'_{\lambda,\mu}(u,w),(u,w)\rangle = 0\big\} = \big\{(u,w)\in Y : \phi'_{u,w}(1)=0\big\}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality between Y and its dual space. Thus $(u,w)\in\mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,w)\|^2 - \int_\Omega \Big(\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q}\Big)\,dx - \int_\Omega b(x)u_+^\alpha w_+^\beta\,dx = 0.$$

We note that $\mathcal{N}_{\lambda,\mu}$ contains every solution of (2). Now as we know that the Nehari manifold is closely related to the behavior of the functions $\phi_{u,w}:\mathbb{R}^+\to\mathbb{R}$ defined as $\phi_{u,w}(t) = J_{\lambda,\mu}(tu,tw)$. Such maps are called fiber maps and were introduced by Drabek and Pohozaev in [11]. For $(u,w)\in Y$, we have

$$\begin{gathered}\phi_{u,w}(t) = \frac{t^2}{2}\|(u,w)\|^2 - \frac{t^{1-q}}{1-q}K_{\lambda,\mu}(u,w) - \frac{t^{\alpha+\beta}}{\alpha+\beta}\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx,\\ \phi'_{u,w}(t) = t\|(u,w)\|^2 - t^{-q}K_{\lambda,\mu}(u,w) - t^{\alpha+\beta-1}\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx,\\ \phi''_{u,w}(t) = \|(u,w)\|^2 + qt^{-q-1}K_{\lambda,\mu}(u,w) - (\alpha+\beta-1)t^{\alpha+\beta-2}\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx.\end{gathered}$$

Then it is easy to see that $(tu,tw)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\phi'_{u,w}(t)=0$ and in particular, $(u,w)\in\mathcal{N}_{\lambda,\mu}$ if and only if $\phi'_{u,w}(1)=0$. Thus it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection. For this we set

$$\begin{gathered}\mathcal{N}^\pm_{\lambda,\mu} := \big\{(u,w)\in\mathcal{N}_{\lambda,\mu} : \phi''_{u,w}(1) \gtrless 0\big\} = \big\{(tu,tw)\in Y : \phi'_{u,w}(t)=0,\ \phi''_{u,w}(t)\gtrless 0\big\},\\ \mathcal{N}^0_{\lambda,\mu} := \big\{(u,w)\in\mathcal{N}_{\lambda,\mu} : \phi''_{u,w}(1) = 0\big\} = \big\{(tu,tw)\in Y : \phi'_{u,w}(t)=0,\ \phi''_{u,w}(t)=0\big\}.\end{gathered}$$

We also observe that if $(u,w)\in\mathcal{N}_{\lambda,\mu}$ then

$$\phi''_{u,w}(1) = \begin{cases}(1+q)\|(u,w)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx\\ \quad\text{(when we solve for } K \text{ in } \phi'_{u.w}(1)=0),\\ (2-\alpha-\beta)\|(u,w)\|^2 + (\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)\\ \quad\text{(when we solve for } \int_\Omega \text{ in } \phi'_{u,w}(1)=0).\end{cases}$$

Inspired by [6], we show that how variational methods can be used to established some existence and multiplicity results for (2). Our results are as follows.

Theorem 2.2. *Suppose* $(\lambda,\mu)\in\Gamma$*, where*

$$\begin{aligned}\Gamma := \Big\{&(\lambda,\mu)\in\mathbb{R}^2\setminus\{(0,0)\} :\\ &0 < \Lambda := (|\lambda|\|f\|_{q^*})^{\frac{2}{1+q}} + (|\mu|\|g\|_{q^*})^{\frac{2}{1+q}} < C(n,\alpha,\beta,q,S)\Big\},\end{aligned}\tag{2.3}$$

and

$$C(n,\alpha,\beta,q,S) = \Big(\frac{(1+q)}{(\alpha+\beta-1+q)}\Big)^{\frac{2}{\alpha+\beta-2}}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{2}{1+q}}$$

$$\times \Big(\frac{1}{\|b\|_\infty}\Big)^{\frac{2}{\alpha+\beta-2}} S^{\frac{2(\alpha+\beta-1+q)}{(1+q)(\alpha+\beta-2)}}.$$

Then problem (1.1) *has at least two solutions*

$$(u,w) \in \mathcal{N}_{\lambda,\mu}^{+}, (U,W) \in \mathcal{N}_{\lambda,\mu}^{-} \quad \textit{with } \|(U,W)\| > \|(u,w)\|.$$

3. Fibering map analysis

In this section, we show that $\mathcal{N}_{\lambda,\mu}^{\pm}$ is nonempty and $\mathcal{N}_{\lambda,\mu}^{0} = \{(0,0)\}$. Moreover, $J_{\lambda,\mu}$ is bounded below and coercive.

Lemma 3.1. *Let $(\lambda,\mu) \in \Gamma$. Then for each $(u,w) \in Y$ with $K_{\lambda,\mu}(u,w) > 0$, we have the following:*

- (i) *$\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx \le 0$, then there exists a unique $0 < t_1 < t_{\max}$ such that $(t_1u,t_1w) \in \mathcal{N}_{\lambda,\mu}^{+}$ and $J_{\lambda,\mu}(t_1u,t_1w) = \inf_{t>0} J_{\lambda,\mu}(tu,tw)$,*
- (ii) *$\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx > 0$, then there exists unique t_1 and t_2 with $0 < t_1 < t_{\max} < t_2$ such that $(t_1u,t_1w) \in \mathcal{N}_{\lambda,\mu}^{+}$, $(t_2u,t_2w) \in \mathcal{N}_{\lambda,\mu}^{-}$ and $J_{\lambda,\mu}(t_1u,t_1w) = \inf_{0\le t\le t_{\max}} J_{\lambda,\mu}(tu,tw)$, $J_{\lambda,\mu}(t_2u,t_2w) = \sup_{t\ge t_1} J_{\lambda,\mu}(tu,tw)$.*

Proof. For $t > 0$, we define

$$\psi_{u,w}(t) = t^{2-\alpha-\beta}\|(u,w)\|^2 - t^{-\alpha-\beta+1-q}K_{\lambda,\mu}(u,w) - \int_\Omega b(x)u_+^\alpha w_+^\beta\,dx.$$

One can easily see that $\psi_{u,w}(t) \to -\infty$ as $t \to 0^+$. Now

$$\begin{aligned}
\psi'_{u,w}(t) &= (2-\alpha-\beta)t^{1-\alpha-\beta}\|(u,w)\|^2 + (\alpha+\beta-1+q)t^{-\alpha-\beta-q}K_{\lambda,\mu}(u,w),\\
\psi''_{u,w}(t) &= (2-\alpha-\beta)(1-\alpha-\beta)t^{-\alpha-\beta}\|(u,w)\|^2\\
&\quad - (\alpha+\beta-1+q)(\alpha+\beta+q)t^{-\alpha-\beta-q-1}K_{\lambda,\mu}(u,w).
\end{aligned}$$

Then $\psi'_{u,w}(t) = 0$ if and only if

$$t = t_{\max} := \Big[\frac{(\alpha+\beta-2)\|(u,w)\|^2}{(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)}\Big]^{-\frac{1}{1+q}}.$$

Also

$$\begin{aligned}
&\psi''_{u,w}(t_{\max})\\
&= (2-\alpha-\beta)(1-\alpha-\beta)\Big[\frac{(\alpha+\beta-2)\|(u,w)\|^2}{(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)}\Big]^{\frac{\alpha+\beta}{1+q}}\|(u,w)\|^2\\
&\quad - (\alpha+\beta-1+q)(\alpha+\beta+q)\Big[\frac{(\alpha+\beta-2)\|(u,w)\|^2}{(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)}\Big]^{\frac{\alpha+\beta+q+1}{1+q}}K_{\lambda,\mu}(u,w)\\
&= -\|(u,w)\|^2(\alpha+\beta-2)(1+q)\Big[\frac{(\alpha+\beta-2)\|(u,w)\|^2}{(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)}\Big]^{\frac{\alpha+\beta}{1+q}} < 0.
\end{aligned}$$

Thus $\psi_{u,w}$ achieves its maximum at $t = t_{\max}$. Now using the Hölder's inequality and fractional Sobolev inequality, we obtain

$$\begin{aligned} K_{\lambda,\mu}(u,w) &\le |\lambda| \int_\Omega |f(x)||u|^{1-q} dx + |\mu| \int_\Omega |g(x)||w|^{1-q} dx \\ &\le |\lambda| \|f\|_{q^*} \|u\|_{\alpha+\beta}^{1-q} + |\mu| \|g\|_{q^*} \|w\|_{\alpha+\beta}^{1-q} \\ &\le \Big((|\lambda| \|f\|_{q^*})^{\frac{2}{1+q}} + (|\mu| \|g\|_{q^*})^{\frac{2}{1+q}} \Big)^{\frac{1+q}{2}} \Big(\frac{\|(u,w)\|}{\sqrt{S}} \Big)^{1-q} \\ &= \Lambda^{\frac{1+q}{2}} \Big(\frac{\|(u,w)\|}{\sqrt{S}} \Big)^{1-q}, \end{aligned} \tag{3.1}$$

where $\Lambda := (|\lambda| \|f\|_{q^*})^{\frac{2}{1+q}} + (|\mu| \|g\|_{q^*})^{\frac{2}{1+q}}$.

$$\begin{aligned} \int_\Omega b(x) u_+^\alpha w_+^\beta \, dx &\le \|b\|_\infty \Big(\frac{\alpha}{\alpha+\beta} \int_\Omega |u|^{\alpha+\beta} dx + \frac{\beta}{\alpha+\beta} \int_\Omega |v|^{\alpha+\beta} dx \Big) \\ &\le \|b\|_\infty \Big(\frac{\|(u,w)\|}{\sqrt{S}} \Big)^{\alpha+\beta}. \end{aligned} \tag{3.2}$$

Using (3.1) and (3.2) we obtain,

$$\begin{aligned} \psi_{u,w}(t_{\max}) &= \frac{(1+q)}{(\alpha+\beta-1+q)} \Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q} \Big)^{\frac{\alpha+\beta-2}{1+q}} \frac{\|(u,w)\|^{\frac{2(\alpha+\beta-1+q)}{(1+q)}}}{[K_{\lambda,\mu}(u,w)]^{\frac{\alpha+\beta-2}{1+q}}} \\ &\quad - \int_\Omega b(x) u_+^\alpha w_+^\beta \, dx \\ &\ge \Big[\frac{(1+q)}{(\alpha+\beta-1+q)} \Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q} \Big)^{\frac{\alpha+\beta-2}{1+q}} \Big(\frac{(\sqrt{S})^{(1-q)}}{\Lambda^{\frac{1+q}{2}}} \Big)^{\frac{(\alpha+\beta-2)}{(1+q)}} \\ &\quad - \|b\|_\infty \big(\frac{1}{\sqrt{S}} \big)^{\alpha+\beta} \Big] \|(u,w)\|^{\alpha+\beta} \\ &\equiv E_{\lambda,\mu} \|(u,w)\|^{\alpha+\beta}. \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} E_{\lambda,\mu} &= \Big[\frac{(1+q)}{(\alpha+\beta-1+q)} \Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q} \Big)^{\frac{\alpha+\beta-2}{1+q}} \Big(\frac{(\sqrt{S})^{(1-q)}}{\Lambda^{\frac{1+q}{2}}} \Big)^{\frac{\alpha+\beta-2}{1+q}} \\ &\quad - \|b\|_\infty \big(\frac{1}{\sqrt{S}} \big)^{\alpha+\beta} \Big] \end{aligned}$$

Then we see that $E_{\lambda,\mu} = 0$ if and only if $\Lambda = C(n, \alpha, \beta, q, S)$, where

$$\begin{aligned} C(n,\alpha,\beta,q,S) &= \Big(\frac{(1+q)}{(\alpha+\beta-1+q)} \Big)^{\frac{2}{\alpha+\beta-2}} \Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q} \Big)^{\frac{2}{1+q}} \\ &\quad \times \Big(\frac{1}{\|b\|_\infty} \Big)^{\frac{2}{\alpha+\beta-2}} S^{\frac{2(\alpha+\beta-1+q)}{(1+q)(\alpha+\beta-2)}}. \end{aligned}$$

Thus for $(\lambda,\mu) \in \Gamma$, we have $E_{\lambda,\mu} > 0$, and therefore it follows from (3.3) that $\psi_{u,w}(t_{\max}) > 0$.

(i) If $\int_\Omega b(x) u_+^\alpha w_+^\beta \, dx \ge 0$, then $\psi_{u,w}(t) \to -\int_\Omega b(x) u_+^\alpha w_+^\beta \, dx < 0$ as $t \to \infty$. Consequently, $\psi_{u,w}(t)$ has exactly two points $0 < t_1 < t_{\max} < t_2$ such that

$$\psi_{u,w}(t_1) = 0 = \psi_{u,w}(t_2) \quad \text{and} \quad \psi'_{u,w}(t_1) > 0 > \psi'_{u,w}(t_2).$$

Now we show that if $\psi_{u,w}(t)=0$ and $\psi'_{u,w}(t)>0$, then $(tu,tw)\in\mathcal{N}^+_{\lambda,\mu}$.

$$\begin{aligned}\psi_{u,w}(t)=0 &\Leftrightarrow \|(tu,tw)\|^2=K_{\lambda,\mu}(tu,tw)+\int_\Omega b(x)(tu)_+^\alpha(tw)_+^\beta dx\\ &\Leftrightarrow (tu,tw)\in\mathcal{N}_{\lambda,\mu},\end{aligned}$$

and therefore

$$\begin{aligned}&\psi'_{u,w}(t)>0\\ &\Rightarrow (2-\alpha-\beta)t^{1-\alpha-\beta}\|(u,w)\|^2-(-\alpha-\beta+1-q)t^{-\alpha-\beta-q}K_{\lambda,\mu}(u,w)>0\\ &\Rightarrow (2-\alpha-\beta)\|(tu,tw)\|^2+(\alpha+\beta-1+q)\Big[\|(tu,tw)\|^2\\ &\quad-\int_\Omega b(x)(tu)_+^\alpha(tw)_+^\beta dx\Big]>0,\\ &\Rightarrow (1+q)\|(tu,tw)\|^2-(\alpha+\beta-1+q)\int_\Omega b(x)(tu)_+^\alpha(tw)_+^\beta dx>0\\ &\Rightarrow (tu,tw)\in\mathcal{N}^+_{\lambda,\mu}.\end{aligned}$$

Similarly one can show that if $\psi_{u,w}(t)=0$ and $\psi'_{u,w}(t)<0$, then $(tu,tw)\in\mathcal{N}^-_{\lambda,\mu}$.

Now $\phi'_{u,w}(t)=t^{\alpha+\beta-1}\psi_{u,w}(t)$. Thus $\phi'_{u,w}(t)<0$ in $(0,t_1)$, $\phi'_{u,w}(t)>0$ in (t_1,t_2) and $\phi'_{u,w}(t)<0$ in (t_2,∞). Hence $J_{\lambda,\mu}(t_1u,t_1w)=\inf_{0\le t\le t_{\max}}J_{\lambda,\mu}(tu,tw)$, $J_{\lambda,\mu}(t_2u,t_2w)=\sup_{t\ge t_1}J_{\lambda,\mu}(tu,tw)$. Moreover $(t_1u,t_1w)\in\mathcal{N}^+_{\lambda,\mu}$ and $(t_2u,t_2w)\in\mathcal{N}^-_{\lambda,\mu}$.

(ii) If $\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx<0$ and $\psi_{u,w}(t)\to-\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx>0$ as $t\to\infty$. Consequently, $\psi_{u,w}(t)$ has exactly one point $0<t_1<t_{\max}$ such that

$$\psi_{u,w}(t_1)=0 \text{ and } \psi'_{u,w}(t_1)>0.$$

Using $\phi'_{u,w}(t)=t^{\alpha+\beta-1}\psi_{u,w}(t)$, we have $\phi'_{u,w}(t)<0$ in $(0,t_1)$, $\phi'_{u,w}(t)>0$ in (t_1,∞). So, $J_{\lambda,\mu}(t_1u,t_1w)=\inf_{t\ge0}J_{\lambda,\mu}(tu,tw)$. Hence, it follows that $(t_1u,t_1w)\in\mathcal{N}^+_{\lambda,\mu}$. □

Corollary 3.2. *Suppose that $(\lambda,\mu)\in\Gamma$, then $\mathcal{N}^\pm_{\lambda,\mu}\neq\emptyset$.*

Proof. From assumptions (A1) and (A2), we can choose $(u,w)\in Y\setminus\{(0,0)\}$ such that $K_{\lambda,\mu}(u,w)>0$ and $\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx>0$. By (ii) of Lemma 3.1, there exists unique t_1 and t_2 such that $(t_1u,t_1w)\in\mathcal{N}^+_{\lambda,\mu}$, $(t_2u,t_2w)\in\mathcal{N}^-_{\lambda,\mu}$. In conclusion, $\mathcal{N}^\pm_{\lambda,\mu}\neq\emptyset$. □

Lemma 3.3. *For $(\lambda,\mu)\in\Gamma$, we have $\mathcal{N}^0_{\lambda,\mu}=\{(0,0)\}$.*

Proof. We prove this by contradiction. Assume that there exists $(0,0)\not\equiv(u,w)\in\mathcal{N}^0_{\lambda,\mu}$. Then it follows from $(u,w)\in\mathcal{N}^0_{\lambda,\mu}$ that

$$(1+q)\|(u,w)\|^2=(\alpha+\beta-1+q)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx$$

and consequently

$$\begin{aligned}0&=\|(u,w)\|^2-K_{\lambda,\mu}(u,w)-\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx\\ &=\frac{(\alpha+\beta-2)}{(\alpha+\beta-1+q)}\|(u,w)\|^2-K_{\lambda,\mu}(u,w).\end{aligned}$$

Therefore, as $(\lambda,\mu)\in\Gamma$ and $(u,w)\not\equiv(0,0)$, we use similar arguments as those in (3.3) to obtain

$$\begin{aligned}
0 &< E_{\lambda,\mu}\|(u,w)\|^{\alpha+\beta}\\
&\le \frac{(1+q)}{(\alpha+\beta-1+q)}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{\alpha+\beta-2}{1+q}}\frac{\|(u,w)\|^{\frac{2(\alpha+\beta-1+q)}{1+q}}}{[K_{\lambda,\mu}(u,w)]^{\frac{\alpha+\beta-2}{1+q}}}\\
&\quad-\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx\\
&= \frac{(1+q)}{(\alpha+\beta-1+q)}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{\alpha+\beta-2}{1+q}}\frac{\|(u,w)\|^{\frac{2(\alpha+\beta-1+q)}{1+q}}}{\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\|(u,w)\|^2\Big)^{\frac{\alpha+\beta-2}{1+q}}}\\
&\quad-\frac{(1+q)}{(\alpha+\beta-1+q)}\|(u,w)\|^2=0,
\end{aligned}$$

a contradiction. Hence $(u,w)=(0,0)$. That is, $\mathcal{N}^0_{\lambda,\mu}=\{(0,0)\}$. □

We note that Γ is also related to a gap structure in $\mathcal{N}_{\lambda,\mu}$.

Lemma 3.4. *Suppose that $(\lambda,\mu)\in\Gamma$, then there exist a gap structure in $\mathcal{N}_{\lambda,\mu}$,*

$$\|(U,W)\|>A_0>A_{\lambda,\mu}>\|(u,w)\|\quad\text{for all } (u,w)\in\mathcal{N}^+_{\lambda,\mu},(U,W)\in\mathcal{N}^-_{\lambda,\mu},$$

where

$$\begin{gathered}
A_0=\Big[\frac{(1+q)}{(\alpha+\beta-1+q)\|b\|_\infty}(\sqrt{S})^{\alpha+\beta}\Big]^{\frac{1}{\alpha+\beta-2}},\\
A_{\lambda,\mu}=\Big[\frac{(\alpha+\beta-1+q)}{(\alpha+\beta-2)}\big(\frac{1}{\sqrt{S}}\big)^{1-q}\Big]^{\frac{1}{1+q}}\Lambda^{1/2}.
\end{gathered}$$

Proof. If $w\in\mathcal{N}^+_{\lambda,\mu}\subset\mathcal{N}_{\lambda,\mu}$, then

$$\begin{aligned}
0&<(1+q)\|(u,w)\|^2-(\alpha+\beta-1+q)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx\\
&=(2-\alpha-\beta)\|(u,w)\|^2+(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w).
\end{aligned}$$

Hence it follows from (3.1)

$$\begin{aligned}
&(\alpha+\beta-2)\|(u,w)\|^2\\
&<(\alpha+\beta-1+q)K_{\lambda,\mu}(u,w)\\
&\le(\alpha+\beta-1+q)\Big((|\lambda|\|f\|_{q^*})^{\frac{2}{1+q}}+(|\mu|\|g\|_{q^*})^{\frac{2}{1+q}}\Big)^{\frac{1+q}{2}}\Big(\frac{\|(u,w)\|}{\sqrt{S}}\Big)^{1-q}
\end{aligned}$$

which yields

$$\begin{aligned}
\|(u,w)\|&<\Big[\frac{(\alpha+\beta-1+q)}{(\alpha+\beta-2)}\big(\frac{1}{\sqrt{S}}\big)^{1-q}\Big]^{\frac{1}{1+q}}\Big(\Big(|\lambda|\|f\|_{q^*}\Big)^{\frac{2}{1+q}}+\Big(|\mu|\|g\|_{q^*}\Big)^{\frac{2}{1+q}}\Big)^{1/2}\\
&\equiv A_{\lambda,\mu}.
\end{aligned}$$

If $(U,W)\in\mathcal{N}^-_{\lambda,\mu}$, then from (3.2) it follows that

$$(1+q)\|(U,W)\|^2<(\alpha+\beta-1+q)\int_\Omega b(x)U_+^\alpha W_+^\beta dx$$

$$\le (\alpha+\beta-1+q)\|b\|_\infty\Big(\frac{\|(U,W)\|}{\sqrt{S}}\Big)^{\alpha+\beta}$$

which yields

$$\|(U,W)\| > \Big[\frac{(1+q)}{(\alpha+\beta-1+q)\|b\|_\infty}(\sqrt{S})^{\alpha+\beta}\Big]^{\frac{1}{\alpha+\beta-2}} \equiv A_0.$$

Now we show that $A_{\lambda,\mu} = A_0$ if and only if $\Lambda = C(n,\alpha,\beta,q,S)$.

$$\begin{aligned}
&\Lambda = C(n,\alpha,\beta,q,S)\\
&= \Big(\frac{(1+q)}{\|b\|_\infty(\alpha+\beta-1+q)}\Big)^{\frac{2}{\alpha+\beta-2}}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{2}{1+q}} S^{\frac{2(\alpha+\beta-1+q)}{(1+q)(\alpha+\beta-2)}}.\\
&\Leftrightarrow A_{\lambda,\mu} = \Lambda^{1/2}\Big[\frac{(\alpha+\beta-1+q)}{(\alpha+\beta-2)}\Big(\frac{1}{\sqrt{S}}\Big)^{1-q}\Big]^{\frac{1}{1+q}}\\
&= \Big(\frac{(1+q)}{\|b\|_\infty(\alpha+\beta-1+q)}\Big)^{\frac{1}{\alpha+\beta-2}}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{1}{1+q}} S^{\frac{\alpha+\beta-1+q}{(1+q)(\alpha+\beta-2)}}\\
&\quad\times\Big[\frac{(\alpha+\beta-1+q)}{(\alpha+\beta-2)}\Big(\frac{1}{\sqrt{S}}\Big)^{1-q}\Big]^{\frac{1}{1+q}} \equiv A_0.
\end{aligned}$$

Thus for all $(\lambda,\mu)\in\Gamma$, we can conclude that

$$\|(U,W)\| > A_0 > A_{\lambda,\mu} > \|(u,w)\| \quad \text{for all } (u,w)\in\mathcal{N}^+_{\lambda,\mu},\ (U,W)\in\mathcal{N}^-_{\lambda,\mu}.$$

This completes the proof. □

Lemma 3.5. *Suppose that $(\lambda,\mu)\in\Gamma$, then $\mathcal{N}^-_{\lambda,\mu}$ is a closed set in Y-topology.*

Proof. Let $\{(U_k,W_k)\}$ be a sequence in $\mathcal{N}^-_{\lambda,\mu}$ with $(U_k,W_k)\to(U,W)$ in Y. Then we have

$$\begin{aligned}
&\|(U_k,W_k)\|^2\\
&= \lim_{k\to\infty}\|(U_k,W_k)\|^2\\
&= \lim_{k\to\infty}\Big[\int_\Omega(\lambda f(x)(U_k)_+^{1-q}+\mu g(x)(W_k)_+^{1-q})dx + \int_\Omega b(x)(U_k)_+^{\alpha}(W_k)_+^{\beta}dx\Big]\\
&= \int_\Omega(\lambda f(x)U_+^{1-q}+\mu g(x)W_+^{1-q})dx + \int_\Omega b(x)U_+^{\alpha}W_+^{\beta}dx
\end{aligned}$$

and

$$\begin{aligned}
&(1+q)\|(U,W)\| - (\alpha+\beta-1+q)\int_\Omega b(x)U_+^\alpha W_+^\beta dx\\
&= \lim_{k\to\infty}\Big[(1+q)\|(U_k,W_k)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)(U_k)_+^\alpha(W_k)_+^\beta dx\Big] \le 0,
\end{aligned}$$

i.e. $(U,W)\in\mathcal{N}^-_{\lambda,\mu}\cap\mathcal{N}^0_{\lambda,\mu}$. Since $\{(U_k,W_k)\}\subset\mathcal{N}^-_{\lambda,\mu}$, from Lemma 3.4 we have

$$\|(U,W)\| = \lim_{k\to\infty}\|(U_k,W_k)\| \ge A_{\lambda,\mu} > 0;$$

that is, $(U,W)\not\equiv(0,0)$. It follows from Lemma 3.1, that $(U,W)\notin\mathcal{N}^0_{\lambda,\mu}$ for any $(\lambda,\mu)\in\Gamma$. Thus $(U,W)\in\mathcal{N}^-_{\lambda,\mu}$. That is, $\mathcal{N}^-_{\lambda,\mu}$ is a closed set in Y- topology for any $(\lambda,\mu)\in\Gamma$. □

Lemma 3.6. *Let $(u,w)\in\mathcal{N}^{\pm}_{\lambda,\mu}$, then for any $\Phi=(\phi,\psi)\in C_Y$, there exists a number $\epsilon>0$ and a continuous function $f: B_\epsilon(0):=\{v=(v_1,v_2)\in Y:\|v\|<\epsilon\}\to\mathbb{R}^+$ such that*

$$f(v_1,v_2)>0, f(0,0)=1 \quad \text{and} \quad f(v_1,v_2)(u+v_1\phi,w+v_2\psi)\in\mathcal{N}^{\pm}_{\lambda,\mu}$$

for all $v\in B_\epsilon(0)$.

Proof. We give the proof only for the case $(u,w)\in\mathcal{N}^{+}_{\lambda,\mu}$, the case $\mathcal{N}^{-}_{\lambda,\mu}$ may be proved the same way. For any $\Phi=(\phi,\psi)\in C_Y$, we define $F:Y\times\mathbb{R}^+\to\mathbb{R}$ as follows:

$$\begin{aligned}
F(v,t) :=& F((v_1,v_2),t)\\
=& t^{1+q}\|(u+v_1\phi,w+v_2\psi)\|^2-t^{\alpha+\beta-1+q}\int_\Omega b(x)(u+v_1\phi)_+^\alpha(w+v_2\psi)_+^\beta dx\\
&-K_{\lambda,\mu}(u+v_1\phi,w+v_2\psi)
\end{aligned}$$

Since $w\in\mathcal{N}^{+}_{\lambda,\mu}(\subset\mathcal{N}_{\lambda,\mu})$, we have

$$F((0,0),1)=\|(u,w)\|^2-K_{\lambda,\mu}(u,w)-\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx=0,$$

and

$$\frac{\partial F}{\partial t}((0,0),1)=(1+q)\|(u,w)\|^2-(\alpha+\beta-1+q)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx>0.$$

Applying the implicit function theorem at the point $((0,0),1)$, we have that there exists $\bar\epsilon>0$ such that for $\|v\|<\bar\epsilon$, $v\in Y$, the equation $F((v_1,v_2),t)=0$ has a unique continuous solution $t=f(v_1,v_2)>0$. It follows from $F((0,0),1)=0$ that $f(0,0)=1$ and from $F((v_1,v_2),f(v_1,v_2))=0$ for $\|v\|<\bar\epsilon$, $v\in Y$ that

$$\begin{aligned}
0&=f^{1+q}(v)\|(u+v_1\phi,w+v_2\psi)\|^2-K_{\lambda,\mu}(u+v_1\phi,w+v_2\psi)\\
&\quad-f^{\alpha+\beta-1+q}(v)\int_\Omega b(x)(u+v_1\phi)_+^\alpha(x)(w+v_2\psi)_+^\beta(x)dx\\
&=\frac{\|f(v)(u+v_1\phi,w+v_2\psi)\|^2-K_{\lambda,\mu}(f(v)(u+v_1\phi),f(v)(w+v_2\psi))}{f^{1-q}(v)}\\
&\quad-\frac{\int_\Omega b(x)(f(v)(u+v_1\phi))_+^\alpha(x)(f(v)(w+v_2\psi))_+^\beta(x)dx}{f^{1-q}(v)};
\end{aligned}$$

that is,

$$f(v_1,v_2)(u+v_1\phi,w+v_2\psi)\in\mathcal{N}_{\lambda,\mu}\quad\text{for all } v\in Y,\|v\|<\tilde\epsilon.$$

Since $\frac{\partial F}{\partial t}((0,0),1)>0$ and

$$\begin{aligned}
&\frac{\partial F}{\partial t}((v_1,v_2),f(v_1,v_2))\\
&=(1+q)f^q(v)\|(u+v_1\phi,w+v_2\psi\|^2\\
&\quad-(\alpha+\beta-1+q)f^{\alpha+\beta-1+q-1}(v)\int_\Omega b(x)(u+v_1\phi)_+^\alpha(w+v_2\psi)_+^\beta\\
&=\frac{(1+q)\|(f(v)(u+v_1\phi),f(v)(w+v_2\psi))\|^2}{f^{2-q}(v)}\\
&\quad-\frac{(\alpha+\beta-1+q)\int_\Omega b(x)(f(v)(u+v_1\phi))_+^\alpha(f(v)(w+v_2\psi))_+^\beta dx}{f^{2-q}(v)}
\end{aligned}$$

we can take $\epsilon > 0$ possibly smaller ($\epsilon < \bar{\epsilon}$) such that for any $v = (v_1, v_2) \in Y$, $\|v\| < \epsilon$,

$$\begin{aligned}&(1+q)\|(f(v)(u+v_1\phi), f(v)(w+v_2\psi))\|^2\\ &-(\alpha+\beta-1+q)\int_\Omega b(x)(f(v)(u+v_1\phi))_+^\alpha (f(v)(w+v_2\psi))_+^\beta dx > 0;\end{aligned}$$

that is,

$$f(v_1, v_2)(u+v_1\phi, w+v_2\psi) \in \mathcal{N}^+_{\lambda,\mu} \quad \text{for all } v = (v_1, v_2) \in B_\epsilon(0).$$

This completes the proof. □

Lemma 3.7. *J_λ is bounded below and coercive on $\mathcal{N}_{\lambda,\mu}$.*

Proof. For $(u, w) \in \mathcal{N}_{\lambda,\mu}$, from (3.1) we obtain

$$\begin{aligned}&J_{\lambda,\mu}(u,w)\\ &= \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|(u,w)\|^2 - \Big(\frac{1}{1-q} - \frac{1}{\alpha+\beta}\Big)K_{\lambda,\mu}(u,w)\\ &\geq \Big(\frac{1}{2} - \frac{1}{\alpha+\beta}\Big)\|(u,w)\|^2 - \Big(\frac{1}{1-q} - \frac{1}{\alpha+\beta}\Big)\Lambda^{\frac{1+q}{2}}\Big(\frac{\|(u,w)\|}{\sqrt{\mathrm{S}}}\Big)^{1-q},\end{aligned} \tag{3.4}$$

where Λ is given in (2.3). Now consider the function $\rho : \mathbb{R}^+ \to \mathbb{R}$ as $\rho(t) = ct^2 - dt^{1-q}$, where c, d are both positive constants. One can easily show that ρ is convex($\rho''(t) > 0$ for all $t > 0$) with $\rho(t) \to 0$ as $t \to 0$ and $\rho(t) \to \infty$ as $t \to \infty$. ρ achieves its minimum at $t_{min} = [\frac{d(1-q)}{2c}]^{\frac{1}{1+q}}$ and

$$\rho(t_{min}) = c\Big[\frac{d(1-q)}{2c}\Big]^{\frac{2}{1+q}} - d\Big[\frac{d(1-q)}{2c}\Big]^{\frac{1-q}{1+q}} = -\frac{(1+q)}{2}d^{\frac{2}{1+q}}\Big(\frac{1-q}{2c}\Big)^{\frac{1-q}{1+q}}.$$

Applying $\rho(t)$ with $c = \big(\frac{1}{2} - \frac{1}{\alpha+\beta}\big)$, $d = \big(\frac{1}{1-q} - \frac{1}{\alpha+\beta}\big)\Lambda^{\frac{1+q}{2}}\big(\frac{1}{\sqrt{S}}\big)^{1-q}$ and $t = \|(u,w)\|$, $(u, w) \in \mathcal{N}_{\lambda,\mu}$, we obtain from (3.4) that

$$\lim_{\|(u,w)\|\to\infty} J_{\lambda,\mu}(u,w) \geq \lim_{t\to\infty} \rho(t) = \infty,$$

since $0 < q < 1$. That is $J_{\lambda,\mu}$ is coercive on $\mathcal{N}_{\lambda,\mu}$. Moreover it follows from (3.4) that

$$J_{\lambda,\mu}(u,w) \geq \rho(t) \geq \rho(t_{min}) \quad \text{(a constant)}, \tag{3.5}$$

i.e.

$$\begin{aligned}J_{\lambda,\mu}(u,w) &\geq -\frac{(1+q)}{2}d^{\frac{2}{1+q}}\left(\frac{1-q}{2c}\right)^{\frac{1-q}{1+q}}\\ &= -\frac{(1+q)(\alpha+\beta-2)}{(1-q)(\alpha+\beta)}\left(\frac{\alpha+\beta-1+q}{2(\alpha+\beta-2)}\right)^{\frac{2}{1+q}}\Lambda\Big(\frac{1}{\sqrt{S}}\Big)^{\frac{2(1-q)}{1+q}}.\end{aligned}$$

Thus $J_{\lambda,\mu}$ is bounded below on $\mathcal{N}_{\lambda,\mu}$. □

4. Existence of Solutions in $\mathcal{N}^{\pm}_{\lambda,\mu}$

Now from Lemma 3.5, $\mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}$ and $\mathcal{N}^{-}_{\lambda,\mu}$ are two closed sets in Y provided $(\lambda,\mu) \in \Gamma$. Consequently, the Ekeland variational principle can be applied to the problem of finding the infimum of $J_{\lambda,\mu}$ on both $\mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}$ and $\mathcal{N}^{-}_{\lambda,\mu}$. First, consider $\{(u_k,w_k)\} \subset \mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}$ with the following properties:

$$J_{\lambda,\mu}(u_k,w_k) < \inf_{(u,w)\in \mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}} J_{\lambda,\mu}(u,w) + \frac{1}{k}, \tag{4.1}$$

$$\begin{gathered} J_{\lambda,\mu}(u,w) \ge J_{\lambda,\mu}(u_k,w_k) - \frac{1}{k}\|(u-u_k, w-w_k)\| \\ \text{for all } (u,w) \in \mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu} \end{gathered}. \tag{4.2}$$

Lemma 4.1. *The sequence $\{(u_k,w_k)\}$ is bounded in $\mathcal{N}_{\lambda,\mu}$. Moreover, there exists $0 \not\equiv (u,w) \in Y$ such that $(u_k,w_k) \rightharpoonup (u,w)$ weakly in Y.*

Proof. From equations (3.5) and (4.1), we have

$$ct^2 - dt^{1-q} = \rho(t) \le J_{\lambda,\mu}(u,w) < \inf_{(u,w)\in \mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}} J_{\lambda,\mu}(u,w) + \frac{1}{k} \le C_5,$$

for sufficiently large k and a suitable positive constant. Hence, putting $t = \|(u_k,w_k)\|$ in the above equation, we obtain the sequence $\{(u_k,w_k)\}$ is bounded.

Let $\{(u_k,w_k)\}$ is bounded sequence in Y. Then, there exists a subsequence of $\{(u_k,w_k)\}_k$, still denoted by $\{(u_k,w_k)\}_k$ and $(u,w) \in Y$ such that $(u_k,w_k) \rightharpoonup (u,w)$ weakly in Y, $(u_k,w_k)(\cdot) \to (u,w)(\cdot)$ strongly in $(L^r(\Omega))^2$ for $1 \le r < 2^*_s$ and $u_k(\cdot) \to u(\cdot)$, $w_k(\cdot) \to w(\cdot)$ a.e. in Ω.

For any $(u,w) \in \mathcal{N}^{+}_{\lambda,\mu}$, from $0<q<1$, $2 < \alpha+\beta < 2^*_s$ we have

$$\begin{aligned} J_{\lambda,\mu}(u,w) &= \Big(\frac{1}{2} - \frac{1}{1-q}\Big)\|(u,w)\|^2 + \Big(\frac{1}{1-q} - \frac{1}{\alpha+\beta}\Big)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx \\ &< \Big(\frac{1}{2} - \frac{1}{1-q}\Big)\|(u,w)\|^2 + \Big(\frac{1}{1-q} - \frac{1}{\alpha+\beta}\Big)\frac{1+q}{\alpha+\beta-1+q}\|(u,w)\|^2 \\ &= \Big(\frac{1}{\alpha+\beta} - \frac{1}{2}\Big)\frac{(1+q)}{(1-q)}\|(u,w)\|^2 < 0, \end{aligned}$$

which means that $\inf_{\mathcal{N}^{+}_{\lambda,\mu}} J_{\lambda,\mu} < 0$. Now for $(\lambda,\mu) \in \Gamma$, we know from Lemma 3.1, that $\mathcal{N}^{0}_{\lambda,\mu} = \{(0,0)\}$. Together, these imply that $(u_k,w_k) \in \mathcal{N}^{+}_{\lambda,\mu}$ for k large and

$$\inf_{(u,w)\in \mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}} J_{\lambda,\mu}(u,w) = \inf_{(u,w)\in \mathcal{N}^{+}_{\lambda,\mu}} J_{\lambda,\mu}(u,w) < 0.$$

Therefore, by weak lower semi-continuity of the norm,

$$J_{\lambda,\mu}(u,w) \le \liminf_{k\to\infty} J_{\lambda,\mu}(u_k,w_k) = \inf_{\mathcal{N}^{+}_{\lambda,\mu} \cup \mathcal{N}^{0}_{\lambda,\mu}} J_{\lambda,\mu} < 0,$$

that is, $(u,w) \not\equiv 0$ and $(u,w) \in Y$. ∎

Lemma 4.2. *Suppose $(u_k,w_k) \in \mathcal{N}^{+}_{\lambda,\mu}$ such that $(u_k,w_k) \rightharpoonup (u,w)$ weakly in Y. Then for $(\lambda,\mu) \in \Gamma$,*

$$(1+q)\int_\Omega (\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx - (\alpha+\beta-2)\int_\Omega b(x)u_+^\alpha w_+^\beta dx > 0. \tag{4.3}$$

Moreover, there exists a constant $C_2 > 0$ such that

$$(1+q)\|(u_k,w_k)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \ge C_2 > 0. \tag{4.4}$$

Proof. For $\{(u_k,w_k)\} \subset \mathcal{N}^+_{\lambda,\mu} (\subset \mathcal{N}_{\lambda,\mu})$, we have

$$\begin{aligned}
&(1+q)K_{\lambda,\mu}(u,w) - (\alpha+\beta-2)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx \\
&= \lim_{k\to\infty}\Big[(1+q)K_{\lambda,\mu}(u_k,w_k) - (\alpha+\beta-2)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta\Big] \\
&= \lim_{k\to\infty}\Big[(1+q)\|(u_k,w_k)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta\Big] \ge 0.
\end{aligned}$$

Now, we can argue by a contradiction and assume that

$$(1+q)K_{\lambda,\mu}(u,w) - (\alpha+\beta-2)\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx = 0. \tag{4.5}$$

Using $(u_k,w_k) \in \mathcal{N}_{\lambda,\mu}$, the weak lower semi continuity of norm and (4.5) we have that

$$\begin{aligned}
0 &= \lim_{k\to\infty}\Big[\|(u_k,w_k)\|^2 - K_{\lambda,\mu}(u_k,w_k) - \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta\Big] \\
&\ge \|(u,w)\|^2 - K_{\lambda,\mu}(u,w) - \int_\Omega b(x)u_+^\alpha w_+^\beta\,dx \\
&= \begin{cases} \|(u,w)\|^2 - \frac{\alpha+\beta-1+q}{1+q}\int_\Omega b(x)u_+^\alpha w_+^\beta\,dx & \text{(solving for } K \text{ in (3.2))}, \\ \|(u,w)\|^2 - \frac{\alpha+\beta-1+q}{\alpha+\beta-2}K_{\lambda,\mu}(u,w) & \text{(solving for } \int_\Omega \text{ in (3.2))}. \end{cases}
\end{aligned}$$

Thus for any $(\lambda,\mu) \in \Gamma$ and $(u,w) \not\equiv 0$, by similar arguments as those in (3.3) we have that

$$\begin{aligned}
&0 < E_{\lambda,\mu}\|(u,w)\|^{\alpha+\beta} \\
&\le \frac{(1+q)}{(\alpha+\beta-1+q)}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{\alpha+\beta-2}{1+q}} \frac{\|(u,w)\|^{\frac{2(\alpha+\beta-1+q)}{1+q}}}{[K_{\lambda,\mu}(u,w)]^{\frac{\alpha+\beta-2}{1+q}}} - \int_\Omega b(x)u_+^\alpha w_+^\beta\,dx \\
&= \frac{(1+q)}{(\alpha+\beta-1+q)}\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\Big)^{\frac{\alpha+\beta-2}{1+q}} \frac{\|(u,w)\|^{\frac{2(\alpha+\beta-1+q)}{1+q}}}{\Big(\frac{\alpha+\beta-2}{\alpha+\beta-1+q}\|(u,w)\|^2\Big)^{\frac{\alpha+\beta-2}{1+q}}} \\
&\quad - \frac{(1+q)}{(\alpha+\beta-1+q)}\|(u,w)\|^2 = 0,
\end{aligned}$$

which is clearly impossible. Now by (4.3), we have that

$$(1+q)K_{\lambda,\mu}(u_k,w_k) - (\alpha+\beta-2)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta \ge C_2 \tag{4.6}$$

for sufficiently large k and a suitable positive constant C_2. Then (4.6), together with the fact that $(u_k,w_k) \in \mathcal{N}_{\lambda,\mu}$ we obtain equation (4.4). $\square$

Fix $(\phi,\psi) \in C_Y$ with $\phi,\psi \ge 0$. Then we apply Lemma 3.6 with $(u_k,w_k) \in \mathcal{N}^+_{\lambda,\mu}$ (k large enough such that $\frac{(1-q)C_1}{k} < C_2$), we obtain a sequence of functions $f_k : B\epsilon_k(0) \subset Y \to \mathbb{R}$ such that $f_k(0,0) = 1$ and $f_k(s_1,s_2)(u_k+s_1\phi, w_k+s_2\psi) \in \mathcal{N}^+_{\lambda,\mu}$

for all $s = (s_1, s_2) \in B_{\epsilon_k}(0)$. It follows from $(u_k, w_k) \in \mathcal{N}_{\lambda,\mu}$ and $f_k(s_1, s_2)(u_k + s_1\phi, w_k + s_2\psi) \in \mathcal{N}_{\lambda,\mu}$ that

$$\|(u_k, w_k)\|^2 - K_{\lambda,\mu}(u_k, w_k) - \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta dx = 0 \tag{4.7}$$

and

$$\begin{aligned} &f_k^2(s_1, s_2)\|(u_k + s_1\phi, w_k + s_2\psi)\|^2 - f_k^{1-q}(s_1, s_2)K(u_k + s_1\phi, w_k + s_2\psi) \\ &- f_k^{\alpha+\beta}(s_1, s_2)\int_\Omega b(x)(u_k + s_1\phi)_+^\alpha (w_k + s_2\psi)_+^\beta dx = 0. \end{aligned} \tag{4.8}$$

Choose $0 < \rho < \epsilon_k$, and $(s_1, s_2) = (\rho v_1, \rho v_2)$ with $\|v\| < 1$ then we find $f_k(v_1, v_2)$ such that $f_k(0,0) = 1$ and $f_k(v_1, v_2)(u_k + v_1\phi, w_k + v_2\psi) \in \mathcal{N}^+_{\lambda,\mu}$ for all $v \in B_\rho(0)$.

Lemma 4.3. *For $(\lambda, \mu) \in \Gamma$ we have $|\langle f_k'(0,0), (v_1, v_2)\rangle|$ is finite for every $0 \le v = (v_1, v_2) \in C_Y$ with $\|v\| \le 1$.*

Proof. From (4.7) and (4.8) we have

$$\begin{aligned} 0 &= [f_k^2(\rho v_1, \rho v_2) - 1]\|(u_k + \rho v_1\phi, w_k + \rho v_2\psi)\|^2 \\ &\quad + \|(u_k + \rho v_1\phi, w_k + \rho v_2\psi)\|^2 - \|(u_k, w_k)\|^2 \\ &\quad - \left[f_k^{1-q}(\rho v_1, \rho v_2) - 1\right]\int_\Omega \Big(\lambda f(x)(u_k + \rho v_1\phi)_+^{1-q} + \mu g(x)(w_k + \rho v_2\psi)_+^{1-q}\Big) dx \\ &\quad - \lambda\int_\Omega f(x)\left[(u_k + \rho v_1\phi)_+^{1-q} - (u_k)_+^{1-q}\right] dx \\ &\quad - \mu\int_\Omega g(x)\left[(w_k + \rho v_2\psi)_+^{1-q} - (w_k)_+^{1-q}\right] dx \\ &\quad - \left[f_k^{\alpha+\beta}(\rho v_1, \rho v_2) - 1\right]\int_\Omega b(x)(u_k + \rho v_1\phi)_+^\alpha(x)(w_k + \rho v_2\psi)_+^\beta(x) dx \\ &\quad - \int_\Omega b(x)[(u_k + \rho v_1\phi)_+^\alpha(w_k + \rho v_2\psi)_+^\beta(x) - (u_k)_+^\alpha(w_k)_+^\beta(x)]dx, \\ &\le \left[f_k^2(\rho v_1, \rho v_2) - 1\right]\|(u_k + \rho v_1\phi, w_k + \rho v_2\psi)\|^2 + \|(u_k + \rho v_1\phi, w_k + \rho v_2\psi)\|^2 \\ &\quad - \|(u_k, w_k)\|^2 - \Big[f_k^{1-q}(\rho v_1, \rho v_2) - 1\Big]\int_\Omega \Big(\lambda f(x)(u_k + \rho v_1\phi)_+^{1-q}(x) \\ &\quad + \mu g(x)(w_k + \rho v_2\psi)_+^{1-q}(x)\Big) dx \\ &\quad - \left[f_k^{\alpha+\beta}(\rho v_1, \rho v_2) - 1\right]\int_\Omega b(x)(u_k + \rho v_1\phi)_+^\alpha(x)(w_k + \rho v_2\psi)_+^\beta(x) dx \\ &\quad - \int_\Omega b(x)\left[(u_k + \rho v_1\phi)_+^\alpha(w_k + \rho v_2\psi)_+^\beta(x) - (u_k)_+^\alpha(w_k)_+^\beta(x)\right] dx. \end{aligned}$$

Since

$$\begin{aligned} &(u_k + \rho v_1\phi)_+^{1-q}(x) - (u_k)_+^{1-q}(x) \\ &= \begin{cases} (u_k + \rho v_1\phi)^{1-q}(x) - (u_k)^{1-q}(x) & \text{if } u_k \ge 0 \\ 0 & \text{if } u_k \le 0, u_k + \rho v_1\phi \le 0 \\ (u_k + \rho v_1\phi)^{1-q}(x) & \text{if } u_k \le 0, u_k + \rho v_1\phi \ge 0, \end{cases} \end{aligned} \tag{4.9}$$

we have

$$\int_\Omega f(x)[((u_k + \rho v_1\phi)_+^{1-q} - (u_k)_+^{1-q})(x)]dx \ge 0.$$

Similarly, one can see that

$$\int_\Omega g(x)[((w_k+\rho v_2\psi)_+^{1-q}-(w_k)_+^{1-q})(x)]dx \geq 0.$$

Now dividing by $\rho>0$ and passing to the limit $\rho\to 0$, we derive that

$$\begin{aligned}
0 &\leq \langle f_k'(0,0),(v_1,v_2)\rangle\Big[2\|(u_k,w_k)\|^2-(1-q)K_{\lambda,\mu}(u_k,w_k)\\
&\quad -(\alpha+\beta)\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta dx\Big]\\
&\quad +2\int_Q\Big[(u_k(x)-u_k(y))\big((v_1\phi)(x)-(v_1\phi)(y)\big)\\
&\quad +(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy\\
&\quad -\alpha\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx-\beta\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^{\beta-1}v_2\psi dx\\
&= \langle f_k'(0,0),(v_1,v_2)\rangle\Big[(1+q)\|(u_k,w_k)\|^2\\
&\quad -(\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta dx\Big]\\
&\quad +2\int_Q\Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad +(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy\\
&\quad -\alpha\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx-\beta\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^{\beta-1}v_2\psi dx.
\end{aligned} \tag{4.10}$$

From (4.4) and (4.10), we know immediately that $\langle f_k'(0,0),(v_1,v_2)\rangle\neq-\infty$. Now we show that $\langle f_k'(0,0),(v_1,v_2)\rangle\neq+\infty$. Arguing by contradiction, we assume that $\langle f_k'(0,0),(v_1,v_2)\rangle=+\infty$. Since

$$|f_k(\rho v_1,\rho v_2)-1|\|(u_k,w_k)\|+\rho f_k(\rho v_1,\rho v_2)\|(v_1\phi,v_2\psi)\| \tag{4.11}$$
$$\geq \|[f_k(\rho v_1,\rho v_2)-1]\|(u_k,w_k\|+f_k(\rho v_1,\rho v_2)\|(\rho v_1\phi,\rho v_2\psi)\| \tag{4.12}$$
$$=\|f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi,w_k+\rho v_2\psi)-(u_k,w_k)\| \tag{4.13}$$

and

$$f_k(\rho v_1,\rho v_2)>f_k(0,0)=1$$

for sufficiently large k. From the definition of derivative $\langle f_k'(0,0),(v_1,v_2)\rangle$, applying equation (4.2) with $(u,w)=f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi,w_k+\rho v_2\psi)\in\mathcal{N}_{\lambda,\mu}^+$, we clearly have

$$\begin{aligned}
&[f_k(\rho v_1,\rho v_2)-1]\frac{\|(u_k,w_k)\|}{k}+\rho f_k(\rho v_1,\rho v_2)\frac{\|(v_1\phi,v_2\psi)\|}{k}\\
&\geq \frac{1}{k}\|f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi,w_k+\rho v_2\psi)-(u_k,w_k)\|\\
&\geq J_{\lambda,\mu}(u_k,w_k)-J_{\lambda,\mu}(f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi,w_k+\rho v_2\psi))\\
&=\Big(\frac12-\frac{1}{1-q}\Big)\|(u_k,w_k)\|^2+\Big(\frac{1}{1-q}-\frac12\Big)f_k^2(\rho v_1,\rho v_2)\|(u_k+\rho v_1\phi,w_k+\rho v_2\psi)\|^2
\end{aligned}$$

$$
\begin{aligned}
&+\Big(\frac{1}{1-q}-\frac{1}{\alpha+\beta}\Big)\Big(\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta \\
&-f_k^{\alpha+\beta}(\rho v_1,\rho v_2)\int_\Omega b(x)(u_k+\rho v_1\phi)_+^\alpha(w_k+\rho v_2\psi)_+^\beta dx\Big)\\
&=\frac{1+q}{1-q}\Big(\|(u_k+\rho v_1\phi,w_k+\rho v_2\psi)\|^2-\|(u_k,w_k)\|^2+[f_k^2(\rho v_1,\rho v_2)-1]\\
&\times\|(u_k+\rho v_1\phi,w_k+\rho v_2\psi)\|^2\Big)\\
&-\Big(\frac{1}{1-q}-\frac{1}{\alpha+\beta}\Big)f_k^{\alpha+\beta}(\rho v_1,\rho v_2)\int_\Omega b(x)\Big[(u_k+\rho v_1\phi)_+^\alpha(w_k+\rho v_2\psi)_+^\beta\\
&-(u_k)_+^\alpha(w_k)_+^\beta\Big]dx\\
&-\Big(\frac{1}{1-q}-\frac{1}{\alpha+\beta}\Big)\big[f_k^{\alpha+\beta}(\rho v_1,\rho v_2)-1\big]\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta dx.
\end{aligned}
$$

Dividing by $\rho>0$ and passing to the limit as $\rho\to 0$, we obtain

$$
\begin{aligned}
&\langle f_k'(0,0),(v_1,v_2)\rangle\frac{\|(u_k,w_k)\|}{k}+\frac{\|(v_1\phi,v_2\psi)\|}{k}\\
&\geq\Big(\frac{1+q}{1-q}\Big)\langle f_k'(0,0),(v_1,v_2)\rangle\|(u_k,w_k)\|^2\\
&\quad-\Big(\frac{\alpha+\beta-1+q}{1-q}\Big)\langle f_k'(0,0),(v_1,v_2)\rangle\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta\\
&+\Big(\frac{1+q}{1-q}\Big)\int_Q\Big(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad+(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy\\
&-\Big(\frac{\alpha+\beta-1+q}{1-q}\Big)\Big(\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx\\
&\quad+\frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^{\beta-1}v_2\psi dx\Big)\\
&=\frac{\langle f_k'(0,0),(v_1,v_2)\rangle}{1-q}\Big[(1+q)\|(u_k,w_k)\|^2-(\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^\beta\Big]\\
&\quad+\Big(\frac{1+q}{1-q}\Big)\int_Q\Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad+(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy\\
&-\Big(\frac{\alpha+\beta-1+q}{1-q}\Big)\Big[\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx\\
&\quad+\frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^\alpha(w_k)_+^{\beta-1}v_2\psi dx\Big];
\end{aligned}
$$

that is,

$$
\begin{aligned}
&\frac{\|(v_1\phi,v_2\psi)\|}{k}\\
&\geq\frac{\langle f_k'(0,0),(v_1,v_2)\rangle}{1-q}\Big[(1+q)\|(u_k,w_k)\|^2
\end{aligned}
$$

$$
\begin{aligned}
&\quad - (\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta - \frac{(1-q)\|(u_k,w_k)\|}{k}\Big]\\
&+\Big(\frac{1+q}{1-q}\Big)\int_Q \Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad + (w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&-\Big(\frac{\alpha+\beta-1+q}{1-q}\Big)\Big[\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx\\
&\quad + \frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2\psi\Big]
\end{aligned}
\tag{4.14}
$$

which is impossible because $\langle f_k'(0,0),(v_1,v_2)\rangle = +\infty$ and

$$
\begin{aligned}
&(1+q)\|(u_k,w_k)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta - \frac{(1-q)\|(u_k,w_k)\|}{k}\\
&\ge C_2 - \frac{(1-q)C_1}{k} > 0.
\end{aligned}
$$

In conclusion, $|\langle f_k'(0,0),(v_1,v_2)\rangle| < +\infty$. Furthermore (4.4) with $\|(u_k,w_k)\| \le C_1$ and two inequalities (4.10) and (4.14) also imply that

$$
|\langle f_k'(0,0),(v_1,v_2)\rangle| \le C_3
$$

for k sufficiently large and a suitable constant C_3. $\square$

Lemma 4.4. *For each $0 \le (\phi,\psi) \in C_Y$ and for every $0 \le v = (v_1,v_2) \in Y$ with $\|v\| \le 1$, we have $\lambda f(x)u_+^{-q}v_1\phi + \mu g(x)w_+^{-q}v_2\psi \in L^1(\Omega)$ and*

$$
\begin{aligned}
&\int_Q \frac{(u(x)-u(y))((v_1\phi)(x)-(v_1\phi)(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&+\int_Q \frac{(w(x)-w(y))((v_2\psi)(x)-(v_2\psi)(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&-\int_\Omega \big(\lambda f(x)u_+^{-q}v_1\phi + \mu g(x)w_+^{-q}v_2\psi\big)\,dx - \int_\Omega b(x)u_+^{\alpha-1}v_+^\beta v_1\phi dx\\
&-\int_\Omega b(x)u_+^\alpha w_+^{\beta-1}v_2\psi dx \ge 0.
\end{aligned}
\tag{4.15}
$$

Proof. Applying (4.11) and (4.2) again, we have that

$$
\begin{aligned}
&[f_k(\rho v_1,\rho v_2)-1]\frac{\|(u_k,w_k)\|}{k} + \rho f_k(\rho v_1,\rho v_2)\frac{\|(v_1\phi,v_2\psi)\|}{k}\\
&\ge \frac{1}{k}\|f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi, w_k+\rho v_2\psi) - (u_k,w_k)\|\\
&\ge J_{\lambda,\mu}(u_k,w_k) - J_{\lambda,\mu}(f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi, w_k+\rho v_2\psi))\\
&= \frac12\|(u_k,w_k)\|^2 - \frac12\|f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi, w_k+\rho v_2\psi)\|^2 dx\\
&\quad - \frac{1}{1-q}\int_\Omega \Big(\lambda f(u_k)_+^{1-q} + \mu g(w_k)_+^{1-q}\Big)\\
&\quad + \frac{1}{1-q}\int_\Omega \Big(\lambda f(x)(f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi))_+^{1-q}\\
&\quad + \mu g(x)(f_k(\rho v_1,\rho v_2)(w_k+\rho v_2\psi))_+^{1-q}\Big)dx
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{\alpha+\beta}\int_\Omega b(x)\Big[(f_k(\rho v_1,\rho v_2)(u_k+\rho v_1\phi))_+^\alpha (f_k(\rho v_1,\rho v_2)(w_k+\rho v_2\psi))_+^\beta \\
&- (u_k)_+^\alpha (w_k)_+^\beta\Big]dx \\
&= -\frac{f_k^2(\rho v_1,\rho v_2)-1}{2}\|(u_k,w_k)\|^2 - \frac{f_k^2(\rho v_1,\rho v_2)}{2}(\|(u_k+\rho v_1\phi, w_k+\rho v_2\psi)\|^2 \\
&- \|(u_k,w_k)\|^2) \\
&+ \frac{f_k^{1-q}(\rho v_1,\rho v_2)-1}{1-q}\int_\Omega \Big(\lambda f(x)(u_k+\rho v_1\phi)_+^{1-q} + \mu g(x)(w_k+\rho v_2\psi)_+^{1-q}\Big)\,dx \\
&+ \frac{1}{1-q}\int_\Omega \Big(\lambda f(x)\Big((u_k+\rho v_1\phi)_+^{1-q} - (u_k)_+^{1-q}\Big)(x) \\
&+ \mu g(x)\Big((w_k+\rho v_2\psi)_+^{1-q} - (w_k)_+^{1-q}\Big)(x)\Big)dx \\
&+ \frac{f_k^{\alpha+\beta}(\rho v_1,\rho v_2)-1}{\alpha+\beta}\int_\Omega b(x)(u_k+\rho v_1\phi)_+^\alpha(x)(w_k+\rho v_2\psi)_+^\beta(x)dx \\
&+ \frac{1}{\alpha+\beta}\int_\Omega b(x)\Big[\Big((u_k+\rho v_1\phi)_+^\alpha (w_k+\rho v_2\psi)_+^\beta - (u_k)_+^\alpha (w_k)_+^\beta\Big)(x)\Big]\,dx.
\end{aligned}
$$

Dividing by $\rho > 0$ and passing to the limit $\rho \to 0^+$, we obtain

$$
\begin{aligned}
&|\langle f_k'(0,0),(v_1,v_2)\rangle| \frac{\|(u_k,w_k)\|}{k} + \frac{\|(v_1\phi, v_2\psi)\|}{k} \\
&\geq -\langle f_k'(0,0),(v_1,v_2)\rangle\Big[\|(u_k,w_k)\|^2 - K_{\lambda,\mu}(u_k,w_k) - \int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^\beta dx\Big] \\
&\quad - \int_Q \Big(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y)) \\
&\quad + (w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx + \frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2\psi dx \\
&\quad + \frac{1}{1-q}\liminf_{\rho\to 0^+}\Big[\int_\Omega \frac{\lambda f(x)((u_k+\rho v_1\phi)_+^{1-q} - (u_k)_+^{1-q})}{\rho} \\
&\quad + \int_\Omega \frac{\mu g(x)((w_k+\rho v_2\psi)_+^{1-q} - (w_k)_+^{1-q})}{\rho}\Big] \\
&= -\int_Q \Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y)) \\
&\quad + (w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&\quad + \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^\beta v_1\phi dx + \frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^\alpha (w_k)_+^{\beta-1} v_2\psi dx \\
&+ \frac{1}{1-q}\liminf_{\rho\to 0^+}\Big[\int_\Omega \frac{\lambda f(x)((u_k+\rho v_1\phi)_+^{1-q} - (u_k)_+^{1-q})}{\rho} \\
&\quad + \int_\Omega \frac{\mu g(x)((w_k+\rho v_2\psi)_+^{1-q} - (w_k)_+^{1-q})}{\rho}\Big].
\end{aligned}
$$

Then by the above inequality, one can see that

$$\liminf_{\rho\to 0^+}\Big[\int_\Omega \frac{\lambda f(x)((u_k+\rho v_1\phi)_+^{1-q}-(u_k)_+^{1-q})}{\rho}dx + \int_\Omega \frac{\mu g(x)((w_k+\rho v_2\psi)_+^{1-q}-(w_k)_+^{1-q})}{\rho}dx\Big]$$

is finite. Now, using (4.9), we have

$$f(x)\Big((u_k+\rho v_1\phi)_+^{1-q}-(u_k)_+^{1-q}\Big)\geq 0.$$

Similarly we have

$$g(x)\Big((w_k+\rho v_2\psi)_+^{1-q}-(w_k)_+^{1-q}\Big)\geq 0,\quad \text{for all } x\in\Omega,\ \text{for all } t>0.$$

Then by Fatou's Lemma, we have

$$\begin{aligned}
&\int_\Omega \big(\lambda f(x)(u_k)_+^{-q}v_1\phi+\mu g(x)(w_k)_+^{-q}v_2\psi\big)\,dx\\
&\leq \frac{1}{1-q}\liminf_{\rho\to 0^+}\Big[\lambda\int_\Omega \frac{f(x)((u_k+\rho v_1\phi)_+^{1-q}-(u_k)_+^{1-q})}{\rho}\\
&\quad +\mu\int_\Omega \frac{g(x)((w_k+\rho v_2\psi)_+^{1-q}-(w_k)_+^{1-q})}{\rho}dx\Big]\\
&\leq \frac{|\langle f_k'(0,0),(v_1,v_2)\rangle|\|(u_k,w_k)\|+\|(v_1\phi,v_2\psi)\|}{k}\\
&\quad +\int_Q\Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad +(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy\\
&\quad -\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^{\beta}v_1\phi dx-\frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha}(w_k)_+^{\beta-1}v_2\psi dx\\
&\leq \frac{C_1C_3\|(v_1,v_2)\|+\|(v_1\phi,v_2\psi)\|}{k}-\frac{\alpha}{\alpha+\beta}\int_\Omega b(u_k)_+^{\alpha-1}(w_k)_+^{\beta}v_1\phi\\
&\quad -\frac{\beta}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha}(w_k)_+^{\beta-1}v_2\psi\\
&\quad +\int_Q\Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y))\\
&\quad +(w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy
\end{aligned}$$

Again using the Fatou's Lemma and the above relation, we have

$$\begin{aligned}
&\lambda\int_\Omega f(x)u_+^{-q}v_1\phi dx+\mu\int_\Omega g(x)w_+^{-q}v_2\psi dx\\
&\leq \int_\Omega\Big[\liminf_{k\to\infty}\big(\lambda f(x)u_+^{-q}v_1\phi+\mu g(x)w_+^{-q}v_2\psi\big)\Big]dx\\
&\leq \liminf_{k\to\infty}\int_\Omega\big(\lambda f(x)(u_k)_+^{-q}v_1\phi+\mu g(x)(w_k)_+^{-q}v_2\psi\big)\,dx\\
&\leq \frac{C_1C_3\|(v_1,v_2)\|+\|(v_1\phi,v_2\psi)\|}{k}-\frac{\alpha}{\alpha+\beta}\int_\Omega b(x)(u_k)_+^{\alpha-1}(w_k)_+^{\beta}v_1\phi
\end{aligned}$$

$$
\begin{aligned}
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(u_k)_+^{\alpha}(w_k)_+^{\beta-1}v_2\psi \\
&+ \int_Q \Big[(u_k(x)-u_k(y))((v_1\phi)(x)-(v_1\phi)(y)) \\
&+ (w_k(x)-w_k(y))((v_2\psi)(x)-(v_2\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy
\end{aligned}
$$

which completes the proof. □

Corollary 4.5. *For every $0 \le (\phi,\psi) \in Y$, we have $\big(\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi\big) \in L^1(\Omega)$, u_+, $w_+ > 0$ in Ω and*

$$
\begin{aligned}
&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&+ \int_Q \frac{(w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy - \lambda\int_\Omega f(x)u_+^{-q}\phi dx \\
&- \mu\int_\Omega g(x)w_+^{-q}\psi dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx \\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx \ge 0.
\end{aligned}
\tag{4.16}
$$

Proof. Choosing $v = (v_1, v_2) \in Y$ such that $v \ge 0$, $v \equiv l$ in the neighborhood of support of ϕ and $\|v\| \le 1$, for some $l > 0$ is a constant. Then we note that $\lambda\int_\Omega f(x)u_+^{-q}\phi dx + \mu\int_\Omega g(x)w_+^{-q}\psi dx < \infty$, for every $0 \le (\phi,\psi) \in C_Y$ which guarantees that u_+, $w_+ > 0$ a.e in Ω. Putting this choice of v in (4.15), for every $0 \le (\phi,\psi) \in C_Y$ we have

$$
\begin{aligned}
&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy + \int_Q \frac{(w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&- \lambda\int_\Omega f(x)u_+^{-q}\phi dx - \mu\int_\Omega g(x)w_+^{-q}\psi dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx \\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx \ge 0.
\end{aligned}
$$

Hence by density argument, (4.16) holds for every $0 \le (\phi,\psi) \in Y$, which completes the proof. □

Lemma 4.6. *We have that $u > 0$, $w > 0$ and $(u,w) \in \mathcal{N}^+_{\lambda,\mu}$.*

Proof. Using (4.16) with $\phi = u^-$, $\psi = w^-$, we obtain that

$$
\begin{aligned}
0 &\le \int_Q \frac{(u(x)-u(y))(u^-(x)-u^-(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_Q \frac{(w(x)-w(y))(w^-(x)-w^-(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\le -\|u^-\|^2 - \|w^-\|^2 - 2\int_Q \frac{u^-(x)u^+(y) + w^-(x)w^+(y)}{|x-y|^{n+2s}}\,dx\,dy \\
&\le -\|u^-\|^2 - \|w^-\|^2 \le 0.
\end{aligned}
$$

i.e, $u^- = w^- = 0$ a.e. So, $u = u^+ > 0$, $w = w^+ > 0$ a.e. by Corollary 4.5. Hence $u, w > 0$ in Ω. Now using (4.16) with $\phi = u$, $\psi = w$, we obtain that

$$\|(u,w)\|^2 \geq \int_\Omega \Big(\lambda f(x) u_+^{1-q} + \mu g(x) w_+^{1-q}\Big)\,dx + \int_\Omega b(x) u_+^\alpha w_+^\beta\,dx.$$

On the other hand, by the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|(u,w)\|^2 &\leq \liminf_{k\to\infty} \|(u_k,w_k)\|^2 \leq \limsup_{k\to\infty} \|(u_k,w_k)\|^2 \\ &= \int_\Omega \Big(\lambda f(x) u_+^{1-q} + \mu g(x) w_+^{1-q}\Big)\,dx + \int_\Omega b(x) u_+^\alpha w_+^\beta\,dx.\end{aligned}$$

Thus

$$\|(u,w)\|^2 = \int_\Omega \Big(\lambda f(x) u_+^{1-q} + \mu g(x) w_+^{1-q}\Big)\,dx + \int_\Omega b(x) u_+^\alpha w_+^\beta\,dx. \tag{4.17}$$

Consequently, $(u_k, w_k) \to (u, w)$ in Y and $(u, w) \in \mathcal{N}_{\lambda,\mu}$. Now from (4.3) it follows that

$$\begin{aligned}&(1+q)\|(u,w)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x) u_+^\alpha w_+^\beta\,dx \\ =&(1+q)\int_\Omega \Big(\lambda f(x) u_+^{1-q} + \mu g(x) w_+^{1-q}\Big)\,dx - (\alpha+\beta-2)\int_\Omega b(x) u_+^\alpha w_+^\beta\,dx > 0,\end{aligned}$$

that is, $(u, w) \in \mathcal{N}^+_{\lambda,\mu}$. □

Lemma 4.7. *the pair (u, w) is a positive weak solution of problem* (1.1).

Proof. Let $(u, w) = (u_1, u_2)$, $(\phi_1, \phi_2) \in Y$ and $\epsilon > 0$, then we define

$$\Psi(x) = (\Psi_1, \Psi_2) = ((u_1 + \epsilon\phi_1)_+, (u_2 + \epsilon\phi_2)_+)$$

For $i = 1, 2$, let $\Omega = \Omega_i \times \Gamma_i$ with

$$\Omega_i := \{x \in \Omega : u_i(x) + \epsilon\phi_i(x) > 0\}, \quad \Gamma_i := \{x \in \Omega : u_i(x) + \epsilon\phi_i(x) \leq 0\}.$$

Then $\Psi_i|_{\Omega_i}(x) = (u_i + \epsilon\phi_i)_+(x)$, and $\Psi_i|_{\Gamma_i}(x) = 0$. Decompose

$$\begin{aligned}Q :=&(\Omega_i \times \Omega^c) \cup (\Gamma_i \times \Omega^c) \cup (\Omega^c \times \Omega_i) \cup (\Omega^c \times \Gamma_i) \\ &\cup (\Gamma_i \times \Omega_i) \cup (\Omega_i \times \Gamma_i) \cup (\Omega_i \times \Omega_i) \cup (\Gamma_i \times \Gamma_i).\end{aligned}$$

Let $M_i(x,y) = u_i(x,y)((u_i + \epsilon\phi_i)^-(x) - (u_i + \epsilon\phi_i)^-(y))K(x,y)$, where $u_i(x,y) = (u_i(x) - u_i(y))$ and $K(x,y) = \frac{1}{|x-y|^{n+2s}}$. Then we have

(1) $\int_{\Omega_i \times \Omega^c} M_i(x,y)\,dx\,dy = \int_{\Omega^c \times \Omega_i} M_i(x,y)\,dx\,dy = 0$,
(2) $\int_{\Gamma_i \times \Omega^c} M_i(x,y)\,dx\,dy = -\int_{\Gamma_i \times \Omega^c} u_i(x)(u_i + \epsilon\phi_i)(x)K(x,y)\,dx\,dy$,
(3) $\int_{\Omega^c \times \Gamma_i} M_i(x,y)\,dx\,dy = -\int_{\Omega^c \times \Gamma_i} u_i(x)(u_i + \epsilon\phi_i)(x)K(x,y)\,dx\,dy$,
(4) $\int_{\Gamma_i \times \Omega_i} M_i(x,y)\,dx\,dy = -\int_{\Gamma_i \times \Omega_i} u_i(x,y)(u_i + \epsilon\phi_i)(x)K(x,y)\,dx\,dy$,
(5) $\int_{\Omega_i \times \Gamma_i} M_i(x,y)\,dx\,dy = -\int_{\Omega_i \times \Gamma_i} u_i(x,y)(u_i + \epsilon\phi_i)(x)K(x,y)\,dx\,dy$,
(6) $\int_{\Omega_i \times \Omega_i} M_i(x,y)\,dx\,dy = 0$,
(7) $\int_{\Gamma_i \times \Gamma_i} M_i(x,y)\,dx\,dy = -\int_{\Gamma_i \times \Gamma_i} u_i(x,y)((u_i + \epsilon\phi_i)(x) - (u_i + \epsilon\phi_i)(y))K(x,y)\,dx\,dy$.

Now relabeling $(\Psi_1, \Psi_2) = (\Phi, \Psi)$, $(u_1, u_2) = (u, w)$ and $(\phi_1, \phi_2) = (\phi, \psi)$. Then putting (Φ, Ψ) into (4.15) and using (4.17), we see that

$$0 \leq \int_Q \frac{u(x,y)(\Phi(x) - \Phi(y)) + w(x,y)(\Psi(x) - \Psi(y))}{|x-y|^{n+2s}}\,dx\,dy$$

$$
\begin{aligned}
&- \int_\Omega \big(\lambda f(x)u_+^{-q}\Phi + \mu g(x)w_+^{-q}\Psi\big)\,dx \\
&- \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\Phi dx - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\Psi dx \\
&= \int_Q \Big[u(x,y)((u+\epsilon\phi)(x) - (u+\epsilon\phi)(y)) \\
&\quad + w(x,y)((w+\epsilon\psi)(x) - (w+\epsilon\psi)(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&\quad + \int_Q \Big[u(x,y)((u+\epsilon\phi)^-(x) - (u+\epsilon\phi)^-(y)) \\
&\quad + w(x,y)((w+\epsilon\psi)^-(x) - (w+\epsilon\psi)^-(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&\quad - \int_\Omega (\lambda f(x)u_+^{-q}(u+\epsilon\phi) + \mu g(x)w_+^{-q}(w+\epsilon\psi)) \\
&\quad - \int_\Omega (\lambda f(x)u_+^{-q}(u+\epsilon\phi)^- + \mu g(x)w_+^{-q}(w+\epsilon\psi)^-) \\
&\quad - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi)dx - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi)dx \\
&- \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi)^-dx - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi)^-dx \\
&= \epsilon\Big(\int_Q \frac{u(x,y)(\phi(x)-\phi(y)) + w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \int_\Omega (\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi \\
&\quad - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi\Big) \\
&\quad + \int_Q \frac{|u(x)-u(y)|^2 + |w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad + \int_Q \Big[u(x,y)((u+\epsilon\phi)^-(x) - (u+\epsilon\phi)^-(y)) \\
&\quad + w(x,y)((w+\epsilon\psi)^-(x) - (w+\epsilon\psi)^-(y))\Big]/|x-y|^{n+2s}\,dx\,dy \\
&- \int_\Omega (\lambda f(x)u_+^{1-q} + \mu g(x)w_+^{1-q})dx + \lambda\int_{\Gamma_1} f(x)u_+^{-q}(u+\epsilon\phi)dx \\
&\quad + \mu\int_{\Gamma_2} g(x)w_+^{-q}(w+\epsilon\psi)dx + \frac{\alpha}{\alpha+\beta}\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi)\,dx \\
&\quad + \frac{\beta}{\alpha+\beta}\int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi)\,dx - \int_\Omega b(x)u_+^{\alpha}w_+^{\beta}dx \\
&= \epsilon\Big(\int_Q \frac{u(x,y)(\phi(x)-\phi(y)) + w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy \\
&\quad - \int_\Omega (\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi \\
&\quad - \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi\Big) - 2\int_{\Gamma_1\times\Omega^c} \frac{u(x)(u+\epsilon\phi)(x)}{|x-y|^{n+2s}}
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_{\Gamma_2\times\Omega^c} \frac{w(x)(w+\epsilon\psi)(x)}{|x-y|^{n+2s}}\,dx\,dy - 2\int_{\Gamma_1\times\Omega_1} \frac{u(x,y)(u+\epsilon\phi)(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&- 2\int_{\Gamma_2\times\Omega_2} \frac{w(x,y)(w+\epsilon\psi)(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&- 2\int_{\Gamma_1\times\Gamma_1} \frac{u(x,y)((u+\epsilon\phi)(x)-(u+\epsilon\phi)(y))}{|x-y|^{n+2s}}\\
&- 2\int_{\Gamma_2\times\Gamma_2} \frac{w(x,y)((w+\epsilon\psi)(x)-(w+\epsilon\psi)(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&+ \lambda\int_{\Gamma_1} f(x)u_+^{-q}(u+\epsilon\phi)dx + \mu\int_{\Gamma_2} g(x)w_+^{-q}(w+\epsilon\psi)\\
&+ \frac{\alpha}{\alpha+\beta}\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi) + \frac{\beta}{\alpha+\beta}\int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi)\\
= \epsilon\Big(&\int_Q \frac{u(x,y)(\phi(x)-\phi(y)) + w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&- \int_\Omega (\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx\\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx\Big) - 2\int_{\Gamma_1\times\Omega^c} \frac{|u(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&- 2\int_{\Gamma_2\times\Omega^c} \frac{|w(x)|^2}{|x-y|^{n+2s}}\,dx\,dy - 2\int_{\Gamma_1\times\Omega_1} \frac{u(x,y)u(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&- 2\int_{\Gamma_2\times\Omega_2} \frac{w(x,y)w(x)}{|x-y|^{n+2s}}\,dx\,dy - 2\int_{\Gamma_1\times\Gamma_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\\
&- 2\int_{\Gamma_2\times\Gamma_2} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \lambda\int_{\Gamma_1} f(x)u_+^{-q}(u+\epsilon\phi)dx\\
&+ \mu\int_{\Gamma_2} g(x)w_+^{-q}(w+\epsilon\psi) + \frac{\alpha}{\alpha+\beta}\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi)\\
&+ \frac{\beta}{\alpha+\beta}\int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi) - 2\epsilon\Big(\int_{\Gamma_1\times\Omega^c} \frac{u(x)\phi(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&+ \int_{\Gamma_2\times\Omega^c} \frac{w(x)\psi(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\Gamma_1\times\Omega_1} \frac{u(x,y)\phi(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&+ \int_{\Gamma_2\times\Omega_2} \frac{w(x,y)\psi(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\Gamma_1\times\Gamma_1} \frac{u(x,y)(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\\
&+ \int_{\Gamma_2\times\Gamma_2} \frac{w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy\Big)\\
\le \epsilon\Big(&\int_Q \frac{u(x,y)(\phi(x)-\phi(y)) + w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&- \int_\Omega \big(\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi\big)\,dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx\\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi\Big) - 2\int_{\Gamma_1\times\Omega_1} \frac{u(x,y)u(x)}{|x-y|^{n+2s}}\,dx\,dy
\end{aligned}
$$

$$
\begin{aligned}
&- 2\int_{\Gamma_2\times\Omega_2} \frac{w(x,y)w(x)}{|x-y|^{n+2s}}\,dx\,dy + \lambda\int_{\Gamma_1} f(x)u_+^{-q}(u+\epsilon\phi)\\
&+ \mu\int_{\Gamma_2} g(x)w_+^{-q}(w+\epsilon\psi))dx + \frac{\alpha}{\alpha+\beta}\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}(u+\epsilon\phi)dx\\
&+ \frac{\beta}{\alpha+\beta}\int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}(w+\epsilon\psi)dx\\
&- 2\epsilon\Big(\int_{\Gamma_1\times\Omega^c} \frac{u(x)\phi(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\Gamma_2\times\Omega^c} \frac{w(x)\psi(x)}{|x-y|^{n+2s}}\,dx\,dy\\
&+ \int_{\Gamma_1\times\Omega_1} \frac{u(x,y)\phi(x)}{|x-y|^{n+2s}}\,dx\,dy + \int_{\Gamma_2\times\Omega_2} \frac{w(x,y)\psi(x)}{|x-y|^{n+2s}}\\
&+ \int_{\Gamma_1\times\Gamma_1} \frac{u(x,y)(\phi(x)-\phi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&+ \int_{\Gamma_2\times\Gamma_2} \frac{w(x,y)(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy\Big)\\
\le{}& \epsilon\Big(\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y)) + (w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\,dx\,dy\\
&- \int_\Omega (\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx\\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi\Big)\\
&+ 2\epsilon\Big(\int_{\Gamma_1\times\Omega_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_1\times\Omega_1} \frac{|\phi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ 2\epsilon\Big(\int_{\Gamma_2\times\Omega_2} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_2\times\Omega_2} \frac{|\psi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ 2\epsilon\Big[\Big(\int_{\Gamma_1\times\Omega^c} \frac{|u(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_1\times\Omega^c} \frac{|\phi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ \Big(\int_{\Gamma_2\times\Omega^c} \frac{|w(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_2\times\Omega^c} \frac{|\psi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ \Big(\int_{\Gamma_1\times\Omega_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_1\times\Omega_1} \frac{|\phi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ \Big(\int_{\Gamma_2\times\Omega_2} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_2\times\Omega_2} \frac{|\psi(x)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ \Big(\int_{\Gamma_1\times\Gamma_1} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_1\times\Gamma_1} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\\
&+ \Big(\int_{\Gamma_2\times\Gamma_2} \frac{|w(x)-w(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big(\int_{\Gamma_2\times\Gamma_2} \frac{|\psi(x)-\psi(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\Big)^{1/2}\Big]\\
&+ \epsilon\epsilon^{\alpha}\|b\|_\infty\Big(\int_{\Gamma_1}|\psi|^{\alpha+\beta}dx\Big)^{\frac{\beta}{\alpha+\beta}}\Big(\int_{\Gamma_1}(w_+)^{\alpha+\beta}dx\Big)^{\frac{\beta}{\alpha+\beta}}\\
&+ \epsilon\epsilon^{\beta}\|b\|_\infty\Big(\int_{\Gamma_2}(u_+)^{\alpha+\beta}dx\Big)^{\frac{\alpha}{\alpha+\beta}}\Big(\int_{\Gamma_2}|\phi|^{\alpha+\beta}dx\Big)^{\frac{\beta}{\alpha+\beta}}
\end{aligned}
$$

$$+ \frac{\epsilon\alpha}{\alpha+\beta}\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx + \frac{\epsilon\beta}{\alpha+\beta}\int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx.$$

Since the measure of $\Gamma_i = \{x \in \Omega | (u_i + \epsilon\phi_i)(x) \le 0\}$ tend to zero as $\epsilon \to 0$, it follows that

$$\int_{\Gamma_i\times\Omega_i} \frac{|\phi_i(x)|^2}{|x-y|^{n+2s}}\, dx\, dy \to 0, \quad \text{as } \epsilon \to 0,$$

and similarly

$$\int_{\Gamma_i\times\Omega^c} \frac{|\phi_i(x)|^2}{|x-y|^{n+2s}}\, dx\, dy, \quad \int_{\Gamma_i\times\Gamma_i} \frac{|\phi_i(x)-\phi_i(y)|^2}{|x-y|^{n+2s}}\, dx\, dy,$$
$$\int_{\Gamma_1} b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx, \quad \int_{\Gamma_2} b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx,$$

all are tend to 0 as $\epsilon \to 0$. Dividing by ϵ and letting $\epsilon \to 0$, we obtain

$$\begin{aligned}
&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y)) + (w(x)-w(y))(\psi(x)-\psi(y))}{|x-y|^{n+2s}}\, dx\, dy \\
&- \int_\Omega (\lambda f(x)u_+^{-q}\phi + \mu g(x)w_+^{-q}\psi)dx - \frac{\alpha}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha-1}w_+^{\beta}\phi dx \\
&- \frac{\beta}{\alpha+\beta}\int_\Omega b(x)u_+^{\alpha}w_+^{\beta-1}\psi dx \ge 0.
\end{aligned}$$

Since this holds equally well for $(-\phi, -\psi)$, it follows that (u, w) is indeed a positive weak solution of problem (2) and hence a positive solution of (1.1). □

Lemma 4.8. *There exists a minimizing sequence $\{(U_k, W_k)\}$ in $\mathcal{N}_{\lambda,\mu}^-$ such that $(U_k, W_k) \to (U, W)$ strongly in $\mathcal{N}_{\lambda,\mu}^-$. Moreover (U, W) is a positive weak solution of* (1.1).

Proof. Using the Ekeland variational principle again, we may find a minimizing sequence $\{(U_k, W_k)\} \subset \mathcal{N}_{\lambda,\mu}^-$ for the minimizing problem $\inf_{\mathcal{N}_{\lambda,\mu}^-} J_{\lambda,\mu}$ such that for $(U_k, W_k) \rightharpoonup (U, W)$ weakly in Y and pointwise a.e. in Q. We can repeat the argument used in Lemma 4.2 to derive that when $(\lambda, \mu) \in \Gamma$

$$\begin{aligned}
&(1+q)\int_\Omega \Big(\lambda f(x)U_+^{1-q} + \mu g(x)W_+^{1-q}\Big)\, dx \\
&- (\alpha+\beta-2)\int_\Omega b(x)U_+^{\alpha}W_+^{\beta}dx < 0
\end{aligned} \tag{4.18}$$

which yields

$$\begin{aligned}
&(1+q)\int_\Omega \Big(\lambda f(x)(U_k)_+^{1-q} + \mu g(x)(W_k)_+^{1-q}\Big)\, dx \\
&- (\alpha+\beta-2)\int_\Omega b(x)(U_k)_+^{\alpha}(W_k)_+^{\beta}dx \le -C_4
\end{aligned}$$

for k sufficiently large and a suitable positive constant C_4. At this point we may proceed exactly as in Lemmas 4.3, 4.4, 4.6, 4.7 and corollary 4.5 to conclude that U, $W > 0$ and (U, W) is the required positive weak solution of problem (2). In particular, $(U, W) \in \mathcal{N}_{\lambda,\mu}$. Moreover from (4.18) it follows that

$$(1+q)\|(U,W)\|^2 - (\alpha+\beta-1+q)\int_\Omega b(x)U_+^{\alpha}W_+^{\alpha}dx$$

$$= (1+q)\Big[K_{\lambda,\mu}(U,W) + \int_\Omega b(x)U_+^\alpha W_+^\beta dx\Big] - (\alpha+\beta-1+q)\int_\Omega b(x)U_+^\alpha W_+^\beta dx$$
$$= (1+q)K_{\lambda,\mu}(U,W) - (\alpha+\beta-2)\int_\Omega b(x)U_+^\alpha W_+^\beta dx < 0,$$

that is $(U,W)\in\mathcal{N}_{\lambda,\mu}^-$. □

Proof of the Theorem 2.2. From Lemmas 4.7, 4.8 and 3.4, we can conclude that the problem (1.1) has at least two positive weak solutions $(u,w)\in\mathcal{N}_{\lambda,\mu}^+$, $(U,W)\in\mathcal{N}_{\lambda,\mu}^-$ with $\|(U,W)\| > \|(u,w)\|$ for any $(\lambda,\mu)\in\Gamma$. □

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