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# EXISTENCE OF SOLUTIONS FOR KIRCHHOFF TYPE EQUATIONS WITH UNBOUNDED POTENTIAL

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Abstract. In this article, we study the Kirchhoff type equation

$$\Big(a+\lambda\int_{\mathbb{R}^3}|\nabla u|^2+\lambda b\int_{\mathbb{R}^3}u^2\Big)[-\Delta u+bu]=K(x)|u|^{p-1}u,\quad \text{in } \mathbb{R}^3,$$

where $a,b>0$, $p\in(2,5)$, $\lambda\geq 0$ is a parameter, and $K$ may be an unbounded potential function. By using variational methods, we prove the existence of nontrivial solutions for the above equation. A cut-off functional and some estimates are used to obtain the bounded Palais-Smale sequences.

## 1. Introduction

In this article, we consider the Kirchhoff type problem

$$\begin{gathered}\Big(a+\lambda\int_{\mathbb{R}^3}|\nabla u|^2+\lambda b\int_{\mathbb{R}^3}u^2\Big)[-\Delta u+bu]=K(x)|u|^{p-1}u,\quad \text{in } \mathbb{R}^3,\\ u\in H^1_r(\mathbb{R}^3),\end{gathered}\tag{1.1}$$

where $a>0$, $b>0$, $p\in(2,5)$, $\lambda\geq 0$ is a parameter and $K(x)$ is a given potential satisfying the following conditions:

- (H1) $K:\mathbb{R}^3\to\mathbb{R}$ is a nonnegative continuous function, $K$ is radial (that is $k(x)=k(|x|)$) and $K\not\equiv 0$;
- (H2) there exists $C_0>0$ and $0\leq l<p-2$ such that $K(x)\leq C_0(1+|x|^l)$ for all $x\in\mathbb{R}^3$.

When $\Omega$ is a bounded domain in $\mathbb{R}^N$, the Kirchhoff type problem

$$\begin{gathered}-\Big(a+b\int_\Omega|\nabla u|^2\Big)\Delta u=f(x,u),\quad \text{in } \Omega,\\ u=0,\quad \text{on } \partial\Omega,\end{gathered}\tag{1.2}$$

has attracted a lot of attention in recent years. Equation (1.2) is a nonlocal problem because of the appearance of the term $\int_\Omega|\nabla u|^2$. This causes some mathematical difficulties but makes the study of (1.2) very interesting. One can refer to [1, 4, 3, 19, 15, 25, 27, 23, 28, 7, 8, 13] and references therein for the existence and multiplicity of positive solutions to (1.2). Mao and Luan [21], Mao and Luan [20],

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Zhang and Perera [29] have investigated the existence of sign-changing solutions of (1.2).

When $\Omega = \mathbb{R}^N$, the existence and multiplicity of solutions to (1.2) have been treated in [16, 12, 14, 26, 6, 5]. In particular, Nie and Wu [22] have studied the Schrödinger-Kirchhoff type problem

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u + V(|x|)u = Q(|x|)f(u), \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.3}$$

where $N \geq 2$, $a, b > 0$, $f \in C(\mathbb{R}, \mathbb{R})$, $V$ and $Q$ are both radial functions in $\mathbb{R}^N$. Under some conditions on $V$ and $Q$, the existence of nontrivial solutions and a sequence of high energy solutions for problem (1.3) has been proved by the Mountain Pass theorem and symmetric Mountain Pass theorem. From [22, Remarks 1 and 2], we see that the potential $Q$ is bounded in $\mathbb{R}^N$.

In [17], Li, Li and Shi showed the existence of nontrivial solutions of the following problem with zero mass

$$\begin{gathered} -\Big(a + b\int_{\mathbb{R}^N} |\nabla u|^2\Big)\Delta u = K(x)f(u), \quad \text{in } \mathbb{R}^N, \\ u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \end{gathered}$$

where the potential function $K(x)$ is a nonnegative continuous function, $K \in [L^s(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)] \backslash \{0\}$ for some $s \geq 2N/(N+2)$ and $|x \cdot \nabla K(x)| \leq \alpha K(x)$ for a.e. $x \in \mathbb{R}^N$ and some $\alpha \in (0, 2)$.

To be best of our knowledge, there is no result about the Kirchhoff type problems with unbounded potential at infinity. Motivated by the above work, in this article, we are lead to study problem (1.1) with unbounded potential $K$. Our main results are as follows:

**Theorem 1.1.** *Assume* (H1) *and* (H2) *hold. Then there exists* $\lambda_0 > 0$ *such that for any* $\lambda \in [0, \lambda_0)$*, problem* (1.1) *has at least one nontrivial solution in* $H_r^1(\mathbb{R}^3)$.

**Corollary 1.2.** *Assume* (H1) *and* (H2) *hold. Then the local problem*

$$-\Delta u + bu = K(x)|u|^{p-1}u, \quad \text{in } \mathbb{R}^3 \tag{1.4}$$

*has at least one nontrivial solution in* $H_r^1(\mathbb{R}^3)$.

We remark that in [17], a cut-off functional and Pohozaev type identity are utilized to obtain the bounded Palais-Smale sequences. However, we find that when $f(u) = |u|^{p-1}u$ can be expressed explicitly, it is not necessary to establish the Pohozaev type identity. In the case that $K$ is unbounded, we need the Radial Lemma (see Lemma 2.1) to overcome the loss of compactness, which is different from [17].

The remaining of this paper is organized as follows. In Section 2, we give some notations and elementary lemmas which will be used in the paper. In Section 3, we are devoted to the proof of our main results.

## 2. Preliminaries

In this paper, we shall use the following notation:
- $C$ stands for different positive constants.

- For $r > 0$, $B_r(x)$ is an open ball in $\mathbb{R}^3$ with radius $r$ centered at $x$.
- $H^1(\mathbb{R}^3)$ denotes the usual Sobolev space equipped with the inner product

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + buv) dx,$$

and the corresponding norm $\|u\| = (u, u)^{1/2}$.
- $H_r^1(\mathbb{R}^3)$ is the set of all radial functions in $H^1(\mathbb{R}^3)$.
- For $\Omega \subset \mathbb{R}^3$ and $1 \le q < \infty$, $L^q(\Omega)$ denotes the Lebesgue space with the norm

$$|u|_{L^q(\Omega)} = (\int_\Omega |u|^q dx)^{\frac{1}{q}}.$$

When $\Omega = \mathbb{R}^3$, we write $|u|_q = |u|_{L^q(\mathbb{R}^3)}$ for simplicity of notations. Furthermore, when $q = \infty$, we write

$$|u|_\infty = \operatorname{ess\,sup}_{x \in \mathbb{R}^3} |u(x)|.$$

- $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ continuously for $q \in [2, 6]$, while $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ compactly for $q \in (2, 6)$.
- The dual space of $H_r^1(\mathbb{R}^3)$ is denoted by $H_r^{-1}(\mathbb{R}^3)$.
- Let $\langle \cdot, \cdot \rangle$ be the duality pairing between $H_r^1(\mathbb{R}^3)$ and $H_r^{-1}(\mathbb{R}^3)$.

Before establishing the variational setting for (1.1), we need the following lemma.

**Lemma 2.1** ([2]). *Let $N \ge 2$. Then for any radial function $u \in H_r^1(\mathbb{R}^N)$,*

$$|u(r)| \le C_1 \|u\| r^{\frac{1-N}{2}}, \quad \text{for } r \ge 1,$$

*where $C_1$ is a positive constant and only depends on $N$.*

**Remark 2.2.** Since $H^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ continuously for $q \in [2, 6]$, for any $u \in H_r^1(\mathbb{R}^3)$, by (H1) and (H2), we have

$$\begin{aligned}
0 &\le \int_{\mathbb{R}^3} K(x)|u|^{p+1} \mathrm{d}x \\
&\le C_0 \int_{\mathbb{R}^3} (1 + |x|^l)|u|^{p+1} \mathrm{d}x \\
&= C_0 \int_{B_1(0)} (1 + |x|^l)|u|^{p+1} \mathrm{d}x + C_0 \int_{\mathbb{R}^3 \setminus B_1(0)} (1 + |x|^l)|u|^{p+1} \mathrm{d}x \\
&\le C(\int_{B_1(0)} |u|^{p+1} \mathrm{d}x + \int_1^\infty r^{l+2} |u(r)|^{p+1} \mathrm{d}r) \\
&\le C\|u\|^{p+1} + C \int_1^\infty r^{l+2} |u(r)|^{p+1} \mathrm{d}r.
\end{aligned}$$

Note that $0 \le l < p - 2$. Then by Lemma 2.1, we deduce

$$\int_1^\infty r^{l+2} |u(r)|^{p+1} \mathrm{d}r \le C\|u\|^{p+1} \int_1^\infty r^{l-p+1} \mathrm{d}r \le \frac{C}{p-2-l} \|u\|^{p+1}.$$

Thus,

$$0 \le \int_{\mathbb{R}^3} K(x)|u|^{p+1} \mathrm{d}x < C\|u\|^{p+1}.$$

In view of Remark 2.2, we consider the Sobolev space $H_r^1(\mathbb{R}^3)$. As usual, the energy functional $J_\lambda : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ associated with (1.1) is well defined by

$$J_\lambda(u) = \frac{1}{2} a\|u\|^2 + \frac{1}{4} \lambda \|u\|^4 - \frac{1}{p+1} \int_{\mathbb{R}^3} K(x)|u|^{p+1} \mathrm{d}x.$$

It is easy to check that $J_\lambda$ is $C^1$-functional, whose Gateaux derivative is given by

$$\langle J_\lambda'(u), v\rangle = a(u,v) + \lambda\|u\|^2(u,v) - \int_{\mathbb{R}^3} K(x)|u|^{p-1}uv\mathrm{d}x \quad u,v \in H_r^1(\mathbb{R}^3).$$

In the weak sense, solutions to (1.1) correspond to the critical points of the functional $J_\lambda$.

## 3. Proof of our main results

In this section, we prove the existence of nontrivial solutions to (1.1). We first have the following lemma.

**Lemma 3.1.** *If $u$ is a nontrivial weak solution of* (1.1)*, then $\|u\| \geq r$ for some $r > 0$.*

*Proof.* Since $u$ is a nontrivial weak solution of (1.1), by Remark 2.2, we have

$$a\|u\|^2 \leq (a + \lambda\|u\|^2)\|u\|^2 = \int_{\mathbb{R}^3} K(x)|u|^{p+1}\mathrm{d}x \leq C\|u\|^{p+1}.$$

Because $p + 1 > 2$, there exists $r > 0$ such that $\|u\| > r$. □

To find a bounded Palais-Smale sequence for the energy functional $J_\lambda$, by following [9, 11] (also see [17, 18]), we introduce a cut-off function $\phi \in C^\infty(\mathbb{R}_+, \mathbb{R})$ such that

$$\begin{gathered}
\phi(t) = 1, \quad t \in [0,1], \\
0 \leq \phi(t) \leq 1, \quad t \in (1,2), \\
\phi(t) = 0, \quad t \in [2,\infty), \\
|\phi'|_\infty \leq 2.
\end{gathered}$$

So we can define the functional $J_\lambda^T : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$J_\lambda^T(u) = \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \frac{1}{p+1}\int_{\mathbb{R}^3} K(x)|u|^{p+1}\mathrm{d}x, \quad u \in H_r^1(\mathbb{R}^3),$$

where for every $T > 0$,

$$h_T(u) = \phi(\frac{\|u\|^2}{T^2}).$$

Moreover, for every $u, v \in H_r^1(\mathbb{R}^3)$, we have

$$\begin{aligned}
\langle (J_\lambda^T)'(u), v\rangle = a(u,v) &+ \lambda h_T(u)\|u\|^2(u,v) + \frac{\lambda}{2T^2}\phi'(\frac{\|u\|^2}{T^2})\|u\|^4(u,v) \\
&- \int_{\mathbb{R}^3} K(x)|u|^{p-1}uv\mathrm{d}x.
\end{aligned} \tag{3.1}$$

It is easy to see that if $u$ is a critical point of $J_\lambda^T$ such that $\|u\| \leq T$, then $\langle (J_\lambda^T)'(u), v\rangle = \langle J_\lambda'(u), v\rangle$. So the arbitrary of $v$ yields $J_\lambda'(u) = 0$ and thus $u$ is also a critical point of $J_\lambda$.

Next we recall a theorem, for which a corollary was proved by Struwe [24].

**Theorem 3.2** ([10])**.** *Let $(X, \|\cdot\|)$ be a Banach space and $I \subset \mathbb{R}_+$ an interval. Consider the family of $C^1-$ functional on $X$*

$$J_\mu(u) = A(u) - \mu B(u), \quad \mu \in I,$$

*with $B$ nonnegative and either $A(u) \to \infty$ or $B(u) \to \infty$ as $\|u\| \to \infty$ and such that $J_\mu(0) = 0$.*

*For every $\mu \in I$, we set*

$$\Gamma_\mu = \{\gamma \in C([0,1], X) : \gamma(0) = 0, J_\mu(\gamma(1)) < 0\}.$$

*If for every $\mu \in I$, the set $\Gamma_\mu$ is nonempty and*

$$c_\mu = \inf_{\gamma \in \Gamma_\mu} \max_{t \in [0,1]} J_\mu(\gamma(t)) > 0,$$

*then for almost every $\mu \in I$, there is a sequence $\{u_n\} \subset X$ such that*

(i) *$\{u_n\}$ is bounded;*
(ii) *$J_\mu(u_n) \to c_\mu$;*
(iii) *$J'_\mu(u_n) \to 0$ in the dual space $X^{-1}$ of $X$.*

In our case, $X = H_r^1(\mathbb{R}^3)$,

$$\begin{aligned} A(u) &= \frac{1}{2} a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4, \\ B(u) &= \frac{1}{p+1} \int_{\mathbb{R}^3} K(x)|u|^{p+1} \mathrm{d}x, \end{aligned}$$

and the associated perturbed functional we study is

$$J_{\lambda,\mu}^T(u) = \frac{1}{2} a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \frac{\mu}{p+1} \int_{\mathbb{R}^3} K(x)|u|^{p+1} \mathrm{d}x, \quad u \in H_r^1(\mathbb{R}^3).$$

It is clear that this functional is $C^1$-functional and for every $u, v \in H_r^1(\mathbb{R}^3)$,

$$\begin{aligned} \langle (J_{\lambda,\mu}^T)'(u), v\rangle &= a(u,v) + \lambda h_T(u)\|u\|^2(u,v) + \frac{\lambda}{2T^2}\phi'(\frac{\|u\|^2}{T^2})\|u\|^4(u,v) \\ &\quad - \mu \int_{\mathbb{R}^3} K(x)|u|^{p-1}uv \mathrm{d}x. \end{aligned} \tag{3.2}$$

Notice that $J_{\lambda,\mu}^T(0) = 0$, $B$ is nonnegative in $H_r^1(\mathbb{R}^3)$ and $A(u) \to +\infty$ as $\|u\| \to \infty$. Next we shall prove the following two lemmas which show that the functional $J_{\lambda,\mu}^T$ satisfies the other conditions of Theorem 3.2.

**Lemma 3.3.** *We have $\Gamma_\mu \neq \emptyset$ for all $\lambda \geq 0$ and $\mu \in I := [\frac{1}{2}, 1]$.*

*Proof.* By (H1), there exist $R, C_2 > 0$ and $\varphi \in C_0^\infty(\mathbb{R}^3)$ such that

$$\begin{gathered} \varphi \geq 0, \quad \|\varphi\| = 1, \\ \operatorname{supp}(\varphi) \subset B_R(0), \\ \int_{B_R(0)} K(x)\varphi^{p+1} \mathrm{d}x \geq C_2. \end{gathered} \tag{3.3}$$

Then for $t^2 \geq 2T^2$,

$$\begin{aligned} J_{\lambda,\mu}^T(t\varphi) &= \frac{1}{2} at^2\|\varphi\|^2 + \frac{1}{4}\lambda\phi(\frac{t^2\|\varphi\|^2}{T^2})t^4\|\varphi\|^4 - \frac{\mu}{p+1}|t|^{p+1} \int_{\mathbb{R}^3} K(x)\varphi^{p+1} \mathrm{d}x \\ &= \frac{1}{2} at^2 - \frac{\mu}{p+1}|t|^{p+1} \int_{\mathbb{R}^3} K(x)\varphi^{p+1} \mathrm{d}x \\ &\leq \frac{1}{2} at^2 - \frac{1}{2(p+1)}|t|^{p+1} \int_{B_R(0)} K(x)\varphi^{p+1} \mathrm{d}x \end{aligned}$$

$$\le \frac{1}{2}at^2 - \frac{C_2}{2(p+1)}|t|^{p+1}.$$

Since $p+1>2$, we can choose $t_0>0$ large such that $J^T_{\lambda,\mu}(t_0\varphi)<0$. We set

$$\gamma(t) = tt_0\varphi,\ t\in[0,1].$$

So $\gamma\in\Gamma_\mu$ and $\Gamma_\mu\neq\emptyset$. The proof is complete. $\square$

**Lemma 3.4.** *There exists a positive constant $\alpha$ such that $c_\mu\ge\alpha$ for all $\mu\in I$.*

*Proof.* For any $\mu\in I$ and $u\in H^1_r(\mathbb{R}^3)$, by Remark 2.2, we have that

$$\begin{aligned} J^T_{\lambda,\mu}(u) &= \frac{1}{2}a\|u\|^2 + \frac{1}{4}\lambda h_T(u)\|u\|^4 - \frac{\mu}{p+1}\int_{\mathbb{R}^3} K(x)|u|^{p+1}\mathrm{d}x \\ &\ge \frac{1}{2}a\|u\|^2 - \frac{\mu}{p+1}C\|u\|^{p+1} \\ &\ge \frac{1}{2}a\|u\|^2 - \frac{C}{p+1}\|u\|^{p+1}. \end{aligned}$$

Take $\rho := \left(\frac{ap+a}{4C}\right)^{\frac{1}{p-1}}$ and $\alpha := \frac{1}{2}a\rho^2 - \frac{C}{p+1}\rho^{p+1}>0$. Then $J^T_{\lambda,\mu}(u)>0$ for any $\mu\in I$ and $0<\|u\|\le\rho$. Moreover, $J^T_{\lambda,\mu}(u)\ge\alpha$ if $\|u\|=\rho$. By the definition of $\Gamma_\mu$, for each $\gamma\in\Gamma_\mu$, there is $t^*\in(0,1)$ such that $\|\gamma(t^*)\|=\rho$. Therefore, the arbitrary of $\mu\in I$ yields that

$$c_\mu \ge \inf_{\gamma\in\Gamma_\mu} J^T_{\lambda,\mu}(\gamma(t^*)) \ge \alpha.$$

The proof is complete. $\square$

Next we need some compactness on Palais-Smale sequences of the functional $J^T_{\lambda,\mu}$ in order to prove our main results.

**Lemma 3.5.** *Assume $4\lambda T^2<a$. For any $\mu\in I$, any bounded Palais-Smale sequence of $J^T_{\lambda,\mu}$ contains a convergent subsequence.*

*Proof.* Fix $\mu\in I$. We assume $\{u_n\}$ be a bounded Palais-Smale sequence of $J^T_{\lambda,\mu}$, namely

$$\begin{gathered} \{u_n\} \text{ and } \{J^T_{\lambda,\mu}(u_n)\} \text{ are bounded}, \\ (J^T_{\lambda,\mu})'(u_n)\to 0 \quad \text{in } H^{-1}_r(\mathbb{R}^3). \end{gathered}$$

Up to a subsequence, there exists $u\in H^1_r(\mathbb{R}^3)$ such that

$$\begin{gathered} u_n \rightharpoonup u, \quad \text{in } H^1_r(\mathbb{R}^3), \\ u_n \to u, \quad \text{in } L^s(\mathbb{R}^3),\ s\in(2,6), \\ u_n \to u, \quad \text{in } L^s_{\rm loc}(\mathbb{R}^3),\ s\in[1,6), \\ u_n(x)\to u(x), \quad \text{a.e. } x\in\mathbb{R}^3. \end{gathered}$$

For $R > 1$, from (H2) we obtain

$$
\begin{aligned}
\Big|\int_{\mathbb{R}^3} K(x)|u_n|^{p-1}u_n(u_n-u)\mathrm{d}x\Big| &\le \int_{\mathbb{R}^3} K(x)|u_n|^p|u_n-u|\mathrm{d}x \\
&\le C_0\int_{\mathbb{R}^3}(1+|x|^l)|u_n|^p|u_n-u|\mathrm{d}x \\
&= C_0\int_{B_R(0)}(1+|x|^l)|u_n|^p|u_n-u|\mathrm{d}x \\
&\quad + C_0\int_{\mathbb{R}^3\setminus B_R(0)}(1+|x|^l)|u_n|^p|u_n-u|\mathrm{d}x \\
&=: I_1+I_2.
\end{aligned}
\tag{3.4}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
I_1 &= C_0\int_{B_R(0)}(1+|x|^l)|u_n|^p|u_n-u|\mathrm{d}x \\
&\le C(R)(\int_{B_R(0)}|u_n|^{p+1}\mathrm{d}x)^{\frac{p}{p+1}}\Big(\int_{B_R(0)}|u_n-u|^{p+1}\mathrm{d}x\Big)^{\frac{1}{p+1}} \\
&= C(R)|u_n|^p_{L^{p+1}(B_R(0))}|u_n-u|_{L^{p+1}(B_R(0))} \\
&\le C(R)\|u_n\|^p|u_n-u|_{L^{p+1}(B_R(0))}.
\end{aligned}
\tag{3.5}
$$

Moreover, from Lemma 2.1 we deduce that

$$
\begin{aligned}
I_2 &= C_0\int_{\mathbb{R}^3\setminus B_R(0)}(1+|x|^l)|u_n|^p|u_n-u|\mathrm{d}x \\
&\le 2C_0\int_{\mathbb{R}^3\setminus B_R(0)}|x|^l(|u_n|^{p+1}+|u_n|^p|u|)\mathrm{d}x \\
&\le C(\|u_n\|^{p+1}+\|u_n\|^p\|u\|)\int_R^\infty r^{l-p+1}\mathrm{d}r \\
&= C\frac{R^{l-p+2}}{p-2-l}(\|u_n\|^{p+1}+\|u_n\|^p\|u\|).
\end{aligned}
\tag{3.6}
$$

Since $3 < p+1 < 6$, $u_n \to u$ in $L^{p+1}(B_R(0))$. Therefore, by (3.4)-(3.6), we can find $R > 1$ large enough such that

$$
\int_{\mathbb{R}^3} K(x)|u_n|^{p-1}u_n(u_n-u)\mathrm{d}x \to 0 \quad \text{as } n\to\infty. \tag{3.7}
$$

Note that $(J^T_{\lambda,\mu})'(u_n) \to 0$. Then by (3.7), we deduce that

$$
\begin{aligned}
0 &\leftarrow \langle (J^T_{\lambda,\mu})'(u_n), u_n-u\rangle \\
&= a(u_n,u_n-u)+\lambda h_T(u_n)\|u_n\|^2(u_n,u_n-u) \\
&\quad + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4(u_n,u_n-u) - \mu\int_{\mathbb{R}^3}K(x)|u_n|^{p-1}u_n(u_n-u)\mathrm{d}x \\
&= [a+\lambda h_T(u_n)\|u_n\|^2+\frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4](u_n,u_n-u)+o(1).
\end{aligned}
$$

So

$$
(a+\lambda h_T(u_n)\|u_n\|^2+\frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4)(u_n,u_n-u) \to 0. \tag{3.8}
$$

Next, we claim that

$$(u_n, u_n - u) \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Note that if $\|u_n\|^2 > 2T^2$, then $h_T(u_n) = 0$ and $\phi'(\frac{\|u_n\|^2}{T^2}) = 0$. If $\|u_n\|^2 \le 2T^2$, then

$$|\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4| \le 8T^4.$$

Since $4\lambda T^2 < a$, we have

$$\begin{aligned} a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4 &\ge a + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4 \\ &\ge a + \frac{\lambda}{2T^2}(-8T^4) \\ &= a - 4\lambda T^2 > 0 \end{aligned}$$

and

$$\begin{aligned} a + \lambda h_T(u_n)\|u_n\|^2 + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4 &\le a + \lambda\|u_n\|^2 + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4 \\ &\le a + 2\lambda T^2 + \frac{\lambda}{2T^2}(8T^4) \\ &= a + 6\lambda T^2 \le \frac{5a}{2}. \end{aligned}$$

This combined with (3.8) yields that (3.9) holds.

Since $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, we infer from the above claim that up to a subsequence,

$$u_n \to u \quad \text{in } H_r^1(\mathbb{R}^3).$$

The proof is complete. □

**Lemma 3.6.** *Let $4\lambda T^2 < a$. For almost every $\mu \in I$, there exists $u^\mu \in H_r^1(\mathbb{R}^3)\backslash\{0\}$ such that*

$$(J_{\lambda,\mu}^T)'(u^\mu) = 0 \quad \text{and} \quad J_{\lambda,\mu}^T(u^\mu) = c_\mu. \tag{3.10}$$

*Proof.* By Lemma 3.3, Lemma 3.4 and Theorem 3.2, for almost every $\mu \in I$, we can find a bounded sequence $\{u_n^\mu\} \subset H_r^1(\mathbb{R}^3)$ satisfying

$$\begin{gathered} J_{\lambda,\mu}^T(u_n^\mu) \to c_\mu, \\ (J_{\lambda,\mu}^T)'(u_n^\mu) \to 0. \end{gathered}$$

Furthermore, by using Lemma 3.5, we conclude that there is $u^\mu \in H_r^1(\mathbb{R}^3)$ such that $u_n^\mu \to u^\mu$ in $H_r^1(\mathbb{R}^3)$. So the continuity of $J_{\lambda,\mu}^T$ and $(J_{\lambda,\mu}^T)'$ imply that (3.10) holds. This completes the proof. □

According to Lemma 3.6, we obtain that there exist sequences $\{\mu_n\} \subset I$ with $\mu_n \to 1^-$ and $\{u_n\} \subset H_r^1(\mathbb{R}^3)$ such that

$$J_{\lambda,\mu_n}^T(u_n) = c_{\mu_n}, \quad (J_{\lambda,\mu_n}^T)'(u_n) = 0. \tag{3.11}$$

In what follows, we shall show $\|u_n\| \le T$, which is a critical key in the proof of existence of solutions to (1.1).

**Lemma 3.7.** *Assume $\{\mu_n\} \subset I$ with $\mu_n \to 1^-$ and $\{u_n\}$ satisfies (3.11). Then for $T > 0$ sufficiently large, there exists $\lambda_0 = \lambda_0(T)$ with $4\lambda_0 T^2 < a$ such that for any $\lambda \in [0, \lambda_0)$, up to a subsequence,*

$$\|u_n\| \le T \quad \forall n.$$

*Proof.* The proof consists of three steps.

**Step 1**. By the definition of $c_\mu$ and Lemma 3.4, we conclude that

$$
\begin{aligned}
\alpha \le c_{\mu_n} &\le \sup_{t\in[0,\infty)} J^T_{\lambda,\mu_n}(t\varphi)\\
&= \sup_{t\in[0,\infty)} \big[\frac{1}{2}at^2 + \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4 - \frac{\mu_n}{p+1}|t|^{p+1}\int_{\mathbb{R}^3} K(x)\varphi^{p+1}\mathrm{d}x\big]\\
&\le \max_{t\in[0,\infty)} \big[\frac{1}{2}at^2 - \frac{1}{2(p+1)}|t|^{p+1}\int_{B_R(0)} K(x)\varphi^{p+1}\mathrm{d}x\big]\\
&\quad + \sup_{t\in[0,\infty)} \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4\\
&= \frac{a(p-1)}{2(p+1)}\Big(\frac{2a}{\int_{B_R(0)} K(x)\varphi^{p+1}\mathrm{d}x}\Big)^{\frac{2}{p-1}} + \sup_{t\in[0,\infty)} \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4\\
&=: A + \sup_{t\in[0,\infty)} \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4,
\end{aligned}
\tag{3.12}
$$

where $\varphi$ is defined in Lemma 3.3.

**Step 2**. It is easy to see that if $t^2 \ge 2T^2$, then $\phi(\frac{t^2}{T^2}) = 0$ and so

$$
\sup_{t\in[\sqrt{2}T,\infty)} \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4 = 0,
$$

while if $t^2 < 2T^2$, then

$$
\sup_{t\in[0,\sqrt{2}T)} \frac{1}{4}\lambda\phi(\frac{t^2}{T^2})t^4 < \lambda T^4.
$$

This, combined with (3.12), yields

$$
0 < \alpha \le c_{\mu_n} \le A + \lambda T^4.
$$

**Step 3**. Note that

$$
c_{\mu_n} = J^T_{\lambda,\mu_n}(u_n) = \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}\lambda h_T(u_n)\|u_n\|^4 - \frac{\mu_n}{p+1}\int_{\mathbb{R}^3} K(x)|u_n|^{p+1}\mathrm{d}x
$$

and

$$
\begin{aligned}
0 &= \langle (J^T_{\lambda,\mu_n})'(u_n), u_n\rangle\\
&= a(u_n,u_n) + \lambda h_T(u_n)\|u_n\|^2(u_n,u_n)\\
&\quad + \frac{\lambda}{2T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^4(u_n,u_n) - \mu_n\int_{\mathbb{R}^3} K(x)|u_n|^{p+1}\mathrm{d}x.
\end{aligned}
$$

Thus, we have that

$$
\begin{aligned}
c_{\mu_n} &= J^T_{\lambda,\mu_n}(u_n) - \frac{1}{p+1}\langle (J^T_{\lambda,\mu_n})'(u_n), u_n\rangle\\
&= a(\frac{1}{2} - \frac{1}{p+1})\|u_n\|^2 + (\frac{1}{4} - \frac{1}{p+1})\lambda h_T(u_n)\|u_n\|^4\\
&\quad - \frac{\lambda}{2(p+1)T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^6.
\end{aligned}
\tag{3.13}
$$

We claim: there exists a subsequence of $\{u_n\}$ which is uniformly bounded by $T$. By way of contradiction, we distinguish two cases to prove the claim.

**Case 1.** Up to a subsequence, $\|u_n\|^2 > 2T^2$ for all $n$. It is easy to check that $h_T(u_n) = 0$ and $\phi'(\frac{\|u_n\|^2}{T^2}) = 0$. Then we deduce from (3.13) and Step 2 that

$$\Big(\frac{1}{2} - \frac{1}{p+1}\Big)a\|u_n\|^2 = c_{\mu_n} \leq A + \lambda T^4.$$

Since $4\lambda T^4 < a$,

$$2T^2 < \|u_n\|^2 \leq \Big[\Big(\frac{1}{2} - \frac{1}{p+1}\Big)a\Big]^{-1}(A + \lambda T^4) \leq \Big[\Big(\frac{1}{2} - \frac{1}{p+1}\Big)a\Big]^{-1}(A + \frac{a}{4}),$$

which contradicts with the assumption that $T$ is sufficiently large.

**Case 2.** Up to a subsequence, $T^2 < \|u_n\|^2 \leq 2T^2$ for all $n$. By (3.13) and Step 2, we have $c_{\mu_n} \leq A + \lambda T^4$. Furthermore, since $p \in (2, 5)$, we obtain

$$\begin{aligned}
&(\frac{1}{2} - \frac{1}{p+1})aT^2 \\
&< (\frac{1}{2} - \frac{1}{p+1})a\|u_n\|^2 \\
&= c_{\mu_n} + \frac{\lambda}{2(p+1)T^2}\phi'(\frac{\|u_n\|^2}{T^2})\|u_n\|^6 - (\frac{1}{4} - \frac{1}{p+1})\lambda h_T(u_n)\|u_n\|^4 \\
&\leq A + \lambda T^4 + \frac{\lambda}{2(p+1)T^2}\big|\phi'(\frac{\|u_n\|^2}{T^2})\big|\|u_n\|^6 + (\frac{1}{4} + \frac{1}{p+1})\lambda h_T(u_n)\|u_n\|^4 \\
&\leq A + \lambda T^4 + \frac{8}{p+1}\lambda T^4 + (\frac{1}{4} + \frac{1}{p+1}) \times 4\lambda T^4 \\
&\leq A + \frac{a(p+14)}{4(p+1)},
\end{aligned}$$

which contradicts the assumption that $T$ is sufficiently large. Then the claim holds and we complete the proof. $\square$

*Proof of Theorem 1.1.* We assume $T, \lambda_0$ are defined as in Lemma 3.7 and for each $\mu_n \in I$, $u_n$ is a critical point for $J^T_{\lambda,\mu_n}$ at level $c_{\mu_n}$. By using Lemma 3.7, up to a subsequence, we can also assume $\|u_n\| \leq T$. Hence, $h_T(u_n) = 1$ and

$$J^T_{\lambda,\mu_n}(u_n) = \frac{1}{2}a\|u_n\|^2 + \frac{1}{4}\lambda\|u_n\|^4 - \frac{\mu_n}{p+1}\int_{\mathbb{R}^3} K(x)|u_n|^{p+1}\mathrm{d}x.$$

Next, we claim that $\{u_n\}$ is a Palais-Smale sequence of $J_\lambda$. Indeed, since $\|u_n\|$ is bounded for all $n$, we conclude that $|J_\lambda(u_n)|$ is bounded for all $n$. Moreover, we have

$$\langle J'_\lambda(u_n), v\rangle = \langle (J^T_{\lambda,\mu_n})'(u_n), v\rangle + (\mu_n - 1)\int_{\mathbb{R}^3} K(x)|u_n|^{p-1}u_n v\mathrm{d}x,$$

for any $v \in H^1_r(\mathbb{R}^3)$. Note that $(J^T_{\lambda,\mu_n})'(u_n) = 0$ and $\mu_n \to 1^-$. Then we obtain $J'_\lambda(u_n) \to 0$ and the claim holds. By Lemma 3.5, $\{u_n\}$ has a convergent subsequence. We may assume that $u_n \to \bar{u}$ in $H^1_r(\mathbb{R}^3)$. Therefore, $J'_\lambda(\bar{u}) = 0$ and by Lemma 3.4,

$$J_\lambda(\bar{u}) = \lim_{n\to\infty} J_\lambda(u_n) = \lim_{n\to\infty} J^T_{\lambda,\mu_n}(u_n) = c_{\mu_n} \geq \alpha > 0.$$

Consequently, $\bar{u}$ is nontrivial solution of (1.1). The proof is complete. $\square$

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