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APPROXIMATE CONTROLLABILITY OF A SEMILINEAR ELLIPTIC PROBLEM WITH ROBIN CONDITION IN A PERIODICALLY PERFORATED DOMAIN

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Abstract. In this article, we study the approximate controllability and homegenization results of a semi-linear elliptic problem with Robin boundary condition in a periodically perforated domain. We prove the existence of minimal norm control using Lions constructive approach, which is based on Fenchel-Rockafeller duality theory, and by means of Zuazua's fixed point arguments. Then, as the homogenization parameter goes to zero, we link the limit of the optimal controls (the limit of fixed point of the controllability problems) with the optimal control of the corresponding homogenized problem.

1. Introduction

Periodic homogenization (without holes) has been studied during late 1960's, we refer to the reader the classical works of Spagnolo [24], Bensoussan et al. [1] and Sánchez-Palencia [23]. For the further developments concerning the perforated domains and periodic structures, we refer to Lions [19], Cioranescu and Saint Jean Paulin [5]. Let us now describe the setting of the problem.

Let Ω be a bounded, connected open set in $\mathbb{R}^N$, with smooth boundary $\partial\Omega$. From the geometrical point of view, we shall consider the periodic structures obtained by removing periodically from Ω, with period εY (where Y is a given hyperrectangle in $\mathbb{R}^N$). The reference hole T which has been appropriately rescaled and is strictly included in Y. Precisely, let $Y = (0, l_1) \times \cdots \times (0, l_N)$ be the reference cell, with $l_1, \dots, l_N > 0$. The reference hole T is an open set such that $T \Subset Y$. We denote by ε a positive parameter taking its values in a decreasing positive sequence which tends to zero. Set

$$\tau(\varepsilon\overline{T}) = \{\varepsilon(k(l) + \overline{T}), k \in \mathbb{Z}^N, \quad k(l) = (k_1 l_1, \dots, k_N l_N)\}.$$

Assume that for any ε there exists a subset $\mathcal{K}_\varepsilon$ of $\mathbb{Z}^N$ such that

$$T_\varepsilon = \Omega \cap \tau(\varepsilon\overline{T}) = \cup_{k\in\mathcal{K}_\varepsilon}\big(\varepsilon(k(l) + \overline{T})\big).$$

Then for any $\varepsilon > 0$, we define the perforated domain Ω_ε by

$$\Omega_\varepsilon = \Omega \setminus \tau(\varepsilon\overline{T})$$

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and thus we obtain

$$\partial\Omega_\varepsilon = \partial\Omega \cup \partial T_\varepsilon.$$

Hence, Ω_ε is a periodic domain with periodically distributed holes of the size of the same order as the period. We introduce two nonempty sub-domains of Ω, which are the control region ω and the observable region S, where the error between the obtained and the desired state has to be minimized, respectively.

We let ω and S to be two open subsets of Ω, with S compactly contained in ω and set

$$\omega_\varepsilon = \omega \cap \Omega_\varepsilon, \quad S_\varepsilon = S \cap \Omega_\varepsilon.$$

For the constants $0 < \alpha_m \le \alpha_M$, let $A(y) = (a_{ij}(y))_{1\le i,j\le N}$ be $N \times N$ matrix valued function lying in the space $\mathcal{M}(\alpha_m, \alpha_M, \Omega)$, which is defined as:

$$\begin{aligned} &\mathcal{M}(\alpha_m, \alpha_M, \Omega) \\ &:= \begin{cases} A \in L^\infty(\Omega)^{N\times N}, & \text{a.e. on } \Omega, \\ A \text{ is } Y\text{-periodic}, \\ \big(A(x)\lambda, \lambda\big) \ge \alpha_m(|\lambda|^2) \text{ and } |A(x)\lambda| \le \alpha_M|\lambda|, & \forall \lambda \in \mathbb{R}^N. \end{cases} \end{aligned} \tag{1.1}$$

Let us denote, for any $\varepsilon > 0$,

$$A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big) \quad \text{a.e. in } \Omega.$$

Then for each $\varepsilon > 0$, we consider the state equation

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla y_\varepsilon(v_\varepsilon)) + f(y_\varepsilon(v_\varepsilon)) = \chi_{\omega_\varepsilon} v_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon \nabla y_\varepsilon(v_\varepsilon)) \cdot n_\varepsilon + h\varepsilon y_\varepsilon(v_\varepsilon) = \varepsilon g^\varepsilon \quad \text{on } \partial T^\varepsilon, \\ y_\varepsilon(v_\varepsilon) = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.2}$$

where f is real valued continuous function for which we assume that

$$f(0) = 0 \quad \text{ and } \exists \gamma > 0, \quad 0 \le \frac{f(s)}{s} \le \gamma, \ \forall s \in \mathbb{R} \setminus \{0\} \tag{1.3}$$

and h is a real, positive number, $g^\varepsilon(x) = g(\frac{x}{\varepsilon})$, where g is Y-periodic function in $L^2(\partial T)$, v_ε is the control supported in ω_ε and $y_\varepsilon(v_\varepsilon)$ is the associated state. Let us now consider the control problem to be addressed.

Control Problem. Given $\varepsilon > 0$, $\alpha > 0$, $y_{1\varepsilon} \in L^2(S_\varepsilon)$, find a control v_ε with support in ω_ε such that

$$\|y_\varepsilon(v_\varepsilon)|_{S_\varepsilon} - y_{1\varepsilon}\|_{0,S_\varepsilon} \le \alpha, \tag{1.4}$$

where $y_\varepsilon(v_\varepsilon)|_{S_\varepsilon}$ is just the restriction of $y_\varepsilon(v_\varepsilon)$ to S_ε.

By the approximate controllability (1.4), we mean that the L^2-distance between the obtained state observed on S_ε $\big(y_\varepsilon(v_\varepsilon)|_{S_\varepsilon}\big)$ and the desired state $(y_{1\varepsilon})$ will be approximated by the given prescribed precision α.

A real life application of the above control problem (in a fixed domain) is the following: consider a polluted sand filter occupying some domain Ω (with a fixed flow rate of pollutant). In Ω there is thus a granular / porous medium where, once the situation is idealized, the parameter ε represents both the characteristic pore length and the distance between adjacent grains. We add a suitable chemical reactant with concentration v (a control), to the control region $\omega \subseteq \Omega$ of the filter. Let $y(v)$ be the resulting concentration of the pollutant (which satisfies some elliptic boundary value problem in Ω). The problem is to find the optimal concentration of reagent to control the contaminant (altering its chemical state) throughout the region Ω.

In this article, we first study for each $\varepsilon > 0$, the approximate internal controllability of the ε-problem (1.2) with the Robin boundary condition, in a periodically perforated domain. Among these controls, we obtain the optimal one, which minimizes the given cost-functional, see for instance [20, 21]. We refer [10] for the similar result in a fixed domain and, [8] in a perforated domain respectively. The existence of the optimal control is established by means of a combination of Fenchel-Rockefellar duality theory [22] and the Zuazua's fixed point argument [27], introduced in the context of wave equation. Later this technique has been adapted in Fabre et al. [14] to deal with the semilinear heat equation.

Our second main result consists of proving the convergence of the optimal controls associated to the linearized ε-problem of (1.2). In the process, we pass to the limit in the cost-functional, homogenize the state and the adjoint equation. We end with identifying the weak limit v_0 of the optimal controls v_ε^* with the optimal element of the homogenized problem, which minimizes the limit cost-functional. Using the techniques by Zuazua [28], we observe that the minimizers of the cost-functionals are uniformly bounded. Thus we were able to apply the results of Donato and Näbil [11] to obtain the weak convergence of the minimizers. This allows us to pass to the limit in the cost-functional. The homogenization results using the periodic unfolding method, for the equations of the form (1.2) are given by Cioranescu and Donato [3, Section 6]. However, here we use the classical energy method introduced by Tartar [25, 26] for the homogenization. It consists of constructing the suitable test functions that are used in the variational problems. Such test-functions were also used in [6], where the authors have studied the homogenization of certain nonlinear models involving chemical reactive flows.

Approximate internal control problems were introduced by Lions [16, 17], also see [28]. The approximate controllability and homogenization results for the parabolic equations has been studied by Donato and Näbil [11, 12] for the periodically perforated domain. Later, Conca et al. studied the L^2-approximate controllability and homogenization of an elliptic boundary value problem in [9] for a fixed domain and, in [8] for a perforated domain with Neumann conditions on the boundary of holes. Then it was natural to look at the same problem in a periodically perforated domain. We have considered this problem with Robin boundary conditions on the boundary of holes, which is even more general condition. Thus our paper generalizes these results for the elliptic equations in a periodically perforated domain.

The organization of this paper is as follows: In Section 2, we introduce certain notations, a functional space, recall extension operators and some convergence results of the solutions in a periodically perforated domain. In subsection 2.1, we linearize the problem (1.2) and introduce the adjoint problem of the linearized one. In subsection 2.2, using Fenchel-Rockafellar's duality theory we obtain an expression for optimal control, in terms of dual variable. In Section 3, we state two main results of the paper. In Section 4, we prove our first main result Theorem 3.1 and the second main result, Theorem 3.3 is proved in Section 5.

2. Preliminaries

In this section, we recall some definitions, lemmas and other preparatory results to be used in the sequel. First we mention certain notation:

- $Y^* = Y \setminus \overline{T}$.
- $|E|$ is the Lebesgue measure of the measurable set E.
- $\theta = |Y^*|/|Y|$ the proportion of the material.
- $\mathcal{M}_Y(v) =$ the mean value of v over the measurable set Y.
- χ_E is the characteristic function of the set E.
- δ_E is the Dirac mass concentrated on the set E.
- $\tilde{u}$ is the extension by zero on E of any function u defined on $E_\varepsilon = E \cap \Omega_\varepsilon$.
- $(n_\varepsilon) = (n_\varepsilon^i)_{i=1}^N$ the unit external normal vector with respect to $Y \setminus T$ or Ω_ε.
- $\langle \cdot, \cdot \rangle_E$ denotes the inner product $\langle \cdot, \cdot \rangle_{L^2(E)}$.
- $\| \cdot \|_{0,E}$, $\| \cdot \|_{1,E}$, represents the L^2 and H^1-norms defined over the set E respectively.

The constants at the various places are denoted by C, which are independent of ε. Let us recall that

$$\chi_{\Omega_\varepsilon} \rightharpoonup \theta = |Y^*|/|Y| \quad \text{weak}^* \text{ in } L^\infty(\Omega).$$

Let us now introduce the functional space

$$V^\varepsilon = \{v \in H^1(\Omega_\varepsilon) : v = 0 \text{ on } \partial\Omega\},$$

equipped with norm $\|v\|_{V^\varepsilon} := \|\nabla v\|_{[L^2(\Omega_\varepsilon)]^N}$.

The weak formulation of (1.2) is: find $y_\varepsilon \in V_\varepsilon$, such that

$$\begin{aligned}
&\int_{\Omega_\varepsilon} A^\varepsilon \nabla y_\varepsilon \nabla \varphi dx + \int_{\Omega_\varepsilon} f(y_\varepsilon)\varphi dx + h\varepsilon \int_{\partial T\varepsilon} y_\varepsilon \varphi d\sigma(x) \\
&= \int_{\omega_\varepsilon} v_\varepsilon \varphi + \varepsilon \int_{\partial T_\varepsilon} g^\varepsilon \varphi d\sigma(x), \quad \text{for all } \varphi \in V^\varepsilon.
\end{aligned} \tag{2.1}$$

In the next lemma, we introduce a linear extension operator on $H^1(Y^*)$ and V^ε.

Lemma 2.1 ([4]). *For any $\varepsilon > 0$, we obtain*

(a) *There exist an extension operator $P \in \mathcal{L}\big(H^1(Y^*); H^1(Y)\big)$ such that*
$$\|\nabla(P\varphi)\|_{[L^2(Y)]^N} \le C\|\nabla\varphi\|_{[L^2(Y^*)]^N}, \quad \textit{for all } \varphi \in H^1(Y^*).$$
(b) *There exists an extension operator $P^\varepsilon \in \mathcal{L}(V^\varepsilon, H_0^1(\Omega))$ such that*
  (i) *$P^\varepsilon u = u$ in Ω_ε,*
  (ii) $\|P^\varepsilon u\|_{L^2(\Omega)} \le C\|u\|_{L^2(\Omega_\varepsilon)}$,
  (iii) $\|\nabla P^\varepsilon u\|_{[L^2(\Omega)]^N} \le C\|\nabla u\|_{[L^2(\Omega_\varepsilon)]^N}$.

Note that Lemma 2.1 provides a Poincaré inequality in V^ε with a constant independent of ε, that is

$$\|u\|_{V^\varepsilon} \le C\|\nabla u\|_{[L^2(\Omega_\varepsilon)]^N}.$$

Let us consider the elliptic boundary value problem

$$\begin{gathered}
-\operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f \quad \text{in } \Omega_\varepsilon, \\
(A^\varepsilon \nabla u_\varepsilon) \cdot n_\varepsilon = 0 \quad \text{on } \partial T^\varepsilon, \\
u_\varepsilon = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{2.2}$$

Now we recall the homogenization results for (2.2), its proof is available in [4].

Theorem 2.2. *Let $f \in L^2(\Omega)$. Under the hypotheses* (1.1)-(1.3), *the solution u_ε of* (2.2) *satisfies*

(i) *$P^\varepsilon u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$,*

(ii) $\widetilde{(A^\varepsilon \nabla u_\varepsilon)} \rightharpoonup (A^0 \nabla u)$ *weakly in* $[L^2(\Omega)]^N$,

where u *is the solution of the problem*

$$\begin{gathered} -\operatorname{div}(A^0 \nabla u) = \theta f \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

and A^0 *is the same matrix as obtained in* [4].

Remark 2.3 ([4, Theorem 2])**.** The homogenized operator A^0 and the limit function u do not depend on the extension operators.

2.1. **Linearized version and the adjoint problem.** In this section we linearize the nonlinear problem (1.2) and also introduce the adjoint problem of (1.2). It is very useful to follow a dual approach introduced by Lions [17]. We conclude this section by finding an expression for optimal control in terms of dual variable.

We assume that $f \in C^1(\mathbb{R})$ and define the function

$$p(s) := \begin{cases} f(s)/s & \text{if } s \neq 0, \\ f'(0) & \text{if } s = 0. \end{cases} \tag{2.3}$$

The assumptions on f (see (1.3)), implies that

$$p \in C^0(\mathbb{R}) \quad \text{and} \quad 0 \le p(s) \le \gamma, \quad \text{for all } s \in \mathbb{R}. \tag{2.4}$$

To the function p, we associate the linearized problem

$$\begin{gathered} -\operatorname{div}(A^\varepsilon \nabla y_\varepsilon(z, v_\varepsilon)) + p(z) y_\varepsilon(z, v_\varepsilon) = \chi_{\omega_\varepsilon} v_\varepsilon \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon \nabla y_\varepsilon) \cdot n_\varepsilon = \varepsilon g^\varepsilon - h\varepsilon y_\varepsilon \quad \text{on } \partial T_\varepsilon, \\ y_\varepsilon(z, v_\varepsilon) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.5}$$

Let us define the operators L_ε and L_ε^* as follows

$$L_\varepsilon : L^2(\omega_\varepsilon) \to L^2(S_\varepsilon) : \quad (v_\varepsilon \mapsto y_\varepsilon(z, v_\varepsilon)|_{S_\varepsilon}) \tag{2.6}$$

$$L_\varepsilon^* : L^2(S_\varepsilon) \to L^2(\omega_\varepsilon) : \quad (\varphi_{1\varepsilon} \mapsto \varphi_\varepsilon|_{\omega_\varepsilon}), \tag{2.7}$$

where $\varphi_\varepsilon = \varphi_\varepsilon(z, \varphi_{1\varepsilon})$ satisfies the adjoint of (2.5), which is given by

$$\begin{gathered} -\operatorname{div}({}^t A_\varepsilon \nabla \varphi_\varepsilon(z, \varphi_{1\varepsilon})) + p(z) \varphi_\varepsilon(z, \varphi_{1\varepsilon}) = \delta_{S_\varepsilon} \varphi_{1\varepsilon} \quad \text{in } \Omega_\varepsilon, \\ ({}^t A_\varepsilon \nabla \varphi_\varepsilon) \cdot n_\varepsilon = -h\varepsilon \varphi_\varepsilon \quad \text{on } \partial T_\varepsilon, \\ \varphi_\varepsilon(z, \varphi_{1\varepsilon}) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{2.8}$$

Now we shall give a detailed calculation for the adjoint (2.8) of the problem (2.5). Multiplying (2.5) by $\varphi_\varepsilon \in V_\varepsilon$ and integrating by parts, we obtain

$$\begin{aligned} &- \int_{\partial\Omega} (A^\varepsilon \nabla y_\varepsilon) \cdot n \varphi_\varepsilon - \int_{\partial T_\varepsilon} (A^\varepsilon \nabla y_\varepsilon) \cdot n_\varepsilon \varphi_\varepsilon \\ &+ \int_{\Omega_\varepsilon} A^\varepsilon \nabla y_\varepsilon \nabla \varphi_\varepsilon + \int_{\Omega_\varepsilon} p(z) y_\varepsilon \varphi_\varepsilon \\ &= \int_{\omega_\varepsilon} v_\varepsilon \varphi_\varepsilon \,. \end{aligned} \tag{2.9}$$

By the very definition of the operator L_ε^* we see that the right side in this identity is nothing but the duality pairing $\langle v_\varepsilon, L_\varepsilon^* \varphi_{1\varepsilon} \rangle_{\omega_\varepsilon}$. Thus, the right hand side of (2.9) can also be written as

$$\langle v_\varepsilon, L_\varepsilon^* \varphi_{1\varepsilon}\rangle_{\omega_\varepsilon} = \langle L_\varepsilon v_\varepsilon, \varphi_{1\varepsilon}\rangle_{S_\varepsilon} = \int_{S_\varepsilon} y_\varepsilon \varphi_{1\varepsilon} = \langle \delta_{S_\varepsilon}, y_\varepsilon \varphi_{1\varepsilon}\rangle_{\Omega_\varepsilon}.$$

Since $\varphi_\varepsilon \in V_\varepsilon$, a new integration by parts in (2.9) yields:

$$\int_{\partial T_\varepsilon} (h\varepsilon y_\varepsilon - \varepsilon g^\varepsilon)\varphi_\varepsilon + \int_{\Omega_\varepsilon} \nabla y_\varepsilon ({}^tA^\varepsilon \nabla \varphi_\varepsilon) + \int_{\Omega_\varepsilon} p(z) y_\varepsilon \varphi_\varepsilon = \langle \delta_{S_\varepsilon}, y_\varepsilon \varphi_{1\varepsilon}\rangle_{\Omega_\varepsilon},$$

which can also be written as

$$\begin{aligned} &\int_{\partial T_\varepsilon} h\varepsilon \varphi_\varepsilon y_\varepsilon - \int_{\partial T_\varepsilon} \varepsilon g^\varepsilon \varphi_\varepsilon + \int_{\partial T_\varepsilon} ({}^tA^\varepsilon \nabla \varphi_\varepsilon)\cdot n_\varepsilon y_\varepsilon + \int_{\partial\Omega} ({}^tA^\varepsilon \nabla\varphi_\varepsilon)\cdot n y_\varepsilon \\ &- \int_{\Omega_\varepsilon} \operatorname{div}({}^tA^\varepsilon \nabla \varphi_\varepsilon)\cdot y_\varepsilon + \int_{\Omega_\varepsilon} p(z) y_\varepsilon \varphi_\varepsilon \\ &= \langle \delta_{S_\varepsilon}, y_\varepsilon \varphi_{1\varepsilon}\rangle_{\Omega_\varepsilon}. \end{aligned}$$

Comparing the coefficients of y_ε both sides, we obtain the adjoint problem (2.8).

2.2. **Optimal control.** In this section, we obtain the optimal control by using the Fenchel-Rockafellar's duality theory, and we also establish an interesting relation between the optimal control and a solution of the adjoint (2.8).

We define the cost functional as follows

$$I_z^\varepsilon(v_\varepsilon) = \begin{cases} \frac12 \|v_\varepsilon\|^2_{0,\omega_\varepsilon}, & \text{if } \|y_\varepsilon(z, v_\varepsilon)|_{S_\varepsilon} - y_{1\varepsilon}\|_{0,S_\varepsilon} \le \alpha, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.10}$$

Let us decompose $I_z^\varepsilon(v)$ as

$$I_z^\varepsilon(v_\varepsilon) = F(v_\varepsilon) + G(L_\varepsilon v_\varepsilon),$$

where

$$F(v_\varepsilon) = \frac{1}{2}\|v_\varepsilon\|^2_{0,\omega_\varepsilon} \quad \text{and} \quad G(L_\varepsilon v_\varepsilon) = \begin{cases} 0 & \text{if } \|y_{\varepsilon|_{S_\varepsilon}} - y_{1\varepsilon}\|_{0,S_\varepsilon} \le \alpha, \\ \infty & \text{otherwise.} \end{cases} \tag{2.11}$$

Now we state a lemma, which gives the existence of a unique control minimizing the above cost functional (2.10).

Lemma 2.4. *For a given $z \in L^2(\Omega_\varepsilon)$, let $I_z^\varepsilon(v_\varepsilon^*(z))$ (defined by* (2.10)*), be a cost functional associated to the linearized problem* (2.5)*. Then by classical linear control theory (see* [18]*), it is well known that there exists a unique* minimal norm control *$v_\varepsilon^*(z)$, which minimizes $I_z^\varepsilon(v_\varepsilon)$ in the sense that*

$$I_z^\varepsilon(v_\varepsilon^*(z)) = \min_{v_\varepsilon \in L^2(\omega_\varepsilon)} I_z^\varepsilon(v_\varepsilon) < +\infty. \tag{2.12}$$

Let us denote by $y_\varepsilon^* := y_\varepsilon(z, v_\varepsilon^*(z))$ as the corresponding solution of (2.5). We are now in a position to define the operator $\mathcal{F}_\varepsilon$:

$$\mathcal{F}_\varepsilon : L^2(\Omega_\varepsilon) \to L^2(\Omega_\varepsilon); \quad z \mapsto y_\varepsilon(z, v_\varepsilon^*(z)). \tag{2.13}$$

Let $\overline{z}_\varepsilon$ be a fixed point of the map $\mathcal{F}_\varepsilon$. The existence of a fixed point is proved in the Theorem 3.1. Then the limit $v_0^*(z_0)$ of the optimal controls $v_\varepsilon^*(\overline{z}_\varepsilon)$ is the minimal norm control, among all the controls v satisfying

$$\|y_0(z_0, v)|_S - y_1\|_{0,S} \le \frac{\alpha}{\sqrt{\theta}}.$$

The duality theory of Fenchel and Rockafellar [13, 22] shows that the minimization problem (2.12) is equivalent to minimizing another non-quadratic functional,

$$J_z^\varepsilon(\varphi_{1\varepsilon}) = \frac{1}{2}\int_{\omega_\varepsilon} |\varphi_\varepsilon|^2 dx + \alpha\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon} - \int_{S_\varepsilon} y_{1\varepsilon}\varphi_{1\varepsilon} ds. \tag{2.14}$$

Thus we obtain

$$\inf_{v_\varepsilon \in L^2(\omega_\varepsilon)} I_\varepsilon^z(v_\varepsilon) = - \inf_{\varphi_{1\varepsilon}\in L^2(S_\varepsilon)} J_z^\varepsilon(\varphi_{1\varepsilon}), \tag{2.15}$$

where

$$\begin{gathered} J_z^\varepsilon(\varphi_{1\varepsilon}) = F^*(L_\varepsilon^*\varphi_{1\varepsilon}) + G^*(-\varphi_{1\varepsilon}), \\ F^*(L_\varepsilon^*\varphi_{1\varepsilon}) = \frac{1}{2}\|\varphi_\varepsilon(z,\varphi_{1\varepsilon})\|_{0,\omega_\varepsilon}^2, \\ G^*(L_\varepsilon^*\varphi_{1\varepsilon}) = \alpha\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon} + \langle\varphi_{1\varepsilon}, y_{1\varepsilon}\rangle_{S_\varepsilon}, \end{gathered} \tag{2.16}$$

and F^*, G^* are the conjugate functions of F and G respectively. We have

$$J_z^\varepsilon(\varphi_{1\varepsilon}) = \frac{1}{2}\|\varphi_\varepsilon(z,\varphi_{1\varepsilon})\|_{0,\omega_\varepsilon}^2 + \alpha\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon} - \langle\varphi_{1\varepsilon}, y_{1\varepsilon}\rangle_{S_\varepsilon}. \tag{2.17}$$

From the strict convexity of J_z^ε, we obtain $\varphi_{1\varepsilon}^*(z) \in L^2(S_\varepsilon)$ is the unique optimal element which minimizes $J_z^\varepsilon(\varphi_{1\varepsilon})$ over $L^2(S_\varepsilon)$. Let us denote $\varphi_\varepsilon^* := \varphi_\varepsilon(z,\varphi_{1\varepsilon}^*)$, the solution of (2.8)(with $\varphi_{1\varepsilon} = \varphi_{1\varepsilon}^*$). It is well known that the duality theory provides the extremal relations, which the optimal controls satisfy, namely:

$$\begin{gathered} F(v_\varepsilon^*(z)) + F^*(L_\varepsilon^*\varphi_{1\varepsilon}^*(z)) - \langle L_\varepsilon^*\varphi_{1\varepsilon}^*(z), v_\varepsilon^*(z)\rangle_{\omega_\varepsilon} = 0, \\ G(L_\varepsilon v_\varepsilon^*(z)) + G^*(-\varphi_{1\varepsilon}^*(z)) + \langle\varphi_{1\varepsilon}^*(z), L_\varepsilon v_\varepsilon^*(z)\rangle_{S_\varepsilon} = 0. \end{gathered} \tag{2.18}$$

With the help of the extremal relations satisfied by optimal controls, we derive the desired relation:

$$v_\varepsilon^*(z) = \varphi_\varepsilon(z,\varphi_{1\varepsilon}^*(z))|_{\omega_\varepsilon}. \tag{2.19}$$

3. Statements of main results

In the first main result, we obtain the fixed point of the operator $\mathcal{F}_\varepsilon$ defined by (2.13) by using the Zuazua's fixed point argument [27]. Hence we obtain the existence of the optimal control for our ε-problem (2.5), which minimizes the corresponding cost-functional (2.10). This theorem will be proved in Section 4.

Theorem 3.1. *Assume that for given $\varepsilon > 0$, $A^\varepsilon \in \mathcal{M}(\alpha_m,\alpha_M,\Omega)$ (see (1.1)). Let $f \in C^1(\mathbb{R})$ be a real valued function satisfying (1.3). Then the operator $\mathcal{F}_\varepsilon$, defined by (2.13) has at least one fixed point $\overline{z}_\varepsilon \in L^2(\Omega_\varepsilon)$. Let $v_\varepsilon^*(\overline{z}_\varepsilon)$ be the optimal control minimizing the functional $I_{\overline{z}_\varepsilon}^\varepsilon$, given by (2.10). Then the fixed element $\overline{z}_\varepsilon$, satisfies the equation:*

$$\overline{z}_\varepsilon = y_\varepsilon^*(\overline{z}_\varepsilon, v_\varepsilon^*(\overline{z}_\varepsilon)),$$

where $y_\varepsilon^*(\overline{z}_\varepsilon, v_\varepsilon^*(\overline{z}_\varepsilon))$ is the state solution of the problem (1.2).

Below we state the second main result of this paper, concerning the homogenization of state and adjoint state equations and the convergence of the optimal controls, which will be proved in Section 5. In the following, we shall need certain hypotheses:

- (H1) If $h = 0$ and $g \equiv 0$, we obtain uniform (with respect to ε) Poincaré inequality in V_ε.
- (H2) Given the sequence $\{y_{1\varepsilon}\} \subset L^2(S_\varepsilon)$, we assume that

$$y_{1\varepsilon} \to y_1, \quad L^2(S)\text{-strongly}. \tag{3.1}$$

(H3) For any sequence $\{\varphi_{1\varepsilon}\} \subset L^2(S_\varepsilon)$, we obtain
$$\frac{\widetilde{\varphi}_{1\varepsilon}}{\theta} \rightharpoonup \varphi_1 \quad \text{in } L^2(S)\text{-weakly.}$$

Remark 3.2. Hypothesis (H1) is essential in order to give a-priori estimates in $H^1(\Omega_\varepsilon)$. However if we add a zero order term in equation (1.2), we do not need it, also see Cioranescu et al. [3, Section 6]. The hypothesis (H2) ensures the inequality $\|\varphi^*_{1\varepsilon}\|_{0,S_\varepsilon} \le C$ ($\varphi^*_{1\varepsilon}$ is the minimizer of the cost (2.14)) and the convergence of approximate control inequality (1.4). Moreover (H2) and (H3) are needed, in order to pass to the limit in the adjoint equation (2.8) and in the cost functional (2.14), as $\varepsilon \to 0$.

Since we are interested here in studying the asymptotic behaviour of optimal controls, it is natural to ask a question: whether the limit of the optimal controls v^*_ε, is the same as the the optimal control associated to the homogenized problem (3.5), given below? The following theorem gives a positive answer to this question.

Theorem 3.3. *Let us assume that the hypotheses of Theorem 3.1 and* (H1)–(H3) *hold. Then there exists $z_0 \in L^2(\Omega)$, which is the weak limit of the fixed points $\{\overline{z}_\varepsilon\}$ (obtained in Theorem 3.1). Moreover, there exists $v_0 \in L^2(\omega)$ which is the weak limit of the sequence of optimal controls $\{v^*_\varepsilon(\overline{z}_\varepsilon)\}$ (identified in Theorem 3.1).*

Further we obtain
$$v_0 = v^*_0(z_0),$$
and up to a subsequence
$$P^\varepsilon y_\varepsilon(\overline{z}_\varepsilon, v^*_\varepsilon(\overline{z}_\varepsilon)) \rightharpoonup y_0(z_0, v_0) \quad H^1_0(\Omega)\text{-weakly}, \quad \text{as } \varepsilon \to 0,$$
where $v^*_0(z_0)$ is the minimal norm control among all the controls v satisfying
$$\|y_0(z_0, v)|_S - y_1\|_{0,S} \le \frac{\alpha}{\sqrt{\theta}},$$
with $y_0(z_0, v)$ being the solution of the homogenized system
$$\begin{gathered} -\theta \operatorname{div}(A^0 \nabla y_0(z_0, v)) + \theta f(y_0(z_0, v)) \\ = \theta \chi_\omega v + \frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g) - h\frac{|\partial T|}{|Y|} y_0(z_0, v) \quad \text{in } \Omega, \\ y_0(z_0, v) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{3.2}$$

The homogenized matrix is
$$A^0 = (a^0_{ij}) = \frac{1}{|Y|}\Big(a_{ji} + a_{jk}\frac{\partial \chi_i}{\partial y_k}\Big), \quad \theta = \frac{|Y^*|}{|Y|}, \tag{3.3}$$
and χ_i satisfy the equation
$$\begin{gathered} -\operatorname{div}\big({}^tA^\varepsilon \nabla(\chi_i + y_i)\big) = 0 \quad \text{in } Y^*, \\ \big({}^tA^\varepsilon \nabla(\chi_i + y_i)\big) \cdot n_\varepsilon = 0 \quad \text{on } \partial T, \\ \chi_i \text{ is } Y\text{-periodic}, \\ \mathcal{M}_{Y^*}(\chi_i) = 0. \end{gathered} \tag{3.4}$$

Moreover the adjoint equation (2.8) *can also be homogenized as*
$$\begin{gathered} -\theta \operatorname{div}({}^tA^0 \nabla \varphi_0) + \theta p(z_0)\varphi_0 = \theta \delta_S \varphi_1 - h\frac{|\partial T|}{|Y|}\varphi_0 \quad \text{in } \Omega, \\ \varphi_0 = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.5}$$

where

$$({}^tA^0) = \frac{1}{|Y|}\Big(a_{ij} + a_{ik}\frac{\partial \widehat{\chi}_j}{\partial y_k}\Big), \tag{3.6}$$

and $\widehat{\chi}_j$ *satisfy the equation*

$$\begin{gathered} -\operatorname{div}\big(A^\varepsilon \nabla(\widehat{\chi}_j + y_i)\big) = 0 \quad \text{in } Y^*,\\ \big(A^\varepsilon \nabla(\widehat{\chi}_j + y_i)\big)\cdot n_\varepsilon = 0 \quad \text{on } \partial T,\\ \widehat{\chi}_j \text{ is } Y\text{-periodic},\\ \mathcal{M}_{Y^*}(\widehat{\chi}_j) = 0. \end{gathered} \tag{3.7}$$

Let $\varphi_0^* := \varphi_0(z_0, \varphi_1^*)$ *be the solution of homogenized adjoint equation associated with* $\varphi_1 = \varphi_1^*$*, where*

$$\varphi_1^* = \operatorname{argmin}\Big(\frac{\theta}{2}\int_\omega |\varphi_0|^2 dx + \alpha\sqrt{\theta}\|\varphi_1\|_{0,S} - \theta\int_S y_1\varphi_1 ds\Big).$$

Then the representation of the optimal control $v_0^*(z_0)$ *in terms of this dual variable* φ_0^* *is* $v_0^* = \theta(\varphi_0^*)_{|\omega}$*.*

Remark 3.4. We refer to [11, Theorem 6] for the homogenization results of the parabolic equations.

4. Proof of Theorem 3.1

Proof. Let $\varphi \in V_\varepsilon$ be a test function in (2.8), the variational formulation is given by:

$$\int_{\Omega_\varepsilon} ({}^tA^\varepsilon \nabla\varphi_\varepsilon)\nabla\varphi + \int_{\Omega_\varepsilon} p(z)\varphi_\varepsilon\varphi + h\varepsilon\int_{\partial T_\varepsilon}\varphi_\varepsilon\varphi = \int_{S_\varepsilon}\varphi_{1\varepsilon}\varphi. \tag{4.1}$$

Let $z_n \to z_0$ in $L^2(\Omega_\varepsilon)$, as $n \to \infty$ and denote $\varphi_{\varepsilon,n} = \varphi_\varepsilon(z_n)$. Using $\varphi_{\varepsilon,n}$ as a test function in (2.8) (written for $\varphi_\varepsilon = \varphi_{\varepsilon,n}$), since h is a positive real number, using the regularity of A^ε and a property of the linearized function p, we obtain

$$\alpha_m\|\nabla\varphi_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^N} \le |\varphi_{\varepsilon,n}\|_{0,\Omega_\varepsilon}\||\varphi_{1\varepsilon}\|_{0,S_\varepsilon}. \tag{4.2}$$

Again using Poincaré's inequality on the right hand side,

$$\alpha_m\|\varphi_{\varepsilon,n}\|^2_{1,\Omega_\varepsilon} \le \|\varphi_{\varepsilon,n}\|_{1,\Omega_\varepsilon}\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon}.$$

This gives

$$\|\varphi_{\varepsilon,n}\|_{1,\Omega_\varepsilon} \le C, \tag{4.3}$$

where the constant C depends on the ellipticity constant α_m, on trace, Poincaré constant, but independent of ε. Thanks to (4.3), up to a subsequence (in n), we obtain

$$\begin{gathered} \varphi_{\varepsilon,n} \rightharpoonup \varphi_{\varepsilon,0} \quad V_\varepsilon \text{ -weakly},\\ \varphi_{\varepsilon,n} \rightharpoonup \varphi_{\varepsilon,0} \quad L^2(\Omega_\varepsilon) \text{ -strongly}. \end{gathered} \tag{4.4}$$

To pass to the limit in the variational formulation (4.1) (as $n \to \infty$), let $\varphi \in V_\varepsilon$, and consider the following:

$$\Big(\int_{\Omega_\varepsilon} p(z_n)\varphi_{\varepsilon,n}\varphi dx - \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0}\varphi dx\Big) + h\varepsilon\Big(\int_{\Omega_\varepsilon}\varphi_{\varepsilon,n}\varphi dx - \int_{\Omega_\varepsilon}\varphi_{\varepsilon,0}\varphi dx\Big).$$

Since $\varphi_{\varepsilon,n} \to \varphi_{\varepsilon,0}$, $L^2(\Omega_\varepsilon)$-strongly, as $n \to \infty$, therefore it suffices to consider the limit of first term only.

$$\int_{\Omega_\varepsilon} p(z_n)\varphi_{\varepsilon,n}\varphi dx - \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0}\varphi dx$$

$$= \int_{\Omega_\varepsilon} p(z_n)(\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0})\varphi dx + \int_{\Omega_\varepsilon} \Big(p(z_n) - p(z_0)\Big)\varphi_{\varepsilon,0}\varphi dx.$$

Now $p(z_n)$ is bounded in $L^\infty(\Omega_\varepsilon)$, so we obtain

$$p(z_n) \rightharpoonup p(z_0), \quad L^\infty(\Omega_\varepsilon) \text{ -weakly}^*. \tag{4.5}$$

This gives us,

$$\int_{\Omega_\varepsilon} p(z_n)\varphi_{\varepsilon,n}\varphi dx \to \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0}\varphi dx, \quad \text{for all } \varphi \in V_\varepsilon. \tag{4.6}$$

Thus (4.5) and (4.6) shows that

$$p(z_n)\varphi_{\varepsilon,n} \to p(z_0)\varphi_{\varepsilon,0} \quad V_\varepsilon^*\text{-weakly (as } n \to \infty). \tag{4.7}$$

Now we shall show that the convergence (4.7) is actually V_ε^* -strong (as $n \to \infty$). Let $\varphi \in V_\varepsilon$ and consider the following:

$$\begin{aligned} &\Big| \int_{\Omega_\varepsilon} p(z_n)\varphi_{\varepsilon,n}\varphi dx - p(z_0)\varphi_{\varepsilon,0}\varphi dx \Big| \\ &\le \Big| \int_{\Omega_\varepsilon} p(z_n)(\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0})\varphi dx \Big| + \Big| \int_{\Omega_\varepsilon} \big(p(z_n) - p(z_0)\big)\varphi_{\varepsilon,0}\varphi dx \Big|. \end{aligned} \tag{4.8}$$

Let us first evaluate the norm estimate of the first term on the right hand side of (4.8). We consider the following:

$$\Big| \int_{\Omega_\varepsilon} p(z_n)(\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0})\varphi dx \Big| \le \|p(z_n)\|_{L^{p_1}(\Omega_\varepsilon)} \cdot \|\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0}\|_{L^{p_2}(\Omega_\varepsilon)} \|\varphi\|_{L^{p_3}(\Omega_\varepsilon)}.$$

Observe that $\|p(z_n)\|_{L^{p_1}(\Omega_\varepsilon)} \le \gamma \operatorname{meas}(\Omega_\varepsilon)^{1/p_1}$. We choose $p_1 = N, p_2 = p_3 = \frac{2N}{N-1}$, for $N \ge 2$; otherwise $p_1 = p_3 = 4$ and $p_2 = 2$. Thanks to the choice of p_2, the injection $H^1(\Omega_\varepsilon) \hookrightarrow L^2(\Omega_\varepsilon)$ is compact and we obtain

$$\|\varphi_{\varepsilon,n} - \varphi_{\varepsilon,0}\|_{L^{p_2}} \to 0, \quad \text{as } n \to \infty.$$

Note that the injection i from $V_\varepsilon \hookrightarrow L^{p_3}(\Omega_\varepsilon)$ is continuous, so

$$\|\varphi\|_{L^{p_3}} \le \|i\|_{\mathcal{L}(V_\varepsilon), L^{p_3}(\Omega_\varepsilon)} \cdot \|\varphi\|_{V_\varepsilon},$$

so that the first term in (4.8) goes to zero. On the other hand,

$$\Big\| \int_{\Omega_\varepsilon} (p(z_n) - p(z_0))\varphi_{\varepsilon,0}\varphi dx \Big\| \le \|p(z_n) - p(z_0)\|_{L^{q_1}(\Omega_\varepsilon)} \|\varphi_{\varepsilon,0}\|_{L^{q_2}(\Omega_\varepsilon)} \|\varphi\|_{L^{q_3}},$$

with $q_1 = \frac{N}{2}$, $q_2 = q_3 = \frac{2N}{N-2}$, $N \ge 3$, otherwise $q_1 \ne 2, q_2 = q_3 = 4$. Thanks to this choice, $H^1(\Omega_\varepsilon) \hookrightarrow L^{q_3}(\Omega_\varepsilon)$ is continuous and in a similar way as above, a bound can be obtained. It follows by Lebesgue Dominated convergence theorem and the bounds of $p(z_n)$ (see (2.4)) that

$$\|p(z_n) - p(z_0)\|_{L^{q_1}} \to 0, \quad \text{as } n \to \infty.$$

Thus the second term on the right hand side of (4.8) also vanishes and hence we obtain

$$p(z_n)\varphi_{\varepsilon,n} \to p(z_0)\varphi_{\varepsilon,0} \quad V_\varepsilon^*\text{-strongly}, \quad \text{as } n \to \infty. \tag{4.9}$$

Next we show that $\varphi_{\varepsilon,0} = \varphi_\varepsilon(z_0)$. For that we multiply adjoint (2.8) (written for $\varphi_{\varepsilon,n}$), by the test function $\varphi \in V_\varepsilon$ and integrate by parts,

$$\int_{\Omega_\varepsilon} A_\varepsilon \nabla\varphi_{\varepsilon,n} \cdot \nabla\varphi dx + \int_{\Omega_\varepsilon} p(z_n)\varphi_{\varepsilon,n}\varphi dx = \int_{S_\varepsilon} \varphi_{1\varepsilon}\varphi d\sigma. \tag{4.10}$$

Passing to the limit in (4.10) as $n \to \infty$, we obtain

$$\int_{\Omega_\varepsilon} A^\varepsilon \varepsilon \nabla\varphi_{\varepsilon,0} \cdot \nabla\varphi dx + \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0}\varphi dx = \int_{S_\varepsilon} \varphi_{1\varepsilon}\varphi d\sigma,$$

and this shows that $\varphi_{\varepsilon,0} = \varphi_\varepsilon(z_0)$. Now we shall show that the convergence (4.4) is V_ε-strong. We take $\varphi_{\varepsilon,n} \in V_\varepsilon$ as test function in adjoint (2.8) (written for $z = z_n$ and $\varphi_\varepsilon = \varphi_{\varepsilon,n}$) and pass to the limit in the variational formulation, as $n \to \infty$. In view of (4.4) and (4.9), we obtain

$$\lim_{n\to\infty} \int_{\Omega_\varepsilon} {}^t A_\varepsilon \nabla\varphi_{\varepsilon,n} \nabla\varphi_{\varepsilon,n} dx = \int_{S_\varepsilon} \varphi_{1\varepsilon}\varphi_{\varepsilon,0} d\sigma - \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0} \cdot \varphi_{\varepsilon,0} dx. \tag{4.11}$$

Now, taking $\varphi_{\varepsilon,0}$ as a test function in the adjoint equation (with $\varphi_\varepsilon = \varphi_{\varepsilon,0}$), we obtain the following:

$$\int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla\varphi_{\varepsilon,0} \cdot \nabla\varphi_{\varepsilon,0} dx = \int_{S_\varepsilon} \varphi_{1\varepsilon}\varphi_{\varepsilon,0} d\sigma - \int_{\Omega_\varepsilon} p(z_0)\varphi_{\varepsilon,0} \cdot \varphi_{\varepsilon,0} dx. \tag{4.12}$$

Comparing (4.11) and (4.12), we get

$$\lim_{n\to\infty} \int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla\varphi_{\varepsilon,n} \cdot \nabla\varphi_{\varepsilon,n} dx = \int_{\Omega_\varepsilon} {}^t A_\varepsilon \nabla\varphi_{\varepsilon,0} \cdot \nabla\varphi_{\varepsilon,0} dx.$$

We conclude that $\varphi_{\varepsilon,n} \to \varphi_{\varepsilon,0}$, V_ε -strongly (*energy convergence*), as $n \to \infty$, since we know that $(\int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla v \cdot \nabla v dx)^{1/2}$ is equivalent to the standard H^1-norm defined over V_ε. Now using the coercivity property of the functional, we show the following

$$\|\varphi^*_{1\varepsilon}\|_{0,S_\varepsilon} \le C, \tag{4.13}$$

where C independent of n and ε. For if (4.13) holds, then it follows by Banach-Alaoglu-Bourbaki theorem, that there exists ξ^ε such that $\varphi^*_{1\varepsilon}(z_n) \rightharpoonup \xi^\varepsilon, L^2(S_\varepsilon)$-weakly, which would then imply for another subsequence (in n)

$$\varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n)) \to \varphi_\varepsilon(z_0, \xi^\varepsilon), \quad V_\varepsilon \text{ -strongly.} \tag{4.14}$$

The inequality (4.13) will be proved by contradiction. We assume on contrary that

$$\|\varphi^*_{1\varepsilon}(z_n)\|_{0,S_\varepsilon} \to +\infty, \quad \text{as } n \to \infty.$$

Since $\varphi^*_{1\varepsilon}(z_n)$ is the minimizer of $J^\varepsilon_{z_n}$, for each n, we obtain

$$J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)) \le J^\varepsilon_{z_n}(\varphi_{1\varepsilon}), \quad \text{for all } \varphi_{1\varepsilon} \in L^2(S_\varepsilon). \tag{4.15}$$

On the other hand, thanks to the convergence $\varphi_{\varepsilon,n} \to \varphi_{\varepsilon,0}$ (V_ε-strong), we see that

$$J^\varepsilon_{z_n}(\varphi_{1\varepsilon}) = \frac{1}{2}\int_{\omega_\varepsilon} |\varphi_\varepsilon(z_n)|^2 dx + \alpha\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon} - \int_{S_\varepsilon} y_{1\varepsilon}\varphi_{1\varepsilon} d\sigma,$$

converges to

$$J^\varepsilon_{z_0}(\varphi_{1\varepsilon}) = \frac{1}{2}\int_{\omega_\varepsilon} |\varphi_\varepsilon(z_0)|^2 dx + \alpha\|\varphi_{1\varepsilon}\|_{0,S_\varepsilon} - \int_{S_\varepsilon} y_{1\varepsilon}\varphi_{1\varepsilon} d\sigma.$$

Therefore from (4.15), for fixed $\varphi_{1\varepsilon}$,

$$J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)) \le C, \quad C \text{ is independent of } n \text{ and } \varepsilon.$$

This contradicts the coercivity of the functional $J^\varepsilon_{z_n}$, since

$$\liminf_{\|\varphi^*_{1\varepsilon}(z_n)\|_{0,S_\varepsilon}\to\infty} \frac{J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n))}{\|\varphi^*_{1\varepsilon}(z_n)\|_{0,S_\varepsilon}} \ge \alpha > 0.$$

Next, arguing as we as we did in (4.3), we obtain

$$\|\varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n))\|_{V_\varepsilon} \le C(\varphi^*_1). \tag{4.16}$$

In view of (4.13) (thanks to Banach-Alaoglu-Bourbaki theorem) we obtain

$$\varphi^*_{1\varepsilon}(z_n) \rightharpoonup \xi^\varepsilon \quad L^2(S_\varepsilon)\text{-weakly, as } n \to \infty. \tag{4.17}$$

It remains to identify the limit: $\xi^\varepsilon = \varphi^*_{1\varepsilon}(z_0)$. For that, we show that ξ^ε is the minimizer of $J^\varepsilon_{z_0}$, that is to show

$$J^\varepsilon_{z_0}(\xi^\varepsilon) \le J^\varepsilon_{z_0}(\varphi_{1\varepsilon}), \quad \text{for all } \varphi_{1\varepsilon} \in L^2(S_\varepsilon). \tag{4.18}$$

Since we know that $\varphi^*_{1\varepsilon}$ is optimal element for $J^\varepsilon_{z_n}$, we obtain

$$J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)) \le J^\varepsilon_{z_n}(\varphi_{1\varepsilon}), \quad \text{for all } \varphi_{1\varepsilon} \in L^2(S_\varepsilon),$$

hence

$$\liminf_n J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)) \le \liminf_n J^\varepsilon_{z_n}(\varphi_{1\varepsilon}) = J^\varepsilon_{z_0}(\varphi_{1\varepsilon}), \quad \text{for all } \varphi_{1\varepsilon} \in L^2(S_\varepsilon).$$

Therefore in order to prove (4.18), it remains to prove that

$$J^\varepsilon_{z_0}(\xi^\varepsilon) \le \liminf_n J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)). \tag{4.19}$$

Let us recall that

$$J^\varepsilon_{z_n}(\varphi^*_{1\varepsilon}(z_n)) = \frac{1}{2}\int_{\omega_\varepsilon} |\varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n))|^2 dx + \alpha\|\varphi^*_{1\varepsilon}(z_n)\|_{0,S_\varepsilon} - \int_{S\varepsilon} y_{1\varepsilon}\varphi^*_{1\varepsilon}(z_n) d\sigma.$$

From (4.17) we obtain

$$\liminf_n \alpha\|\varphi^*_{1\varepsilon}(z_n)\|_{0,S_\varepsilon} - \int_{S_\varepsilon} y_{1\varepsilon}\varphi^*_{1\varepsilon}(z_n) d\sigma \ge \liminf_n \alpha\|\xi^\varepsilon\|_{0,S_\varepsilon} - \int_{S_\varepsilon} y_{1\varepsilon}\xi^\varepsilon d\sigma.$$

Also from (4.14) and Fatou's lemma, we obtain

$$\liminf_n \int_{\omega_\varepsilon} |\varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n))|^2 dx \ge \int_{\omega_\varepsilon} |\varphi_\varepsilon(z_0, \xi^\varepsilon)|^2 dx.$$

Thus we obtained (4.19) and hence (4.18). In other words we proved that

$$\xi^\varepsilon = \varphi^*_{1\varepsilon}(z_0).$$

Together with this relation, convergence (4.14) holds for the whole sequence, that is,

$$\varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n)) \to \varphi_\varepsilon(z_0, \varphi^*_{1\varepsilon}(z_0)) \quad V_\varepsilon \text{ - strongly}. \tag{4.20}$$

In view of the relation (2.19), we know that

$$v^*_\varepsilon(z_n) = \varphi_\varepsilon(z_n, \varphi^*_{1\varepsilon}(z_n))|_{\omega_\varepsilon}$$
$$v^*_\varepsilon(z_0) = \varphi_\varepsilon(z_0, \varphi^*_{1\varepsilon}(z_0))|_{\omega_\varepsilon}$$

It follows by (4.20) that $v^*_\varepsilon(z_n) \to v^*_\varepsilon(z_0)$ strongly in $H^1(\omega_\varepsilon)$. Using this convergence in state equation (2.5), and an analogous proof to obtain (4.20) from adjoint problem (2.8), we get the following:

$$y_\varepsilon(z_n, v^*_\varepsilon(z_n)) \to y_\varepsilon(z_0, v^*_\varepsilon(z_0)) \quad V_\varepsilon \text{ -strongly}.$$

Hence $\mathcal{F}_\varepsilon$ is continuous for fixed $\varepsilon > 0$.

Next we show that $\mathcal{F}_\varepsilon$ is compact (uniformly in ε). Since we obtain (2.4), for given $z \in L^2(\Omega_\varepsilon)$, it follows by [11, Theorem 6] that under the hypothesis (H2), the sequence of minimizers $\{\varphi_{1\varepsilon}^*\}$ are uniformly bounded in $L^2(S)$ (also see [28, Lemma 2]) and thus satisfy

$$\frac{1}{\theta}\widetilde{\varphi_{1\varepsilon}^*} \rightharpoonup \varphi_1^* \quad L^2(S)\text{-weakly}, \tag{4.21}$$

where φ_1^* minimizes the functional

$$J(\varphi_1^*) = \frac{1}{2}\theta\int_\omega |\varphi_0|^2 dx + \alpha\sqrt{\theta}\|\varphi_1^*\|_{0,S} - \theta\int_S y_1\varphi_1^* dx.$$

By (4.2) and (4.21), we obtain

$$\|\varphi_\varepsilon(z,\varphi_{1\varepsilon}^*)\|_{1,\Omega_\varepsilon} \le C\|\varphi_1^*\|_{0,S}, \tag{4.22}$$

where C is independent of z and ε. This implies that

$$J_z^\varepsilon(\varphi_{1\varepsilon}^*(z)) \le C(\varphi_1^*).$$

Again using coercivity of J_z^ε, we see that $\|\varphi_{1\varepsilon}^*\|_{0,S}$ and hence $\|\varphi_\varepsilon(z,\varphi_{1\varepsilon}^*)\|_{1,\Omega_\varepsilon}$ is bounded by a constant independent of z and ε. Consequently by (2.19) and (4.22) we obtain

$$\|v_\varepsilon^*(z)\|_{0,\omega_\varepsilon} \le C. \tag{4.23}$$

Using (2.4) and (4.23) in the variational formulation formulation (2.1), it is easy to see that $y_\varepsilon(z,v_\varepsilon^*(z))$ is bounded. Hence by Schauder's fixed point theorem, $\mathcal{F}_\varepsilon$ has at least one fixed point in $L^2(\Omega_\varepsilon)$. □

5. Proof of Theorem 3.3

This proof is completed using several steps.

Proof. Let $\overline{z}_\varepsilon$ be a fixed point of the operator $\mathcal{F}_\varepsilon$. Let $v_\varepsilon^* = v_\varepsilon^*(\overline{z}_\varepsilon)$ be the optimal controls satisfying (1.2)-(1.4). Note that (4.23) holds true for every z, in particular for $z = \overline{z}_\varepsilon$. This implies that there exists $v_0 \in L^2(\omega)$, such that up to a subsequence we obtain

$$\begin{gathered}\widetilde{v_\varepsilon^*(\overline{z}_\varepsilon)} \rightharpoonup \theta v_0 \quad L^2(\omega)\text{-weakly},\\ \chi_{\omega_\varepsilon}\widetilde{v_\varepsilon^*(\overline{z}_\varepsilon)} \to \theta\chi_\omega v_0 \quad H^{-1}(\Omega)\text{-strongly}.\end{gathered} \tag{5.1}$$

Step 1. Let us consider the variational formulation (2.1) (for $v_\varepsilon = v_\varepsilon^* := v_\varepsilon^*(\overline{z}_\varepsilon)$) and take the test function $\varphi = y_\varepsilon(= y_\varepsilon(v_\varepsilon^*)) \in V^\varepsilon$, to obtain

$$\begin{aligned}&\int_{\Omega_\varepsilon} A^\varepsilon\nabla y_\varepsilon\nabla y_\varepsilon dx + h\varepsilon\int_{\partial T\varepsilon} y_\varepsilon y_\varepsilon d\sigma(x)\\ &= \int_{\omega_\varepsilon} v_\varepsilon^* y_\varepsilon + \varepsilon\int_{\partial T_\varepsilon} g^\varepsilon y_\varepsilon d\sigma(x) - \int_{\Omega_\varepsilon} f(y_\varepsilon)y_\varepsilon dx.\end{aligned} \tag{5.2}$$

Using (1.1), the assumption on $(A^\varepsilon(x))$, we get that the left hand side of the equation (5.2) is at least $\alpha_m\|\nabla y_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^N} + h\varepsilon\|y_\varepsilon\|^2_{\partial T_\varepsilon}$. Taking the norm estimates on both the sides of the (5.2), using (4.23), the uniform Poincaré inequality (H1) and [3, Corollary 5.4], we derive that

$$\begin{aligned}\alpha_m\|y_\varepsilon\|^2_{H^1(\Omega_\varepsilon)} &\le \alpha_m\|\nabla y_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^N} + h\varepsilon\|y_\varepsilon\|^2_{\partial T_\varepsilon}\\ &\le C\left(\|v_\varepsilon^*\|_{0,\omega_\varepsilon} + |\mathcal{M}_{\partial T}(g)| + \|f\|_{[L^2(\Omega_\varepsilon)]^N}\right)\|\nabla y_\varepsilon\|_{[L^2(\Omega_\varepsilon)]^N}\\ &\le C'\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)}.\end{aligned}$$

Hence $\|y_\varepsilon\|_{H^1(\Omega_\varepsilon)} \le \tilde{C}$, for some constant $\tilde{C}$ independent of ϵ.

By [4, Lemma 1], there exists a linear continuous extension operator $P^\varepsilon \in L^2(\Omega_\varepsilon, L^2(\Omega)) \cap L(V^\varepsilon, H_0^1(\Omega))$ such that

$$\|P^\varepsilon y_\varepsilon\|_{1,\Omega} \le C\|y_\varepsilon\|_{1,\Omega_\varepsilon} \le C\tilde{C}. \tag{5.3}$$

Thus there exists $y_0(v_0) \in H^1(\Omega)$ such that up to a subsequence,

$$P^\varepsilon y_\varepsilon \rightharpoonup y_0(v_0) \quad H_0^1(\Omega)\text{-weakly}. \tag{5.4}$$

Hereafter we denote by $y_0 := y_0(v_0)$.

Step 2. Now we want to identify the limit equation satisfied by y_0. To get that, we want to pass to the limit in Equation (5.2), as $\varepsilon \to 0$. For that we first define a linear form μ_h^ε on $W_0^{1,s}(\Omega)$, for any $h \in L^{p'}(\partial T_\varepsilon)$, $1 \le p' \le \infty$, as follows:

$$< \mu_h^\varepsilon, \varphi >:= \varepsilon \int_{\partial T_\varepsilon} h\left(\frac{x}{\varepsilon}\right) \varphi \, d\sigma(x), \quad \text{for all } \varphi \in W_0^{1,s}(\Omega).$$

It follows from [2], [7] that

$$\mu_h^\varepsilon \to \mu_h \text{ strongly in } W_0^{1,s}(\Omega)', \tag{5.5}$$

where, $< \mu_h, \varphi >= \mu_h \int_\Omega \varphi \, dx$, and $\mu_h = \frac{1}{|Y|} \int_{\partial T} h(y) \, d\sigma(y)$. In particular, when $h \in L^\infty(\partial T)$, we obtain

$$\mu_h^\varepsilon \to \mu_h \quad \text{strongly in } W^{-1,\infty}(\Omega).$$

For $h \equiv 1$, μ_h becomes $\mu_1 = \frac{|\partial T|}{|Y|}$ and

$$\lim_{\varepsilon \to 0} < \mu^\varepsilon, \varphi h y_\varepsilon >= \lim_{\varepsilon \to 0} \varepsilon \int_{\partial T} \varphi h y_\varepsilon \, d\sigma(x), \quad \text{for all } \varphi \in W_0^{1,s}(\Omega).$$

From (5.5), with $h = 1$, we obtain

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\partial T_\varepsilon} \varphi h y_\varepsilon = \frac{|\partial T|}{|Y|} \int_\Omega \varphi h y_0 dx, \quad \text{for all } \varphi \in \mathcal{D}(\Omega). \tag{5.6}$$

Let $\xi^\varepsilon = A^\varepsilon \nabla y_\varepsilon$ in Ω_ε and let $\widetilde{\xi^\varepsilon}$ be its extension by zero to the whole of Ω. By the property of A^ε and the boundedness of y_ε (see (5.3)), we obtain $\widetilde{\xi^\varepsilon}$ is bounded in $(L^2(\Omega))^N$. Hence there exists $\xi \in [L^2(\Omega)]^N$ such that

$$\widetilde{\xi^\varepsilon} \rightharpoonup \xi \quad [L^2(\Omega)]^N\text{-weakly}. \tag{5.7}$$

To obtain the equation satisfied by ξ, take $\varphi \in \mathcal{D}(\Omega)$ as the test function in the variational formulation (5.2), we get

$$\int_{\Omega_\varepsilon} \xi^\varepsilon \nabla \varphi dx + \int_{\Omega_\varepsilon} f(y_\varepsilon) \varphi dx + h\varepsilon \int_{\partial T\varepsilon} y_\varepsilon \varphi d\sigma(x) = \int_{\omega_\varepsilon} v_\varepsilon^* \varphi + \varepsilon \int_{\partial T_\varepsilon} g^\varepsilon \varphi d\sigma(x). \tag{5.8}$$

It follows by (5.7),

$$\lim_{\varepsilon \to 0} \int_\Omega \widetilde{\xi^\varepsilon} \nabla \varphi = \int_\Omega \xi \nabla \varphi \, dx, \tag{5.9}$$

and since f is uniformly Lipschitz and $P^\varepsilon y_\varepsilon \to y_0$, we get $f(P^\varepsilon y_\varepsilon) \to f(y_0)$. By [15, Lemma 3.1], (also see [8, Theorem 3.5]), we get

$$\lim_{\varepsilon \to 0} \int_\Omega \chi_{\Omega_\varepsilon} f(P^\varepsilon y_\varepsilon) \varphi dx = \frac{|Y^*|}{|Y|} \int_\Omega f(y_0) \, dx \quad \text{strongly in } L^2(\Omega). \tag{5.10}$$

Using (5.1), (5.6), (5.9), (5.10) and [3, corollary 5.4], we pass to the limit in (5.8) (as $\varepsilon \to 0$), and obtain the following:

$$\begin{aligned} &\int_\Omega \xi \nabla\varphi \, dx + \frac{|Y^*|}{|Y|}\int_\Omega f(y_0)\varphi dx + h\frac{|\partial T|}{|Y|}\int_\Omega y_0\varphi \, dx \\ &= \frac{|Y^*|}{|Y|}\int_\omega \chi_\omega v_0 \ \varphi + \mathcal{M}_{\partial T}(g)\frac{|\partial T|}{|Y|}\int_\Omega \varphi \, dx. \end{aligned} \tag{5.11}$$

Hence ξ satisfies

$$-\operatorname{div}(\xi) + \frac{|Y^*|}{|Y|}f(y_0) + h\frac{|\partial T|}{|Y|}y_0 = \frac{|Y^*|}{|Y|}\chi_\omega v_0 + \mathcal{M}_{\partial T}(g)\frac{|\partial T|}{|Y|}, \quad \text{in } \Omega. \tag{5.12}$$

It remains to identify the limit ξ.

Step 3. In this step, we identify the limit equation satisfied by ξ. The idea is to make use of solutions of the cell problems (3.4). For $i = 1, \dots, n$, let us define

$$\Phi_{i\varepsilon} = \varepsilon\Big(\chi_i\Big(\frac{x}{\varepsilon}\Big) + y_i\Big), \quad \text{for all } \xi \in \Omega_\varepsilon,$$

where $y = \frac{x}{\varepsilon}$. By Y-periodicity of $\Phi_{i\varepsilon}$ we obtain,

$$P^\varepsilon \Phi_{i\varepsilon} \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega). \tag{5.13}$$

Let us define $\eta_i^\varepsilon := \nabla\Phi_{i\varepsilon}$ in Ω_ε. Then

$$\Big(\widetilde{{}^tA^\varepsilon\eta_i^\varepsilon}\Big)_j = \frac{\partial}{\partial x_j}\Big({}^tA^\varepsilon\Phi_{i\varepsilon}\Big) = \frac{1}{|Y|}\Big(a_{jk}\frac{\partial\chi_i}{\partial y_k} + a_{jk}\delta_{ki}\Big) = \frac{|Y^*|}{|Y|}q_{ij},$$

where

$$q_{ij} = \frac{1}{|Y|}\Big(a_{jk}\frac{\partial\chi_i}{\partial y_k} + a_{ji}\Big).$$

Hence

$$\Big(\widetilde{{}^tA^\varepsilon\eta_i^\varepsilon}\Big)_j \rightharpoonup \frac{|Y^*|}{|Y|}q_{ij} \quad \text{weakly in } L^2(\Omega), \tag{5.14}$$

and we observe that η_i^ε satisfies

$$\begin{gathered} -\operatorname{div}({}^tA^\varepsilon\eta_i^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon, \\ ({}^tA^\varepsilon\eta_i^\varepsilon)\cdot\nu = 0 \quad \text{on } \partial T_\varepsilon. \end{gathered} \tag{5.15}$$

Let $\varphi \in \mathcal{D}(\Omega)$, multiplying (5.15) by φy_ε, and integrating by parts, we obtain

$$\int_{\Omega_\varepsilon} ({}^tA^\varepsilon\eta_i^\varepsilon)\nabla\varphi y_\varepsilon dx + \int_{\Omega_\varepsilon} ({}^tA^\varepsilon\eta_i^\varepsilon)\nabla y_\varepsilon \varphi dx = 0,$$

which implies that

$$\int_{\Omega_\varepsilon} ({}^tA^\varepsilon\eta_i^\varepsilon)\nabla y_\varepsilon\varphi dx = -\int_\Omega (\widetilde{{}^tA^\varepsilon\eta_i^\varepsilon})\nabla\varphi P^\varepsilon y_\varepsilon dx. \tag{5.16}$$

Now, we take $\varphi\Phi_{i\varepsilon}$ as test function in (5.2), we get

$$\begin{aligned} &\int_{\Omega_\varepsilon} A^\varepsilon\nabla y_\varepsilon\nabla(\varphi\Phi_{i\varepsilon})dx + \int_{\Omega_\varepsilon} f(y_\varepsilon)\varphi\Phi_{i\varepsilon}dx + h\varepsilon\int_{\partial T_\varepsilon} y_\varepsilon\varphi\Phi_{i\varepsilon}d\sigma(x) \\ &= \int_{\omega_\varepsilon} v_\varepsilon^*\varphi\Phi_{i\varepsilon}dx + \varepsilon\int_{\partial T_\varepsilon} g^\varepsilon\varphi\Phi_{i\varepsilon}d\sigma(x). \end{aligned}$$

Expressing the integrals over Ω and using the definition of $\widetilde{\xi}^\varepsilon$, we obtain

$$\begin{aligned}&\int_\Omega \widetilde{\xi}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon}dx + \int_{\Omega_\varepsilon} A^\varepsilon \nabla y_\varepsilon \cdot \eta_i^\varepsilon \varphi dx + h\varepsilon \int_{\partial T_\varepsilon} y_\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x)\\ &+ \int_\Omega \chi_{\Omega_\varepsilon} f(y_\varepsilon)\varphi P^\varepsilon \Phi_{i\varepsilon}dx\\ &= \int_\Omega \chi_{\omega_\varepsilon} v_\varepsilon^* \varphi P^\varepsilon \Phi_{i\varepsilon} + \varepsilon \int_{\partial T_\varepsilon} g^\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x),\end{aligned}$$

which can also be written as

$$\begin{aligned}&\int_\Omega \widetilde{\xi}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon}dx + \int_{\Omega_\varepsilon} A^\varepsilon \nabla y_\varepsilon \cdot \eta_i^\varepsilon \varphi dx + h\varepsilon \int_{\partial T_\varepsilon} y_\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x)\\ &+ \int_\Omega \chi_{\Omega_\varepsilon} f(y_\varepsilon)\varphi P^\varepsilon \Phi_{i\varepsilon}dx\\ &= \int_\Omega \chi_{\omega_\varepsilon} v_\varepsilon^* \varphi P^\varepsilon \Phi_{i\varepsilon} + \varepsilon \int_{\partial T_\varepsilon} g^\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x).\end{aligned}$$

Using the relation (5.16), we obtain

$$\begin{aligned}&\int_\Omega \widetilde{\xi}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Phi_{i\varepsilon}dx - \int_\Omega (\widetilde{{}^tA^\varepsilon \eta_i^\varepsilon}) \nabla\varphi P^\varepsilon y_\varepsilon dx + h\varepsilon \int_{\partial T_\varepsilon} y_\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x)\\ &+ \int_\Omega \chi_{\Omega_\varepsilon} f(y_\varepsilon)\varphi P^\varepsilon \Phi_{i\varepsilon}dx\\ &= \int_\Omega \chi_{\omega_\varepsilon} v_\varepsilon^* \varphi P^\varepsilon \Phi_{i\varepsilon} + \varepsilon \int_{\partial T_\varepsilon} g^\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x).\end{aligned} \tag{5.17}$$

By (5.7) and (5.13), we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \widetilde{\xi}^\varepsilon \nabla\varphi P^\varepsilon \Phi_{i\varepsilon}dx = \int_\Omega \xi \nabla\varphi x_i dx \tag{5.18}$$

and using (2.7) and (5.14), we get

$$\lim_{\varepsilon\to 0} \int_\Omega (\widetilde{{}^tA^\varepsilon \eta_i^\varepsilon}) \nabla\varphi P^\varepsilon y_\varepsilon dx = \frac{|Y^*|}{|Y|} \int_\Omega q_i \cdot \nabla\varphi y_0 dx, \tag{5.19}$$

where

$$(q_i)_j = \frac{1}{|Y^*|} \int_Y \Big(a_{ji} + a_{jk}\frac{\partial \chi_i}{\partial y_k}\Big) dy.$$

Analogously as we get (5.6), we obtain the following using (5.13),

$$\lim_{\varepsilon\to 0} h\varepsilon \int_{\partial T_\varepsilon} y_\varepsilon \varphi \Phi_{i\varepsilon} d\sigma(x) = h\frac{|\partial T|}{|Y|} \int_\Omega y_0 \varphi x_i dx. \tag{5.20}$$

The same arguments used for (5.10), and the convergence (5.13), will give us

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_{\Omega_\varepsilon} f(y_\varepsilon)\varphi P^\varepsilon \Phi_{i\varepsilon}dx = \frac{|Y^*|}{|Y|} \int_\Omega f(y_0)\varphi x_i dx. \tag{5.21}$$

Passing to the limit in (5.17) as $\varepsilon \to 0$, by means of (5.1), (5.13), (5.18), (5.19), (5.20), (5.21) and [3, corollary 5.4], we obtain

$$\begin{aligned}&\int_\Omega \xi \cdot \nabla\varphi x_i dx - \frac{|Y^*|}{|Y|} \int_\Omega q_i \nabla\varphi y_0 dx + h\frac{|\partial T|}{|Y|} \int_\Omega y_0 \varphi x_i dx + \frac{|Y^*|}{|Y|} \int_\Omega f(y_0)\varphi x_i dx\\ &= \frac{|Y^*|}{|Y|} \int_\Omega \chi_\omega v_0 \varphi x_i dx + \frac{|\partial T|}{|Y|} \mathcal{M}_{\partial T}(g) \int_\Omega \varphi x_i dx.\end{aligned}$$

Integrating by parts, using Green's formula and (5.12) we obtain,

$$-\int_\Omega \xi\cdot\nabla x_i\varphi dx+\frac{|Y^*|}{|Y|}\int_\Omega q_i\cdot\nabla y_0\varphi dx=0\quad\text{in }\Omega.$$

Since this is true for any $\varphi\in\mathcal{D}(\Omega)$, we have

$$-\xi\cdot\nabla x_i+\frac{|Y^*|}{|Y|}q_i\cdot\nabla y_0=0\quad\text{in }\Omega.\tag{5.22}$$

Let us write (5.22) component-wise, differentiating with respect to x_i, summing over i, then using (5.12) we conclude that

$$\frac{|Y^*|}{|Y|}\sum_{i,j=1}^n q_{ij}\frac{\partial^2 y_0}{\partial x_i x_j}=\operatorname{div}(\xi)=\frac{|Y^*|}{|Y|}f(y_0)-\frac{|Y^*|}{|Y|}\chi_\omega v_0-\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)+h\frac{|\partial T|}{|Y|}y_0.$$

This implies that y_0 satisfies the equation

$$-\theta\sum_{i,j=1}^n q_{ij}\frac{\partial^2 y_0}{\partial x_i x_j}+\theta f(y_0)=\theta\chi_\omega v_0+\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)-h\frac{|\partial T|}{|Y|}y_0,$$

which can also be written as

$$-\theta\operatorname{div}(A^0\nabla y_0(v_0))+\theta f(y_0(v_0))=\theta\chi_\omega v_0+\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)-h\frac{|\partial T|}{|Y|}y_0(v_0)\text{ in }\Omega,$$

where $A^0=(a^0_{ij})=(q_{ij})$, is given by (3.3).

It follows by (H2) and the convergence

$$\widetilde{y_\varepsilon(v_\varepsilon^*)}|_{S_\varepsilon}\rightharpoonup\theta y_0(v_0)|_S,$$

that v_0 satisfies the approximate controllability inequality

$$\|y_0(v_0)|_S-y_1\|_{0,S}\le\frac{\alpha}{\sqrt{\theta}}.\tag{5.23}$$

Step 4. Existence of optimal control. In this step, we identify the limit v_0 of the optimal controls $v^*_\varepsilon(\overline{z}_\varepsilon)$ appeared in (5.1). A natural question arises: whether v_0 is an optimal solution? Here we answer affirmatively to this question.

We start by writing the fixed point identity

$$\overline{z}_\varepsilon=y_\varepsilon(\overline{z}_\varepsilon,v^*_\varepsilon(\overline{z}_\varepsilon))=y^*_\varepsilon.$$

By (2.7), there exists z_0, such that (up to a subsequence) we obtain

$$\begin{gathered}P^\varepsilon\overline{z}_\varepsilon\rightharpoonup z_0\quad H^1_0(\Omega)\text{-weakly},\\ P^\varepsilon\overline{z}_\varepsilon\to z_0\quad L^2(\Omega)\text{-strongly},\end{gathered}\tag{5.24}$$

as $\varepsilon\to0$. For a fixed control v, let $y_0(z_0,v)$ be the solution of the homogenized linearized problem:

$$\begin{gathered}-\theta\operatorname{div}(A_0\nabla y_0(z_0,v))+\theta p(z_0)y_0(z_0,v)\\ =\theta\chi_\omega v+\frac{|\partial T|}{|Y|}\mathcal{M}_{\partial T}(g)-h\frac{|\partial T|}{|Y|}y_0(v)\quad\text{in }\Omega\\ y_0(z_0,v)=0\quad\text{on }\partial\Omega.\end{gathered}\tag{5.25}$$

With this state equation we associate the cost functional

$$I^0_{z_0}(v) = \frac{1}{2}\|v\|^2_{0,\omega} + \begin{cases} 0 & \text{if } \|y_0(z_0,v)|_S - y_1\|_{0,S} \le \frac{\alpha}{\sqrt{\theta}}, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.26}$$

By classical linear control theory, there exists a unique optimal control $v^*_0(z_0)$ such that

$$I^0_{z_0}(v^*_0(z_0)) = \min_{v\in L^2(\omega)} I^0_{z_0}(v) < \infty.$$

Let $y^*_0 = y_0(z_0, v^*_0(z_0))$ be the corresponding state. Again Fenchel-Rockafellar duality gives the minimum $v^*_0(z_0)$ as the solution of the adjoint problem, the characterization of which is given as follows:

Given $\varphi_1 \in L^2(S)$ (which is the limit of $\varphi_{1\varepsilon}$), we introduce $\varphi_0(z_0,\varphi_1)$ as the solution of the dual problem associated to (5.25), that is

$$\begin{gathered} -\theta \operatorname{div}({}^tA_0\nabla\varphi_0(z_0,\varphi_1)) + \theta p(z_0)\varphi_0(z_0,\varphi_1) \\ = \theta\delta_S\varphi_1 - h\frac{|\partial T|}{|Y|}\varphi_0(z_0,\varphi_1) \quad \text{in } \Omega, \\ \varphi_0(z_0,\varphi_1) = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{5.27}$$

Let us now define the dual functional of $I^0_{z_0}(v)$ (defined by (5.26)), as follows:

$$J^0_{z_0}(\varphi_1) = \frac{\theta}{2}\int_\omega |\varphi_0|^2 dx + \alpha\sqrt{\theta}\|\varphi_1\|_{0,S} - \theta\int_S y_1\varphi_1 ds. \tag{5.28}$$

Since $J^0_{z_0}$ is convex, coercive and lower semicontinuous, by the direct method of calculus of variations, there exists a unique optimal element φ^*_1 minimizing the cost functional $J^0_{z_0}$ in $L^2(S)$. Let $\varphi^*_0 = \varphi_0(z_0,\varphi^*_1)$ be the solution of (5.27) associated with φ^*_1. Then Fenchel's duality theory gives us

$$v^*_0(z_0) = \theta(\varphi^*_0)|_\omega. \tag{5.29}$$

Step 5. Passage to the limit in the adjoint equation. Let $\varphi \in V_\varepsilon$ as a test function in the adjoint equation (2.8) (written for $z = \overline{z}_\varepsilon$) and integrate by parts, we obtain

$$\int_{\Omega_\varepsilon} ({}^tA^\varepsilon\nabla\varphi_\varepsilon)\nabla\varphi dx + \int_{\Omega_\varepsilon} p(\overline{z}_\varepsilon)\varphi_\varepsilon\varphi dx + h\varepsilon\int_{\partial T_\varepsilon}\varphi_\varepsilon\varphi d\sigma(x) = \int_{S_\varepsilon}\varphi_{1\varepsilon}\varphi. \tag{5.30}$$

Using ellipticity of A_ε, property of linearized function p and the fact that h is a real positive constant, evaluating the norm estimate on both the sides of (5.30),

$$\alpha_m\|\varphi_\varepsilon\|_{1,\Omega_\varepsilon} \le \|\varphi_{1\varepsilon}\|_{0,S_\varepsilon},$$

hence we obtain

$$\|P^\varepsilon\varphi_\varepsilon\|_{1,\Omega} \le C.$$

This will imply that there exists $\bar{\varphi}_0$ such that up to a subsequence,

$$P^\varepsilon\varphi_\varepsilon \rightharpoonup \bar{\varphi}_0, \quad H^1_0(\Omega) \text{ weakly.} \tag{5.31}$$

Let us take $\zeta^\varepsilon = ({}^tA^\varepsilon\nabla\varphi_\varepsilon)$ in Ω_ε and $\widetilde{\zeta}^\varepsilon$ be its extension by zero on all of Ω. Then $\widetilde{\zeta}^\varepsilon$ is bounded in $L^2(\Omega)^N$. This implies that there exists ζ such that

$$\widetilde{\zeta}^\varepsilon \rightharpoonup \zeta \quad \text{weakly in } L^2(\Omega). \tag{5.32}$$

To see the equation satisfied by ζ, let us take $\varphi \in \mathcal{D}(\Omega)$ as a test function in the variational formulation (5.30), we obtain

$$\int_{\Omega_\varepsilon} \zeta^\varepsilon \nabla\varphi dx + \int_{\Omega_\varepsilon} p(\overline{z}_\varepsilon)\varphi_\varepsilon\varphi dx + h\varepsilon\int_{\partial T\varepsilon} \varphi_\varepsilon\varphi d\sigma(x) = \int_{S_\varepsilon} \varphi_{1\varepsilon}\varphi. \tag{5.33}$$

Expressing the integrals over Ω,

$$\int_\Omega \widetilde{\zeta}^\varepsilon \nabla\varphi dx + \int_\Omega p(P^\varepsilon\overline{z}_\varepsilon)P^\varepsilon\varphi_\varepsilon\varphi dx + h\varepsilon\int_{\partial T\varepsilon} P^\varepsilon\varphi_\varepsilon\varphi d\sigma(x) = \int_\Omega \delta_{S_\varepsilon}\widetilde{\varphi}_{1\varepsilon}\varphi \tag{5.34}$$

Using (5.24), (5.31), (5.32) and (H3), we pass to the limit in (5.33) (as $\varepsilon \to 0$), to obtain

$$\int_\Omega \zeta\nabla\varphi dx + \frac{|Y^*|}{|Y|}\int_\Omega p(z_0)\bar{\varphi}_0\varphi dx + h\frac{|\partial T|}{|Y|}\int_\Omega \bar{\varphi}_0\varphi dx = \frac{|Y^*|}{|Y|}\int_\Omega \delta_S\varphi_1\varphi dx. \tag{5.35}$$

Hence ζ satisfies

$$-\operatorname{div}(\zeta) + \frac{|Y^*|}{|Y|}p(z_0)\bar{\varphi}_0 + h\frac{|\partial T|}{|Y|}\bar{\varphi}_0 = \frac{|Y^*|}{|Y|}\delta_S\varphi_1 \quad \text{in } \Omega. \tag{5.36}$$

Now, to identify the limit equation satisfied by ζ, we shall use the cell problems (3.7). Let us define for $i = 1, 2, \dots$ the functions

$$\Psi_{i\varepsilon} := \varepsilon\Big(\widehat{\chi}_i(\frac{x}{\varepsilon}) + y_i\Big), \quad \text{for all } x \in \Omega_\varepsilon, \tag{5.37}$$

where $y = \frac{x}{\varepsilon}$ and by Y-periodicity of $\Psi_{i\varepsilon}$ we obtain

$$P^\varepsilon\Psi_{i\varepsilon} \rightharpoonup x_i \quad \text{weakly in } H^1(\Omega). \tag{5.38}$$

Let us define $\mu_i^\varepsilon := \nabla\Psi_{i\varepsilon}$ in Ω_ε. Then

$$(\widetilde{A^\varepsilon\mu_i^\varepsilon})_j = \frac{\partial}{\partial x_j}(A^\varepsilon\Psi_{i\varepsilon}) = \frac{1}{|Y|}\Big(a_{ik}\frac{\partial\chi_j}{\partial y_k} + a_{ik}\delta_{kj}\Big) = \frac{|Y^*|}{|Y|}({}^tq_{ij}),$$

where

$$({}^tq_{ij}) = \frac{1}{|Y|}\Big(a_{ik}\frac{\partial\chi_j}{\partial y_k} + a_{ij}\Big).$$

Hence

$$\Big(\widetilde{A^\varepsilon\mu_i^\varepsilon}\Big)_j \rightharpoonup \frac{|Y^*|}{|Y|}({}^tq_{ij}) \quad \text{weakly in } L^2(\Omega), \tag{5.39}$$

and in view of (3.7), we observe that μ_i^ε satisfies

$$\begin{gathered} -\operatorname{div}(A^\varepsilon\mu_i^\varepsilon) = 0 \quad \text{in } \Omega_\varepsilon, \\ (A^\varepsilon\mu_i^\varepsilon)\cdot\nu = 0 \quad \text{on } \partial T_\varepsilon. \end{gathered} \tag{5.40}$$

Let $\varphi \in \mathcal{D}(\Omega)$, multiplying (5.40) by $\varphi\varphi_\varepsilon$ and integrate by parts,

$$\int_{\Omega_\varepsilon} (A^\varepsilon\mu_i^\varepsilon)\nabla\varphi\varphi_\varepsilon dx + \int_{\Omega_\varepsilon} (A^\varepsilon\mu_i^\varepsilon)\nabla\varphi_\varepsilon\varphi dx = 0,$$

which in turn implies that

$$\int_{\Omega_\varepsilon} (A^\varepsilon\mu_i^\varepsilon)\nabla\varphi_\varepsilon\varphi dx = -\int_\Omega (\widetilde{A^\varepsilon\mu_i^\varepsilon})\nabla\varphi P^\varepsilon\varphi_\varepsilon dx. \tag{5.41}$$

Now, taking $\varphi\Psi_{i\varepsilon}$ as test function in (5.30),

$$\int_{\Omega_\varepsilon} ({}^tA^\varepsilon\nabla\varphi_\varepsilon)\nabla(\varphi\Psi_{i\varepsilon})dx + h\varepsilon\int_{\partial T_\varepsilon} \varphi_\varepsilon\varphi\Psi_{i\varepsilon}d\sigma(x) + \int_{\Omega_\varepsilon} p(\overline{z}_\varepsilon)\varphi_\varepsilon\varphi\Psi_{i\varepsilon}dx$$

$$= \int_\Omega \delta_\varepsilon \varphi_{1\varepsilon} \varphi \Psi_{i\varepsilon} dx.$$

Expressing the integrals over Ω and using the definition of $\widetilde{\zeta}^\varepsilon$, we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{\zeta}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Psi_{i\varepsilon} dx + \int_{\Omega_\varepsilon} {}^t A^\varepsilon \nabla\varphi_\varepsilon \cdot \mu_i^\varepsilon \varphi dx + h\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \varphi \Psi_{i\varepsilon} d\sigma(x) \\
&+ \int_\Omega \chi_{\Omega_\varepsilon} p(P^\varepsilon \overline{z}_\varepsilon) P^\varepsilon \varphi_\varepsilon \varphi P^\varepsilon \Psi_{i\varepsilon} dx \\
&= \int_\Omega \delta_{S_\varepsilon} \widetilde{\varphi}_{1\varepsilon} \varphi P^\varepsilon \Psi_{i\varepsilon},
\end{aligned}$$

which can also be written as

$$\begin{aligned}
&\int_\Omega \widetilde{\zeta}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Psi_{i\varepsilon} dx + \int_{\Omega_\varepsilon} ({}^t A^\varepsilon \nabla\varphi_\varepsilon) \cdot \mu_i^\varepsilon \varphi dx + h\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \varphi \Psi_{i\varepsilon} d\sigma(x) \\
&+ \int_\Omega \chi_{\Omega_\varepsilon} p(P^\varepsilon \overline{z}_\varepsilon) P^\varepsilon \varphi_\varepsilon \varphi P^\varepsilon \Psi_{i\varepsilon} dx \\
&= \int_\Omega \delta_{S_\varepsilon} \widetilde{\varphi}_{1\varepsilon} \varphi P^\varepsilon \Psi_{i\varepsilon}.
\end{aligned}$$

Using the relation (5.41), we obtain

$$\begin{aligned}
&\int_\Omega \widetilde{\zeta}^\varepsilon \cdot \nabla\varphi P^\varepsilon \Psi_{i\varepsilon} dx - \int_\Omega \widetilde{(A^\varepsilon \mu_i^\varepsilon)} \nabla\varphi P^\varepsilon \varphi_\varepsilon dx + h\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \varphi \Psi_{i\varepsilon} d\sigma(x) \\
&+ \int_\Omega \chi_{\Omega_\varepsilon} p(P^\varepsilon \overline{z}_\varepsilon) P^\varepsilon \varphi_\varepsilon \varphi P^\varepsilon \Psi_{i\varepsilon} dx \qquad (5.42)\\
&= \int_{\Omega_\varepsilon} \delta_{S_\varepsilon} \widetilde{\varphi}_{1\varepsilon} \varphi P^\varepsilon \Psi_{i\varepsilon}.
\end{aligned}$$

In a similar way as we get (5.6); using (5.31) and (5.38) we obtain

$$\lim_{\varepsilon\to 0} h\varepsilon \int_{\partial T_\varepsilon} \varphi_\varepsilon \varphi \Psi_{i\varepsilon} d\sigma(x) = h\frac{|\partial T|}{|Y|} \int_\Omega \bar{\varphi}_0 \varphi x_i dx. \qquad (5.43)$$

Using the property of p (see (2.4)), convergences (5.24), (5.31), (5.38) and adapting the same lines of calculations to get [8, eq. (84)], we obtain

$$\lim_{\varepsilon\to 0} \int_\Omega \chi_{\Omega_\varepsilon} p(P^\varepsilon \overline{z}_\varepsilon) P^\varepsilon \varphi_\varepsilon \varphi P^\varepsilon \Psi_{i\varepsilon} dx = \frac{|Y^*|}{|Y|} \int_\Omega p(z_0) \bar{\varphi}_0 \varphi x_i dx. \qquad (5.44)$$

Passing to the limit in (5.42) as $\varepsilon \to 0$, using (5.31), (5.39), (5.43), (5.44) we have

$$\begin{aligned}
&\int_\Omega \zeta \cdot \nabla\varphi x_i dx - \frac{|Y^*|}{|Y|} \int_\Omega ({}^t q_i) \nabla\varphi \bar{\varphi}_0 dx + h\frac{|\partial T|}{|Y|} \int_\Omega \bar{\varphi}_0 \varphi x_i dx \\
&+ \frac{|Y^*|}{|Y|} \int_\Omega p(z_0) \bar{\varphi}_0 \varphi x_i dx \\
&= \frac{|Y^*|}{|Y|} \int_\Omega \delta_S \varphi_1 \varphi x_i dx.
\end{aligned}$$

Integrating by parts, using Green's formula and (5.36) we have

$$-\int_\Omega \zeta \cdot \nabla x_i \varphi dx + \frac{|Y^*|}{|Y|} \int_\Omega ({}^t q_i) \cdot \nabla\bar{\varphi}_0 \varphi dx = 0, \quad \text{in } \Omega.$$

Since this is true for any $\varphi \in \mathcal{D}(\Omega)$, we have

$$-\zeta \cdot \nabla x_i + \frac{|Y^*|}{|Y|}({}^t q_i) \cdot \nabla \bar{\varphi}_0 = 0, \quad \text{in } \Omega. \tag{5.45}$$

Let us write (5.45) component-wise, differentiating with respect to x_i, summing over i, using (5.36), we conclude that

$$\frac{|Y^*|}{|Y|} \sum_{i,j=1}^{n} ({}^t q_{ij}) \frac{\partial^2 \bar{\varphi}_0}{\partial \zeta_i x_j} = \operatorname{div}(\zeta) = \frac{|Y^*|}{|Y|} p(z_0)\bar{\varphi}_0 - \frac{|Y^*|}{|Y|}\delta_S \varphi_1 + h\frac{|\partial T|}{|Y|}\bar{\varphi}_0.$$

This implies that $\bar{\varphi}_0$ satisfies

$$-\theta \sum_{i,j=1}^{n} ({}^t q_{ij}) \frac{\partial^2 \bar{\varphi}_0}{\partial x_i x_j} + \theta p(z_0)\bar{\varphi}_0 = \theta \delta_S \varphi_1 - h\frac{|\partial T|}{|Y|}\bar{\varphi}_0,$$

which can also be written as

$$-\theta \operatorname{div}({}^t A^0 \nabla \bar{\varphi}_0) + \theta p(z_0)\bar{\varphi}_0 = \theta \delta_S \varphi_1 - h\frac{|\partial T|}{|Y|}\bar{\varphi}_0 \text{ in } \Omega. \tag{5.46}$$

Comparing (5.27) and (5.46), we get that

$$\bar{\varphi}_0 = \varphi_0(z_0, \varphi_1). \tag{5.47}$$

Now, we pass to the limit in the cost functional $J^{\varepsilon}_{\tilde{z}_\varepsilon}$, using (H2), (H3), (5.31) and (5.47), we have

$$\lim_{\varepsilon \to 0} J^{\varepsilon}_{\tilde{z}_\varepsilon}(\varphi_{1\varepsilon}) = J^0_{z_0}(\varphi_1), \tag{5.48}$$

where

$$J^0_{z_0}(\varphi_1) = \frac{\theta}{2}\int_\omega |\varphi_0|^2 dx + \alpha\sqrt{\theta}\|\varphi_1\|_{0,S} - \theta \int_S y_1 \varphi_1 ds.$$

Step 6. Convergence of the optimal controls of the state equation. By using the similar techniques as in [28, Lemma 2] one can prove that the minimizers $\{\varphi^*_{1\varepsilon}\}$ of the functional $J^{\varepsilon}_{\tilde{z}_\varepsilon}(\varphi_{1\varepsilon})$ (defined by (2.17)), are uniformly bounded, that is

$$\|\varphi^*_{1\varepsilon}\|_{0,S_\varepsilon} \le C.$$

This implies that up to a subsequence (also see [11, Theorem 6]), there exists an element $\xi^* \in L^2(S)$ such that

$$\widetilde{\varphi^*_{1\varepsilon}} \rightharpoonup \theta \xi^* \quad \text{weakly in } L^2(S). \tag{5.49}$$

Thus up to another subsequence, we have

$$\begin{gathered}\widetilde{\varphi_\varepsilon}(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) \rightharpoonup \theta \varphi_0(z_0, \xi^*) \quad \text{weakly in } H^1_0(\Omega),\\ \widetilde{\varphi_\varepsilon}(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) \to \theta \varphi_0(z_0, \xi^*) \quad \text{strongly in } L^2(\Omega).\end{gathered} \tag{5.50}$$

Next our aim is to show that

$$\xi^* = \varphi^*_1, \tag{5.51}$$

where φ^*_1 is the minimizer of $J^0_{z_0}(\varphi_1)$ defined by (5.28). To show (5.51), it suffices to show that

$$J^0_{z_0}(\xi^*) \le J^0_{z_0}(\varphi_1), \quad \forall \varphi_1 \in L^2(S). \tag{5.52}$$

Thanks to (5.48), we deduce that

$$\liminf_{\varepsilon \to 0} J^{\varepsilon}_{\tilde{z}_\varepsilon}(\varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) \le \lim_{\varepsilon \to 0} J^{\varepsilon}_{\tilde{z}_\varepsilon}(\varphi_{1\varepsilon}) = J^0_{z_0}(\varphi_1).$$

Therefore it suffices to show that

$$J^0_{z_0}(\xi^*) \le \liminf_{\varepsilon\to 0} J^\varepsilon_{\overline{z}_\varepsilon}(\varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)). \tag{5.53}$$

Recall the definition of $J^\varepsilon_{\overline{z}_\varepsilon}$,

$$J^\varepsilon_{\overline{z}_\varepsilon}(\varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) = \frac{1}{2}\|\varphi_\varepsilon(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon))\|^2_{0,\omega_\varepsilon} + \alpha\|\varphi^*_{1\varepsilon}\|_{0,S_\varepsilon} - \langle \varphi^*_{1\varepsilon}, y_{1\varepsilon}\rangle_{L^2(S_\varepsilon)};$$

then we obtain

$$\liminf_{\varepsilon\to 0}\Big(\alpha\|\varphi^*_{1\varepsilon}\|_{0,S_\varepsilon} - \langle \varphi^*_{1\varepsilon}, y_{1\varepsilon}\rangle_{L^2(S_\varepsilon)}\Big) \ge \alpha\sqrt{\theta}\|\xi^*\|_{0,S} - \theta\langle \xi^*, y_1\rangle_{L^2(S)}.$$

Thus we have

$$\begin{aligned} &\liminf_{\varepsilon\to 0} J^\varepsilon_{\overline{z}_\varepsilon}(\varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) \\ &\ge \lim_{\varepsilon\to 0}\Big(\frac{1}{2}\|\varphi_\varepsilon(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon))\|^2_{0,\omega_\varepsilon}\Big) + \alpha\sqrt{\theta}\|\xi^*\|_{0,S} - \theta\langle \xi^*, y_1\rangle_{L^2(S)}. \end{aligned} \tag{5.54}$$

By means of (5.50), the right hand side of (5.54) is $J^0_{z_0}(\xi^*)$, hence (5.53) is proved which in turn implies that (5.51) is proved.

Remark 5.1. Since the minimizer of the functional $J^0_{z_0}(\varphi_1)$ is unique, the convergence (5.49) holds for the whole sequence.

We have the following convergence

$$\begin{gathered} \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon) \rightharpoonup \theta\varphi^*_1(z_0) \quad \text{weakly in } L^2(S), \\ \widetilde{\varphi_\varepsilon}(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon)) \rightharpoonup \theta\varphi_0(z_0, \varphi^*_1(z_0)) \quad \text{weakly in } H^1_0(\Omega). \end{gathered}$$

Hence,

$$\lim_{\varepsilon\to 0} J^\varepsilon_{\overline{z}_\varepsilon}(\varphi^*_{1\varepsilon}) = \frac{\theta}{2}\int_\omega |\varphi_0|^2 dx + \alpha\sqrt{\theta}\|\varphi^*_1\|_{0,S} - \theta\int_S y_1\varphi^*_1 ds.$$

Finally we write (2.19) for $z = \overline{z}_\varepsilon$,

$$v^*_\varepsilon(\overline{z}_\varepsilon) = \widetilde{\varphi}_\varepsilon(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon))|_{\omega_\varepsilon} = \varphi_\varepsilon(\overline{z}_\varepsilon, \varphi^*_{1\varepsilon}(\overline{z}_\varepsilon))|_{\omega_\varepsilon}.$$

Now we have

$$v_0 = \theta\big(\varphi_0(z_0, \varphi^*_1(z_0))\big)|_\omega = v^*_0(z_0). \tag{5.55}$$

This completes the proof of the theorem. □

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