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SEMICLASSICAL SOLUTIONS OF PERTURBED BIHARMONIC EQUATIONS WITH CRITICAL NONLINEARITY

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Abstract. We consider the perturbed biharmonic equations

$$\varepsilon^4\Delta^2 u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N$$

and

$$\varepsilon^4\Delta^2 u + V(x)u = Q(x)|u|^{2^{**}-2}u + f(x,u), \quad x \in \mathbb{R}^N$$

where Δ^2 is the biharmonic operator, $N \geq 5$, $2^{**} = \frac{2N}{N-4}$ is the Sobolev critical exponent, $Q(x)$ is a bounded positive function. Under some mild conditions on V and f, we show that the above equations have at least one nontrivial solution provided that $\varepsilon \leq \varepsilon_0$, where the bound ε_0 is formulated in terms of N, V, Q and f.

1. Introduction

We study the perturbed biharmonic equations with subcritical nonlinearity

$$\begin{gathered}\varepsilon^4\Delta^2 u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N,\\ u \in H^2(\mathbb{R}^N), \quad u(x) \to 0, \quad \text{as } |x| \to \infty,\end{gathered} \tag{1.1}$$

and with critical nonlinearity

$$\begin{gathered}\varepsilon^4\Delta^2 u + V(x)u = Q(x)|u|^{2^{**}-2}u + f(x,u), \quad x \in \mathbb{R}^N,\\ u \in H^2(\mathbb{R}^N), \quad u(x) \to 0, \quad \text{as } |x| \to \infty,\end{gathered} \tag{1.2}$$

where $\varepsilon > 0$ is small, Δ^2 is the biharmonic operator, $N \geq 5$, $2^{**} = 2N/(N-4)$ denotes the Sobolev critical exponent; $V, Q : \mathbb{R}^N \to \mathbb{R} \in C(\mathbb{R}^N, \mathbb{R})$. In this paper, we are interested in the existence of semiclassical solutions for the above equations.

When Ω is a bounded domain of $\mathbb{R}^N$, the problem

$$\begin{gathered}\Delta^2 u + c\Delta u = f(x,u) \quad \text{in } \Omega,\\ u = \Delta u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

has been extensively investigated in recent years. This problem arises in the study of traveling waves in suspension bridges (see [5, 12, 16]) and the study of the static deflection of an elastic plate in a fluid. For results on multiple nontrivial and sign

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changing solutions of problem (1.2) we refer the readers to [1, 2, 3, 11, 19, 20, 23, 24, 30, 34, 35, 36, 41, 42] and the references therein.

Problems in the whole space $\mathbb{R}^N$ have been considered in several works; see for example [4, 7, 15, 17, 18, 21, 22, 27, 31, 32, 33, 37, 38, 39, 40]. To our knowledge, there are only two papers [18, 21] on the singularly perturbation problem. In [18], the authors dealt with the autonomous problem

$$\begin{gathered}\varepsilon^4\Delta^2 u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N,\\ u \in H^2(\mathbb{R}^N),\end{gathered}$$

where $\varepsilon > 0$, $N \geq 5$, and $V : \mathbb{R}^N \to \mathbb{R}$ is such that there exists a bounded domain $\Omega \subset \mathbb{R}^N$ and $x_0 \in \Omega$ with $0 < V(x_0) = \inf_{\mathbb{R}^N} V < \inf_{\partial\Omega} V$. A family of solutions was proved to exist and to be concentrated at a point in the limit. Motivated by Ding and Lin [8], Wang [21] studied the existence of semiclassical solutions of non-autonomous problem (1.2) under the following assumptions:

- (A1) $V \in C(\mathbb{R}^N)$, $0 = \min V \leq V(x)$ and there exists $b > 0$ such that $\mathcal{V}_b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite Lebesgue measure;
- (A2) $Q \in C(\mathbb{R}^N)$ and $0 < Q_1 := \inf Q \leq \sup Q := Q_2 < \infty$;
- (A3) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constants $p_0 \in (2, 2^{**}) > 0$ and $c > 0$ such that
$$|f(x,t)| \leq c(1 + |t|^{p_0 - 1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$
- (A4) $f(x,t) = o(|t|)$, as $|t| \to 0$ uniformly in x.
- (A5) There exist $c_0 > 0$ and $p > 2$ such that $F(x,t) \geq c_0|t|^p$ for all (x,t).
- (A6) There exists $2 < \mu < 2^{**}$ such that
$$\mu F(x,t) \leq f(x,t)t \quad \text{for all } (x,t), \text{ where } F(x,t) = \int_0^t f(x,s)ds$$

It is worth pointing out that a crucial technique from [21] is used in the process of proof: for any $(PS)_c$ sequence $\{u_n\}$ for I_λ with $u_n \rightharpoonup u$, where $\lambda = \varepsilon^{-2}$ and

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\Delta u|^2 + \lambda|u|^2)dx - \frac{\lambda}{2^{**}}\int_{\mathbb{R}^N} Q(x)|u|^{2^{**}}dx - \lambda\int_{\mathbb{R}^N} F(x,u)dx,$$

the author constructed a new sequence $\{v_n\}$ such that I_λ, I'_λ satisfy BL-splits, i.e.,

$$I_\lambda(v_n) \to c - I_\lambda(u), \quad I'_\lambda(v_n) \to 0.$$

With the aid of this property, the author showed that I_λ satisfies the (PS)-condition at the levels less than $\alpha_0\lambda^{1-\frac{N}{4}}$ with some $\alpha_0 > 0$ independent of λ. Based on such arguments, there are many works devote to semilinear Schröinger equations, to quasilinear Schröinger equations and elliptic system, we refer readers to [6, 9, 10, 26, 28, 29, 43] and the references therein.

Inspired by [21, 13], we consider problems (1.1) and (1.2). The main ingredients of our work are two aspects. On the one hand, our aim is to weaken the above conditions to generalize and improve the result in [21]; on the other hand, we will develop a more direct and simpler approach. The novel approach not only makes such an extension possible but also lead to some better results. For example, it enable us to give an explicit upper bound for the parameter ε.

To state our results, we make the following assumptions which are considerably weaker than the ones in the previous work:

(A7) $F(x,t) \ge 0$ and $\lim_{t\to\infty} |F(x,t)|/|t|^2 = \infty$ uniformly in x, and there exist $a_0 > 0, T_0 > 0$ and $q \in (2, 2^{**})$ such that
$$F(x,t) \ge a_0|t|^q, \quad \forall (x,t) \in \mathbb{R}^N \times [-T_0, T_0],$$
$$t^{-2}h^{6-N}\int_{|x|\le h} F(\lambda^{-1/4}x, t/h)dx \ge \frac{(4N^2+2)S_N}{N(1-2^{-N})^2}, \quad \forall h \ge 1,\ \lambda \ge 1,\ t \ge hT_0,$$
where and in the sequel, $S_N = \text{meas}(B_1(0)) = \frac{2\pi N/2}{N\Gamma(N/2)}$;

(A8) $\mathcal{F}(x,t) := \frac{1}{2}tf(x,t) - F(x,t) \ge 0$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, and there exist $R_0 > 0, a_1 > 0$ and $\kappa > \max\{1, \frac{N}{4}\}$ such that
$$tf(x,t) \le \frac{b}{3}|t|^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \le R_0,$$
$$|f(x,t)|^\kappa \le a_1|t|^\kappa \mathcal{F}(x,t), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \ge R_0;$$

(A9) $tf(x,t) \ge 2F(x,t)$ for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ and there exist $a_* > 0$, $T_1 > 0$ and $q \in (2, 2^{**})$ such that
$$\frac{1}{2^{**}}Q(x)|t|^{2^{**}} + F(x,t) \ge a_*|t|^q, \quad \text{for } (x,t) \in \mathbb{R}^N \times [-T_1, T_1].$$

In light of (A3)–(A4), there exist $R_* > 0$ and $a_2 > 0$ such that
$$Q(x)|t|^{2^{**}} + tf(x,t) \le \frac{b}{3}|t|^2, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \le R_*, \tag{1.3}$$
$$tf(x,t) \le a_2Q(x)|t|^{2^{**}}, \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\ |t| \ge R_*. \tag{1.4}$$

Remark 1.1. It is easy to check that the conditions (A7), (A8) and (A9) are weaker than (A5) and (A6). It is well known that many nonlinearities such as
$$f(x,t) = t\ln(1+|t|), \tag{1.5}$$
do not satisfy (A6). A crucial role that (A6) plays is to ensure the boundedness of Palais-Smale sequences.

Now we only show that $f(x,t)$ satisfies (A7) and (A8). Indeed, by a straightforward computation,
$$F(x,t) = \frac{t^2-1}{2}\ln(1+|t|) + \frac{1}{4}|t|(2-|t|),$$
$$\mathcal{F}(x,t) = \frac{1}{2}tf(x,t) - F(x,t) = \frac{1}{2}\ln(1+|t|) + \frac{1}{4}|t|(|t|-2).$$
Observe that, letting $h \ge 1$, $t \ge hT_0$ for some $T_0 \ge 2$, we have
$$\begin{aligned}
&t^{-2}h^{6-N}\int_{|x|\le h} F(\lambda^{-1/4}x, t/h)dx \\
&= t^{-2}h^{6-N}\int_{|x|\le h} \big[\frac{(\frac{t}{h})^2-1}{2}\ln\big(1+\frac{t}{h}\big) + \frac{\frac{t}{h}(2-\frac{t}{h})}{4}\big]dx \\
&= \frac{1}{N}S_Nh^Nt^{-2}h^{6-N}\big[\frac{t^2-h^2}{2h^2}\ln\big(1+\frac{t}{h}\big) + \frac{t(2h-t)}{4h^2}\big] \\
&= \frac{1}{2N}S_Nh^4\big[\big(1-(\frac{h}{t})^2\big)\ln\big(1+\frac{t}{h}\big) + \frac{1}{2}\big(\frac{2h}{t}-1\big)\big] \\
&\ge \frac{1}{2N}S_N\big[(1-T_0^{-2})\ln(1+T_0) + \frac{1}{T_0} - \frac{1}{2}\big]
\end{aligned}$$

$$\geq \frac{1}{2N} S_N \big[\frac{3}{4}\ln(1+T_0) - \frac{1}{2}\big].$$

This implies that

$$t^{-2}h^{6-N}\int_{|x|\leq h} F(\lambda^{-1/4}x, t/h)dx \geq \frac{(4N^2+2)S_N}{N(1-2^{-N})^2}, \quad \forall h \geq 1, t \geq hT_0$$

for suitable large T_0. When $t \in [-T_0, T_0]$, it is easy to see that there exist $\theta > 0$ such that

$$\theta|t| \leq \ln(1+|t|) \leq e^{-1}|t|,$$

then

$$\begin{aligned} F(x,t) &= \frac{t^2-1}{2}\ln(1+|t|) + \frac{1}{4}|t|(2-|t|) \\ &\geq \frac{\theta}{2}|t|^3 - \frac{1}{2}|t|^2 + \big(\frac{1}{2} - \frac{e^{-1}}{2}\big)|t|. \end{aligned}$$

Thus, there exist $a_0 > 0$ and $q \in (2, 2^{**})$ such that

$$F(x,t) \geq a_0|t|^q, \quad t \in [-T_0, T_0].$$

From the above fact, we deduce that (1.5) satisfies (A7). On the other hand, we note that

$$\begin{aligned} \mathcal{F}(x,t) = \frac{1}{2}tf(x,t) - F(x,t) &= \frac{1}{2}\ln(1+|t|) + \frac{1}{4}|t|(|t|-2) \\ &\geq \frac{1}{2}|t| - \frac{1}{4}|t|^2 + \frac{1}{4}|t|^2 - \frac{1}{2}|t| = 0. \end{aligned}$$

By a straightforward computation, there exist $R_0 > 0$, $a_1 > 0$ and $\kappa > \max\{1, \frac{N}{4}\}$ such that

$$tf(x,t) = t^2\ln(1+|t|) \leq \frac{b}{3}|t|^2, \quad |t| \leq R_0,$$

and

$$|\frac{f(x,t)}{t}|^\kappa = \big(\ln(1+|t|)\big)^\kappa \leq a_1\big(\frac{1}{2}\ln(1+|t|) + \frac{1}{4}|t|(|t|-2)\big) = a_1\mathcal{F}(x,t), \quad |t| \geq R_0.$$

This shows that (1.5) satisfies (A8).

The main results of this article are the following theorems.

Theorem 1.2. *Assume that* (A1), (A3), (A4), (A7), (A8) *are satisfied. Then there exists* $\varepsilon_0 > 0$*, such that for* $0 < \varepsilon \leq \varepsilon_0$*, equation* (1.1) *has a solution* u_ε *satisfying*

$$0 < \Phi_{\varepsilon^{-4}}(u_\varepsilon) \leq \frac{b^{(4\kappa-4)/4}}{3^\kappa a_1\,(\gamma_{2^{**}}\gamma_0)^{N/2}}\varepsilon^{N-4},$$

$$\int_{\mathbb{R}^N} \mathcal{F}(x, u_\varepsilon)\mathrm{d}x \leq \frac{b^{(4\kappa-4)/4}}{3^\kappa a_1\big(\gamma_{2^{**}}\gamma_0\big)^{N/2}}\varepsilon^N.$$

Theorem 1.3. *Assume that* (A1)–(A4), (A9) *are satisfied. Then there exists* $\varepsilon_* > 0$*, such that for* $0 < \varepsilon \leq \varepsilon_*$*, equation* (1.2) *has a solution* u_ε *satisfying*

$$0 < \Phi_{\varepsilon^{-4}}(u_\varepsilon) \leq \frac{Q_2}{[3(1+a_2)Q_2]^{N/4}N(\gamma_{2^{**}}\gamma_0)^{N/2}}\varepsilon^{N-4},$$

$$\int_{\mathbb{R}^N} \mathcal{F}(x, u_\varepsilon)\mathrm{d}x + \frac{2}{N}\int_{\mathbb{R}^N} Q(x)|u_\varepsilon|^{2^{**}}\mathrm{d}x \leq \frac{Q_2}{[3(1+a_2)Q_2]^{N/4}N(\gamma_{2^{**}}\gamma_0)^{N/2}}\varepsilon^N.$$

Next, instead of handling (1.1) and (1.2) directly, but handle the equivalent problems. Let $\lambda = \varepsilon^{-4}$, then equations (1.1) and (1.2) are equivalent to the following equations respectively

$$\begin{gathered} \Delta^2 u + \lambda V(x)u = \lambda f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \quad u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.6}$$

and

$$\begin{gathered} \Delta^2 u + \lambda V(x)u = \lambda Q(x)|u|^{2^{**}-2}u + \lambda f(x,u), \quad x \in \mathbb{R}^N, \\ u \in H^2(\mathbb{R}^N), \quad u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.7}$$

Therefore, Theorems 1.2 and 1.3 are equivalent to the following theorems.

Theorem 1.4. *Assume that* (A1), (A3), (A4), (A7), (A8) *are satisfied. Then there exists* $\lambda_0 > 1$*, such that for* $\lambda \geq \lambda_0$*, equation* (1.6) *has a solution* u_λ *satisfying*

$$\begin{gathered} 0 < \Phi_\lambda(u_\lambda) \leq \frac{b^{(4\kappa-4)/4}}{3^\kappa a_1 \left(\gamma_{2^{**}}\gamma_0\right)^{N/2}} \lambda^{1-N/4}, \\ \int_{\mathbb{R}^N} \mathcal{F}(x,u_\lambda)\mathrm{d}x \leq \frac{b^{(4\kappa-4)/4}}{3^\kappa a_1 \left(\gamma_{2^{**}}\gamma_0\right)^{N/2}} \lambda^{-N/4}. \end{gathered}$$

Theorem 1.5. *Assume that* (A1)–(A4), (A9) *are satisfied. Then there exists* $\lambda_* > 1$*, such that for* $\lambda \geq \lambda_*$*, equation* (1.7) *has a solution* u_λ *satisfying*

$$\begin{gathered} 0 < \Phi_\lambda(u_\lambda) \leq \frac{Q_2}{[3(1+a_2)Q_2]^{N/4} N (\gamma_{2^{**}}\gamma_0)^{N/2}} \lambda^{1-N/4}, \\ \int_{\mathbb{R}^N} \mathcal{F}(x,u_\lambda)\mathrm{d}x + \frac{2}{N}\int_{\mathbb{R}^N} Q(x)|u_\lambda|^{2^{**}}\mathrm{d}x \leq \frac{Q_2}{[3(1+a_2)Q_2]^{N/4} N (\gamma_{2^{**}}\gamma_0)^{N/2}} \lambda^{-N/4}. \end{gathered}$$

In the next section, we provide some preliminaries and then prove these theorems.

2. Proof of the main results

To prove our results, first, we introduce the working space

$$E = \left\{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^2\mathrm{d}x < +\infty\right\}$$

and the associated norm

$$\|u\| = \Big(\int_{\mathbb{R}^N} [|\Delta u|^2 + \lambda V(x)|u|^2]\mathrm{d}x\Big)^{1/2}, \quad u \in E.$$

By using (A1) and the Sobolev inequality, we can demonstrate that there exists a constant $\gamma_0 > 0$ independent of λ such that

$$\|u\|_{H^2(\mathbb{R}^N)} \leq \gamma_0\|u\|, \quad \forall u \in E,\ \lambda \geq 1. \tag{2.1}$$

This shows that $(E, \|\cdot\|)$ is a Banach space for $\lambda \geq 1$. Furthermore, by the Sobolev embedding theorem, we have

$$\|u\|_s \leq \gamma_s\|u\|_{H^2(\mathbb{R}^N)} \leq \gamma_s\gamma_0\|u\|, \quad \forall u \in E,\ \lambda \geq 1,\ 2 \leq s \leq 2^{**}, \tag{2.2}$$

where and in the sequel, by $\|\cdot\|_s$ we denote the usual norm in space $L^s(\mathbb{R}^N)$.

Let

$$\Phi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda V(x)|u|^2\right)\mathrm{d}x - \lambda\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x \tag{2.3}$$

and

$$\begin{aligned}\Psi_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\Delta u|^2 + \lambda V(x)|u|^2\right)\mathrm{d}x - \frac{\lambda}{2^{**}}\int_{\mathbb{R}^N} Q(x)|u|^{2^{**}}\mathrm{d}x \\ - \lambda\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x.\end{aligned} \tag{2.4}$$

It is well known that Φ_λ and Ψ_λ are of $C^1(E,\mathbb{R})$, and

$$\langle \Phi'_\lambda(u), v\rangle = \int_{\mathbb{R}^N}(\Delta u\Delta v + \lambda V(x)uv)\,\mathrm{d}x - \lambda\int_{\mathbb{R}^N} f(x,u)v\mathrm{d}x, \quad \forall u,v\in E \tag{2.5}$$

and

$$\begin{aligned}\langle \Psi'_\lambda(u), v\rangle &= \int_{\mathbb{R}^N}(\Delta u\Delta v + \lambda V(x)uv)\,\mathrm{d}x \\ &\quad - \lambda\int_{\mathbb{R}^N}\Big[Q(x)|u|^{2^{**}-2}u + f(x,u)\Big]v\mathrm{d}x, \quad \forall u,v\in E.\end{aligned} \tag{2.6}$$

Observe that, since $(q-2)N-4q<0$, we can let $h_0\geq 1$ and $h_*\geq 1$ be such that

$$\begin{aligned}&\frac{(q-2)S_N}{2Nq(qa_0)^{2/(q-2)}}\Big\{\frac{4N^3+2}{(N+4)(1-2^{-N})^2}\Big\}^{q/(q-2)} h_0^{[(q-2)N-4q]/(q-2)} \\ &= \frac{b^{(4\kappa-N)/4}}{3^\kappa a_1\left(\gamma_{2^{**}}\gamma_0\right)^{N/2}}\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\frac{(q-2)S_N}{2qN(qa_*)^{2/(q-2)}}\Big\{\frac{4N^3+2(N+4)}{(N+4)(1-2^{-N})^2}\Big\}^{q/(q-2)} h_*^{[(q-2)N-4q]/(q-2)} \\ &= \frac{Q_2}{[3(1+a_2)Q_2]^{N/4}N(\gamma_{2^{**}}\gamma_0)^{N/2}}.\end{aligned} \tag{2.8}$$

Let $x_0\in\mathbb{R}^N$ be such that $V(x_0)=0$. From now on, we assume without loss of generality that $x_0=0$, that is $V(0)=0$, then we can choose $\lambda_0>1$ and $\lambda_*>1$ such that

$$\sup_{\lambda^{1/4}|x|\leq 2h_0}|V(x)|\leq h_0^{-4}, \quad \forall\lambda\geq\lambda_0, \tag{2.9}$$

$$\sup_{\lambda^{1/4}|x|\leq 2h_*}|V(x)|\leq h_*^{-4}, \quad \forall\lambda\geq\lambda_*. \tag{2.10}$$

Next, we give the proofs of Theorems 1.2–1.5. Subsection 2.1 considers the subcritical cases Theorems 1.2 and 1.4, while Subsection 2.2 considers the critical cases Theorems 1.3 and 1.5.

2.1. **Subcritical case.** In view of the definition of the norm $\|\cdot\|$, we can re-write Φ_λ in the form

$$\Phi_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x, \quad \forall u\in E. \tag{2.11}$$

Let

$$\vartheta(x) := \begin{cases}\frac{1}{h_0}, & |x|\leq h_0,\\ \frac{h_0^{N-1}}{1-2^{-N}}[|x|^{-N}-(2h_0)^{-N}], & h_0<|x|\leq 2h_0,\\ 0, & |x|>2h_0.\end{cases} \tag{2.12}$$

Then $\vartheta \in H^2(\mathbb{R}^N)$, moreover,

$$\|\Delta\vartheta\|_2^2 = \int_{\mathbb{R}^N} |\Delta\vartheta(x)|^2 \mathrm{d}x \le \frac{4N^2 S_N h_0^{N-6}}{(N+4)(1-2^{-N})^2}, \tag{2.13}$$

$$\|\vartheta\|_2^2 = \int_{\mathbb{R}^N} |\vartheta(x)|^2 \mathrm{d}x \le \frac{2S_N h_0^{N-2}}{(1-2^{-N})^2 N}. \tag{2.14}$$

Let $e_\lambda(x) = \vartheta(\lambda^{1/4}x)$. Then we can prove the following lemma which is very important.

Lemma 2.1. *Suppose that* (A1), (A3), (A4), (A7) *are satisfied. Then*

$$\sup\{\Phi_\lambda(se_\lambda) : s \ge 0\} \le \frac{b^{(4\kappa-N)/4}}{3^\kappa a_1(\gamma_{2^{**}}\gamma_0)^{N/2}} \lambda^{1-N/4}, \quad \forall \lambda \ge \lambda_0. \tag{2.15}$$

Proof. From (A7), (2.3), (2.7), (2.9), (2.12), (2.13) and (2.14), we obtain

$$\begin{aligned}
&\Phi_\lambda(se_\lambda) \\
&= \frac{s^2}{2}\int_{\mathbb{R}^N} \left(|\Delta e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) \mathrm{d}x - \lambda\int_{\mathbb{R}^N} F(x, se_\lambda)\mathrm{d}x \\
&= \lambda^{1-N/4}\Big[\frac{s^2}{2}\int_{\mathbb{R}^N} \Big(|\Delta\vartheta|^2 + V(\lambda^{-1/4}x)|\vartheta|^2\Big)\mathrm{d}x - \int_{\mathbb{R}^N} F(\lambda^{-1/4}x, s\vartheta)\mathrm{d}x\Big] \\
&\le \lambda^{1-N/4}\Big[\frac{s^2}{2}\Big(\|\Delta\vartheta\|_2^2 + \|\vartheta\|_2^2 \sup_{|x|\le 2h_0} |V(\lambda^{-1/4}x)|\Big) \\
&\quad - \int_{|x|\le h_0} F(\lambda^{-1/4}x, s/h_0)\mathrm{d}x\Big] \\
&\le \lambda^{1-N/4}\Big[\frac{s^2}{2}\left(\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2\right) - \int_{|x|\le h_0} F(\lambda^{-1/4}x, s/h_0)\mathrm{d}x\Big],
\end{aligned} \tag{2.16}$$

for all $s \ge 0$ and $\lambda \ge \lambda_0$,

$$\begin{aligned}
&\frac{s^2}{2}\left(\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2\right) - \int_{|x|\le h_0} F(\lambda^{-1/4}x, s/h_0)\mathrm{d}x \\
&\le \frac{s^2}{2}\left[\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2 - \frac{(4N^2+2)S_N}{N\left(1-2^{-N}\right)^2} h_0^{N-6}\right] \le 0,
\end{aligned} \tag{2.17}$$

for all $s \ge h_0 T_0$ and $\lambda \ge \lambda_0$, and

$$\begin{aligned}
&\frac{s^2}{2}\left(\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2\right) - \int_{|x|\le h_0} F(\lambda^{-1/4}x, s/h_0)\mathrm{d}x \\
&\le \frac{s^2}{2}\left(\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2\right) - \frac{a_0 S_N}{N} s^q h_0^{N-q} \\
&\le \frac{(q-2)\left(\|\Delta\vartheta\|_2^2 + h_0^{-4}\|\vartheta\|_2^2\right)^{q/(q-2)}}{2q\left(\frac{qa_0S_N}{N} h_0^{N-q}\right)^{2/(q-2)}} \\
&\le \frac{(q-2)S_N}{2Nq(qa_0)^{2/(q-2)}}\Big\{\frac{4N^3+2}{(N+4)(1-2^{-N})^2}\Big\}^{q/(q-2)} h_0^{[(q-2)N-4q]/(q-2)} \\
&= \frac{b^{(4\kappa-N)/4}}{3^\kappa a_1\left(\gamma_{2^{**}}\gamma_0\right)^{N/2}}, \quad \forall 0 \le s \le h_0 T_0,\ \lambda \ge \lambda_0.
\end{aligned} \tag{2.18}$$

Now the conclusion of Lemma 2.1 follows from (2.16), (2.17) and (2.18). □

Applying the mountain-pass lemma without the (PS) condition, by standard arguments, we can prove the following lemma.

Lemma 2.2. *Suppose that* (A1), (A3), (A4), (A7) *are satisfied. Then there exist a constant* $c_\lambda \in (0, \sup_{s\geq 0} \Phi_\lambda(se_\lambda)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi_\lambda(u_n) \to c_\lambda, \quad \|\Phi'_\lambda(u_n)\|_{E^*}(1 + \|u_n\|) \to 0. \tag{2.19}$$

Lemma 2.3. *Suppose that* (A1), (A3), (A4), (A7), (A8) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying* (2.19) *is bounded in* E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [14] or [25, Lemma 1.21], $v_n \to 0$ in $L^s(\mathbb{R}^N)$ for $2 < s < 2^{**}$. Hence, from (A1), (A8) and the Hölder inequality it follows that

$$\begin{aligned}
&\lambda \int_{|u_n|\leq R_0} |f(x,u_n)v_n|\mathrm{d}x \\
&\leq \frac{\lambda b}{3} \int_{|u_n|\leq R_0} |u_n||v_n|\mathrm{d}x \\
&\leq \frac{\lambda b}{3} \int_{\mathbb{R}^N\setminus\mathcal{V}_b} |u_n||v_n|\mathrm{d}x + \frac{\lambda b}{3} \int_{\mathcal{V}_b} |u_n||v_n|\mathrm{d}x \\
&\leq \frac{\lambda b}{3} \Big(\int_{\mathbb{R}^N\setminus\mathcal{V}_b} |u_n|^2\mathrm{d}x \Big)^{1/2} \Big(\int_{\mathbb{R}^N\setminus\mathcal{V}_b} |v_n|^2\mathrm{d}x \Big)^{1/2} \\
&\quad + \frac{\lambda b[\mathrm{meas}(\mathcal{V}_b)]^{1/(N+1)}}{3} \Big(\int_{\mathcal{V}_b} |u_n|^{2(N+1)/N}\mathrm{d}x \Big)^{N/2(N+1)} \\
&\quad \times \Big(\int_{\mathcal{V}_b} |v_n|^{2(N+1)/N}\mathrm{d}x \Big)^{N/2(N+1)} \\
&\leq \frac{1}{3}\|u_n\| + \frac{\lambda b[\mathrm{meas}(\mathcal{V}_b)]^{1/(N+1)}}{3} \|u_n\|_{2(N+1)/N}\|v_n\|_{2(N+1)/N} \\
&= [\frac{1}{3} + o(1)]\|u_n\|.
\end{aligned} \tag{2.20}$$

From (2.3), (2.5) and (2.19), it holds

$$c_\lambda + o(1) = \lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\mathrm{d}x. \tag{2.21}$$

Let $\kappa' = \kappa/(\kappa-1)$, then $2 < 2\kappa' < 2^{**}$. By (A8), (2.21) and the Hölder inequality, one obtain

$$\begin{aligned}
&\lambda \int_{|u_n|\geq R_0} \frac{|f(x,u_n)v_n|}{\|u_n\|}\mathrm{d}x \\
&= \lambda \int_{|u_n|\geq R_0} \frac{|f(x,u_n)|}{|u_n|}|v_n|^2\mathrm{d}x \\
&\leq \lambda\Big(\int_{|u_n|\geq R_0}\Big(\frac{|f(x,u_n)|}{|u_n|}\Big)^{\kappa}\mathrm{d}x\Big)^{1/\kappa}\Big(\int_{|u_n|\geq R_0}|v_n|^{2\kappa'}\mathrm{d}x\Big)^{1/\kappa'} \\
&\leq \lambda\Big(a_1\int_{|u_n|\geq R_0}\mathcal{F}(x,u_n)\mathrm{d}x\Big)^{1/\kappa}\Big(\int_{|u_n|\geq R_0}|v_n|^{2\kappa'}\mathrm{d}x\Big)^{1/\kappa'} \\
&\leq \lambda^{1-1/\kappa}[a_1c_\lambda + o(1)]^{1/\kappa}\|v_n\|_{2\kappa'}^2 = o(1).
\end{aligned} \tag{2.22}$$

Combining (2.20) with (2.21) and using (2.11) and (2.19), we have

$$\begin{aligned}
1 + o(1) &\leq \frac{\|u_n\|^2 - \langle \Phi'_\lambda(u_n), u_n\rangle}{\|u_n\|^2} = \lambda\int_{\mathbb{R}^N}\frac{|f(x,u_n)v_n|}{\|u_n\|}\mathrm{d}x \\
&= \lambda\int_{|u_n|<R_0}\frac{|f(x,u_n)v_n|}{\|u_n\|}\mathrm{d}x + \lambda\int_{|u_n|\geq R_0}\frac{|f(x,u_n)v_n|}{\|u_n\|}\mathrm{d}x \\
&\leq \frac{1}{3} + o(1).
\end{aligned} \tag{2.23}$$

This contradiction shows that $\delta > 0$.

Going to a subsequence, if necessary, we may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |v_n|^2dx > \frac{\delta}{2}$. Let $w_n(x) = v_n(x + k_n)$. Then

$$\int_{B_{1+\sqrt{N}}(0)} |w_n|^2dx > \frac{\delta}{2}. \tag{2.24}$$

Now we define $\tilde{u}_n(x) = u_n(x + k_n)$, then $\tilde{u}_n/\|u_n\| = w_n$ and $\|w_n\|_2^2 = \|v_n\|_2^2$. Passing to a subsequence, we have $w_n \rightharpoonup w$ in $H^2(\mathbb{R}^N)$, $w_n \to w$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2 \leq s < 2^{**}$ and $w_n \to w$ a.e. on $\mathbb{R}^N$. Obviously, (2.24) implies that $w \neq 0$. For a.e. $x \in \{z \in \mathbb{R}^N : w(z) \neq 0\}$, we have $\lim_{n\to\infty} |\tilde{u}_n(x)| = \infty$. Hence, it follows from (2.11), (2.19), (A7) and Fatou's lemma that

$$\begin{aligned}
0 &= \lim_{n\to\infty}\frac{c_\lambda + o(1)}{\|u_n\|^2} = \lim_{n\to\infty}\frac{\Phi_\lambda(u_n)}{\|u_n\|^2} \\
&= \lim_{n\to\infty}\Big[\frac{1}{2}\|v_n\|^2 - \lambda\int_{\mathbb{R}^N}\frac{F(x+k_n,\tilde{u}_n)}{|\tilde{u}_n|^2}|w_n|^2\mathrm{d}x\Big] \\
&\leq \frac{1}{2} - \lambda\int_{\mathbb{R}^N}\liminf_{n\to\infty}\frac{F(x+k_n,\tilde{u}_n)}{|\tilde{u}_n|^2}|w_n|^2\mathrm{d}x = -\infty.
\end{aligned}$$

This contradiction shows that $\{\|u_n\|\}$ is bounded. □

Proof of Theorem 1.4. Applying Lemmas 2.1, 2.2 and 2.3, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (2.20) with

$$c_\lambda \leq \frac{b^{(4\kappa-N)/4}}{3^\kappa a_1\left(\gamma_{2^{**}}\gamma_0\right)^{N/2}}\lambda^{1-N/4}, \quad \forall\lambda \geq \lambda_0. \tag{2.25}$$

Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u_\lambda$ in $(E, \|\cdot\|)$ and $\Phi'_\lambda(u_n) \to 0$. Next, we prove that $u_\lambda \neq 0$.

Arguing by contradiction, suppose that $u_\lambda = 0$, i.e. $u_n \rightharpoonup 0$ in E, and so $u_n \to 0$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2 \le s < 2^{**}$ and $u_n \to 0$ a.e. on $\mathbb{R}^N$. Since $\mathcal{V}_b$ is a set of finite measure and $u_n \rightharpoonup 0$ in E, it holds

$$\|u_n\|_2^2 = \int_{\mathbb{R}^N\setminus\mathcal{V}_b} |u_n|^2 \mathrm{d}x + \int_{\mathcal{V}_b} |u_n|^2 \mathrm{d}x \le \frac{1}{\lambda b}\|u_n\|^2 + o(1). \tag{2.26}$$

For $s \in (2, 2^{**})$, from (2.2), (2.26) and the Hölder inequality it follows that

$$\begin{aligned} \|u_n\|_s^s &\le \|u_n\|_2^{2(2^{**}-s)/(2^{**}-2)} \|u_n\|_{2^{**}}^{2^{**}(s-2)/(2^{**}-2)} \\ &\le (\gamma_{2^{**}}\gamma_0)^{2^{**}(s-2)/(2^{**}-2)} (\lambda b)^{-(2^{**}-s)/(2^{**}-2)} \|u_n\|^s + o(1). \end{aligned} \tag{2.27}$$

According to (F4) and (2.26), one can obtain

$$\lambda \int_{|u_n|\le R_0} f(x,u_n)u_n \mathrm{d}x \le \frac{\lambda b}{3}\int_{|u_n|\le R_0} |u_n|^2 \mathrm{d}x \le \frac{1}{3}\|u_n\|^2 + o(1). \tag{2.28}$$

By (2.3), (2.5) and (2.19), we have

$$\Phi_\lambda(u_n) - \frac{1}{2}\langle \Phi'_\lambda(u_n), u_n\rangle = \lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u_n)\mathrm{d}x = c_\lambda + o(1). \tag{2.29}$$

Using (A8), (2.25), (2.27) with $s = 2\kappa/(\kappa-1)$ and (2.29), we obtain

$$\begin{aligned} &\lambda \int_{|u_n|\ge R_0} f(x,u_n)u_n \mathrm{d}x \\ &\le \lambda \Big(\int_{|u_n|\ge R_0} \Big(\frac{|f(x,u_n)|}{|u_n|}\Big)^\kappa \mathrm{d}x \Big)^{1/\kappa} \|u_n\|_s^2 \\ &\le a_1^{1/\kappa} (\gamma_{2^{**}}\gamma_0)^{2\cdot 2^{**}(s-2)/s(2^{**}-2)} \lambda^{1-1/\kappa} (\lambda b)^{-2(2^{**}-s)/s(2^{**}-2)} c_\lambda^{1/\kappa} \|u_n\|^2 + o(1) \\ &\le a_1^{1/\kappa} (\gamma_{2^{**}}\gamma_0)^{N/2\kappa} \lambda^{1-1/\kappa} c_\lambda^{1/\kappa} (\lambda b)^{-(4\kappa-N)/4\kappa} \|u_n\|^2 + o(1) \\ &= \frac{a_1^{1/\kappa}(\gamma_{2^{**}}\gamma_0)^{N/2\kappa}}{b^{(4\kappa-N)/4\kappa}} [\lambda^{(N-4)/4} c_\lambda]^{1/\kappa} \|u_n\|^2 + o(1) \\ &\le \frac{1}{3}\|u_n\|^2 + o(1), \end{aligned} \tag{2.30}$$

which, together with (2.5), (2.19) and (2.28), yields

$$o(1) = \langle \Phi'_\lambda(u_n), u_n\rangle = \|u_n\|^2 - \lambda \int_{\mathbb{R}^N} f(x,u_n)u_n \mathrm{d}x \ge \frac{1}{3}\|u_n\|^2 + o(1); \tag{2.31}$$

this results in the fact that $\|u_n\| \to 0$. Consequently, from (A3), (2.11) and (2.19) it follows that

$$0 < c_\lambda = \lim_{n\to\infty} \Phi_\lambda(u_n) = \Phi_\lambda(0) = 0.$$

This contradiction shows $u_\lambda \ne 0$. By a standard argument, we easily certify that $\Phi'_\lambda(u_\lambda) = 0$ and $\Phi_\lambda(u_\lambda) \le c_\lambda$. Then u_λ is a nontrivial solution of (1.7), moreover

$$c_\lambda \ge \Phi_\lambda(u_\lambda) = \Phi_\lambda(u_\lambda) - \frac{1}{2}\langle \Phi'_\lambda(u_\lambda), u_\lambda\rangle = \lambda \int_{\mathbb{R}^N} \mathcal{F}(x,u_\lambda)\mathrm{d}x. \tag{2.32}$$

□

Note that Theorem 1.2 is a direct consequence of Theorem 1.4.

2.2. **Critical case.** In view of the definition of the norm $\|\cdot\|$, we can re-write Ψ_λ in the form

$$\Psi_\lambda(u) = \frac{1}{2}\|u\|^2 - \frac{\lambda}{2^{**}}\int_{\mathbb{R}^N} Q(x)|u|^{2^{**}}\mathrm{d}x - \lambda\int_{\mathbb{R}^N} F(x,u)\mathrm{d}x, \quad \forall u \in E. \tag{2.33}$$

Let $e_\lambda^*(x) = \vartheta^*(\lambda^{1/4}x)$, where

$$\vartheta^*(x) := \begin{cases} \frac{1}{h_*}, & |x| \le h_*, \\ \frac{h_*^{N-1}}{1-2^{-N}}\left[|x|^{-N} - (2h_*)^{-N}\right], & h_* < |x| \le 2h_*, \\ 0, & |x| > 2h_*. \end{cases} \tag{2.34}$$

Then we can prove the following lemma in the same way as the proof of Lemma 2.1.

Lemma 2.4. *Suppose that* (A1), (A3), (A4), (A9) *are satisfied. Then*

$$\sup\{\Psi_\lambda(se_\lambda^*) : s \ge 0\} \le \frac{Q_2}{[3(1+a_2)Q_2]^{N/4}N(\gamma_{2^{**}}\gamma_0)^{\frac{N}{2}}}\lambda^{1-N/4}, \quad \forall \lambda \ge \lambda_*. \tag{2.35}$$

Applying the mountain-pass lemma without the (PS) condition, by standard arguments, we can also prove the following lemma.

Lemma 2.5. *Suppose that* (A1), (A3), (A4), (A9) *are satisfied. Then there exist a constant* $c_\lambda \in (0, \sup_{s\ge 0}\Psi_\lambda(se_\lambda^*)]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Psi_\lambda(u_n) \to c_\lambda, \quad \|\Psi_\lambda'(u_n)\|_{E^*}(1 + \|u_n\|) \to 0. \tag{2.36}$$

Lemma 2.6. *Suppose that* (A1), (A3), (A4), (A9)*t are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying* (2.36) *is bounded in* E*.*

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$. Then $\|v_n\| = 1$. In view of (A2) and (A4), we can choose $R_\lambda \in (0,1)$ such that

$$|Q(x)|t|^{2^{**}-2}t + f(x,t)| \le \frac{1}{3\lambda(\gamma_{2^{**}}\gamma_0)^2}|t|, \quad \forall x \in \mathbb{R}^N,\ |t| \le R_\lambda. \tag{2.37}$$

Hence, by (2.2), (2.37) and the Hölder inequality, it holds

$$\begin{aligned} &\frac{\lambda}{\|u_n\|}\int_{|u_n|\le R_\lambda} |[Q(x)|u_n|^{2^{**}-2} + f(x,u_n)]v_n|\mathrm{d}x \\ &\le \frac{1}{3(\gamma_{2^{**}}\gamma_0)^2\|u_n\|}\int_{|u_n|\le R_\lambda} |u_n||v_n|\mathrm{d}x \\ &\le \frac{1}{3(\gamma_{2^{**}}\gamma_0)^2\|u_n\|}\|u_n\|_2\|v_n\|_2 \le \frac{1}{3}. \end{aligned} \tag{2.38}$$

From (A2), (A9), (2.6), (2.33) and (2.36), one has

$$\begin{aligned} c_\lambda + o(1) &= \lambda\int_{\mathbb{R}^N}\big[\frac{2}{N}Q(x)|u_n|^{2^{**}} + \mathcal{F}(x,u_n)\big]\mathrm{d}x \\ &\ge \frac{2\lambda Q_1}{N}\int_{|u_n|\ge R_\lambda}|u_n|^{2^{**}}\mathrm{d}x. \end{aligned} \tag{2.39}$$

Sing (A3), (A2), (2.39) and the Hölder inequality, we obtain

$$\begin{aligned} &\frac{\lambda}{\|u_n\|}\int_{|u_n|\geq R_\lambda}|\big[Q(x)|u_n|^{2^{**}-2}u_n+f(x,u_n)\big]v_n|\mathrm{d}x\\ &\leq \frac{\lambda C_\lambda Q_2}{\|u_n\|}\int_{|u_n|\geq R_\lambda}|u_n|^{2^{**}-1}|v_n|\mathrm{d}x\\ &\leq \frac{\lambda C_\lambda Q_2}{\|u_n\|}\|v_n\|_{2^{**}}\Big(\int_{|u_n|\geq R_\lambda}|u_n|^{2^{**}}\mathrm{d}x\Big)^{(2^{**}-1)/2^{**}}=o(1), \end{aligned} \tag{2.40}$$

where C_λ is a constant depend on λ. Combining (2.38) with (2.40) and using (2.6) and (2.36), we have

$$\begin{aligned} 1+o(1)&=\frac{\|u_n\|^2-\langle\Psi_\lambda'(u_n),u_n\rangle}{\|u_n\|^2}\\ &=\frac{\lambda}{\|u_n\|}\int_{\mathbb{R}^N}\big[Q(x)|u_n|^{2^{**}-2}u_n+f(x,u_n)\big]v_n\mathrm{d}x\\ &\leq \frac{\lambda}{\|u_n\|}\int_{|u_n|<R_\lambda}|\big[Q(x)|u_n|^{2^{**}-2}u_n+f(x,u_n)\big]v_n|\mathrm{d}x\\ &\quad+\frac{\lambda}{\|u_n\|}\int_{|u_n|\geq R_\lambda}|\big[Q(x)|u_n|^{2^{**}-2}u_n+f(x,u_n)\big]v_n|\mathrm{d}x\\ &\leq \frac{1}{3}+o(1), \end{aligned}$$

which is a contradiction. Thus the sequence $\{u_n\}$ is bounded in E. □

Proof of Theorem 1.5. Applying Lemmas 2.4, 2.5 and 2.6, we deduce that there exists a bounded sequence $\{u_n\}\subset E$ satisfying (2.36) with

$$c_\lambda\leq \frac{Q_2}{[3(1+a_2)Q_2]^{N/4}N(\gamma_{2^{**}}\gamma_0)^{N/2}}\lambda^{1-N/4},\quad \forall\lambda\geq\lambda_*. \tag{2.41}$$

Going to a subsequence,if necessary, we can assume that $u_n\rightharpoonup u_\lambda$ in $(E,\|\cdot\|)$ and $\Psi_\lambda'(u_n)\to 0$. Next, we prove that $u_\lambda\neq 0$.

Arguing by contradiction, suppose that $u_\lambda=0$, i.e. $u_n\rightharpoonup 0$ in E, and so $u_n\to 0$ in $L^s_{\rm loc}(\mathbb{R}^N)$, $2\leq s<2^{**}$ and $u_n\to 0$ a.e. on $\mathbb{R}^N$. Since $\mathcal{V}_b$ is a set of finite measure and $u_n\rightharpoonup 0$ in E,

$$\|u_n\|_2^2=\int_{\mathbb{R}^N\setminus\mathcal{V}_b}|u_n|^2\mathrm{d}x+\int_{\mathcal{V}_b}|u_n|^2\mathrm{d}x\leq\frac{1}{\lambda b}\|u_n\|^2+o(1), \tag{2.42}$$

which, together with (1.3), yields

$$\begin{aligned} &\lambda\int_{|u_n|\leq R_*}\big[Q(x)|u_n|^{2^{**}}+f(x,u_n)u_n\big]\mathrm{d}x\\ &\leq\frac{\lambda b}{3}\int_{|u_n|\leq R_*}|u_n|^2\mathrm{d}x\leq\frac{1}{3}\|u_n\|^2+o(1). \end{aligned} \tag{2.43}$$

By (2.6), (2.33) and (2.36), we have

$$\begin{aligned} \Psi_\lambda(u_n)-\frac{1}{2}\langle\Psi_\lambda'(u_n),u_n\rangle&=\lambda\int_{\mathbb{R}^N}\Big[\frac{2}{N}Q(x)|u_n|^{2^{**}}+\mathcal{F}(x,u_n)\Big]\mathrm{d}x\\ &=c_\lambda+o(1). \end{aligned} \tag{2.44}$$

Using (2.2), (1.4), (2.41), (2.44) and the Hölder inequality, we obtain

$$\begin{aligned}
&\lambda\int_{|u_n|>R_*}\Big[Q(x)|u_n|^{2^{**}}+f(x,u_n)u_n\Big]\,\mathrm{d}x\\
&\le(1+a_2)\lambda\int_{|u_n|>R_*}Q(x)|u_n|^{2^{**}}\mathrm{d}x\\
&\le(1+a_2)\lambda(Q_2)^{2/2^{**}}\Big(\int_{|u_n|>R_*}Q(x)|u_n|^{2^{**}}\mathrm{d}x\Big)^{4/N}\Big(\int_{|u_n|>R_*}|u_n|^{2^{**}}\mathrm{d}x\Big)^{2/2^{**}}\\
&=(1+a_2)(\lambda Q_2)^{2/2^{**}}\Big(\int_{|u_n|>R_*}Q(x)|u_n|^{2^{**}}\mathrm{d}x\Big)^{4/N}\|u_n\|_{2^{**}}^2 \qquad (2.45)\\
&=(1+a_2)Q_2(\gamma_{2^{**}}\gamma_0)^2\Big(\frac{N}{Q_2}\Big)^{4/N}(\lambda^{\frac{N-4}{4}}c_\lambda)^{4/N}\|u_n\|^2+o(1)\\
&\le\frac{1}{3}\|u_n\|^2+o(1),
\end{aligned}$$

which, together with (2.6) and (2.43), yields

$$\begin{aligned}
o(1)&=\langle\Psi'_\lambda(u_n),u_n\rangle\\
&=\|u_n\|^2-\lambda\int_{\mathbb{R}^N}\big[Q(x)|u_n|^{2^{**}}+f(x,u_n)u_n\big]\mathrm{d}x \qquad (2.46)\\
&\ge\frac{1}{3}\|u_n\|^2+o(1);
\end{aligned}$$

this results in the fact that $\|u_n\|\to 0$. The rest proof is the same as one of Theorem 1.4. □

Note that Theorem 1.3 is a direct consequence of Theorem 1.5.

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