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EXISTENCE OF SOLUTIONS TO INFINITE ELASTIC BEAM EQUATIONS WITH UNBOUNDED NONLINEARITIES

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Abstract. This article concerns the existence of unbounded solutions to fourth-order boundary-value problem on the half-line with two-point boundary conditions. One-sided Nagumo condition plays a special role as it allows an asymmetric unbounded behavior on the nonlinearity. The arguments are based on the Schauder fixed point theorem and lower and upper solutions method. As an application, an example is given with non-ordered lower and upper solutions, to prove our results.

1. Introduction

Fourth-order differential equations can model the bending of an elastic beam and, in this sense, we refer them as beam equations. They have received increased interest from several fields of science and engineering, either on bounded domains [3, 5, 8, 20, 21] either on the real line [1, 7, 12, 13, 17, 18].

We study the fully nonlinear beam equation on the half line,

$$u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), \quad t \in [0, +\infty), \tag{1.1}$$

where $f : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is an L^1-Carathéodory function, and the boundary conditions are of Sturm-Liouville type,

$$u(0) = A, u'(0) = B, \quad u''(0) + au'''(0) = C, \quad u'''(+\infty) = D, \tag{1.2}$$

$A, B, C, D \in \mathbb{R}$, $a < 0$ and $u'''(+\infty) := \lim_{t\to+\infty} u'''(t)$.

The non-compactness of the interval requires delicate techniques to obtain sufficient conditions for the solvability of boundary value problems on the half-line. As examples, we refer the extension of continuous solutions on the corresponding finite intervals under a diagonalization process, fixed point theory in some Banach spaces and lower and upper solutions method (see [2, 4, 14, 22] and the references therein). We present here an approach based on an adequate space of weighted functions.

Lower and upper solutions method is a very adequate technique to deal with boundary value problems as it provides not only the existence of bounded or unbounded solutions but also their localization and, from that, some qualitative data about solutions, their variation and behavior (see [6, 9, 15, 16, 19]). In this paper we use not necessarily ordered lower an upper solutions, generalizing, in this way,

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the set of admissible lower and upper functions. As far as we know, it is the first time where such functions are applied to boundary value problems defined on the half line, to obtain unbounded solutions.

An important tool is the Nagumo condition, useful to get *a priori* estimates on some derivatives of the solution. The usual growth condition of the Nagumo type used in the literature is a bilateral one. However the same estimations hold with a similar one-sided assumption, which allows unbounded nonlinearities in boundary value problems. Therefore, it generalizes the two-sided condition, as it is proved in [8, 10].

In short, this work has the following novelties related to the existent literature in this field:

- The nonlinearity f is an L^1-Carathéodory function, allowing discontinuities in time;
- From the unilateral Nagumo growth condition assumed on f, equation (1.1) can deal with unbounded nonlinearities;
- The lower and upper solutions do not need to be well ordered, or even ordered, and, moreover, their boundary conditions are more general, making easier to obtain lower and upper solutions to the problem.
- The non-compactness of the associated operator is overcome by considering an adequate space of weighted functions.

The paper is organized as it follows: In Section 2 some auxiliary result are defined such as the space, the weighted norms, the unilateral Nagumo condition and lower and upper solutions to be used. Section 3 contains the main result: an existence and localization theorem, where it is proved the existence of a solution, and some bounds on the first and second derivatives as well. Finally, an example, which is not covered by the existent results, shows the applicability of the main theorem.

2. Definitions and preliminary results

In this work we consider the space

$$X = \big\{x \in C^3[0,+\infty) : \lim_{t\to+\infty} \frac{x^{(i)}(t)}{1+t^{3-i}} \text{ exists in } \mathbb{R},\ i = 0,1,2,3\big\}$$

with the norm $\|x\|_X := \max\{\|x\|_0, \|x'\|_0, \|x''\|_0, \|x'''\|_0\}$, where

$$\|\omega^{(i)}\|_0 = \sup_{0\le t<+\infty} \Big|\frac{\omega^{(i)}(t)}{1+t^{3-i}}\Big|,\ i = 0,1,2,3.$$

It can be proved that $(X, \|\cdot\|_X)$ is a Banach space (see [16]).

The following definition establishes the assumptions assumed on the nonlinearity.

Definition 2.1. A function $f : [0,+\infty) \times \mathbb{R}^4 \to \mathbb{R}$ is called an L^1-Carathéodory function if it satisfies:

(i) for each $(x, y, z, w) \in \mathbb{R}^4$, $t \mapsto f(t, x, y, z, w)$ is measurable on $[0,+\infty)$;
(ii) for almost every $t \in [0,+\infty), (x, y, z, w) \mapsto f(t, x, y, z, w)$ is continuous in $\mathbb{R}^4$;
(iii) for each $\rho > 0$, there exists a positive function $\varphi_\rho \in L^1[0,+\infty)$ such that for all $(x, y, z, w) \in \mathbb{R}^4$ with $\|(x, y, z, w)\|_X < \rho$, then

$$|f(t, x, y, z, w)| \le \varphi_\rho(t), \quad \text{a.e. } t \in [0,+\infty).$$

Solutions of the linear problem associated with (1.1)-(1.2) are defined with Green's function, which can be obtained by standard calculus.

Lemma 2.2. *Let $t^3\eta \in L^1[0,+\infty)$. Then the linear boundary value problem*

$$u^{(4)}(t) + \eta(t) = 0, \quad t \in [0,+\infty), \tag{2.1}$$

with boundary conditions (1.2)*, has a unique solution in X. Moreover, this solution can be expressed as*

$$u(t) = A + Bt + \frac{C - aD}{2}t^2 + \frac{D}{6}t^3 + \int_0^{+\infty} G(t,s)\eta(s)ds \tag{2.2}$$

where

$$G(t,s) = \begin{cases} \frac{s^3}{6} - \frac{s^2 t}{2} + \frac{st^2}{2} - \frac{at^2}{2}, & 0 \le s \le t \\ -\frac{at^2}{2} + \frac{t^3}{6}, & t \le s < +\infty. \end{cases}$$

To apply a fixed point theorem it is important to have an *a priori* estimation for u'''. In the literature this bound is obtained from a bilateral Nagumo-type growth. In this paper it is used a more general one-sided Nagumo condition, which allows unbounded nonlinearities on (1.1) (for more details see [11, 21, 23]. Remark that, as it is proved in [10] a function can verify an unilateral Nagumo condition but not a two-sided one.

Let $\gamma_i, \Gamma_i \in C[0,+\infty), \gamma_i(t) \le \Gamma_i(t), i = 0,1,2$ and define the set

$$E = \big\{(t, x_0, x_1, x_2, x_3) \in [0,+\infty) \times \mathbb{R}^4 : \gamma_i(t) \le x_i \le \Gamma_i(t), i = 0,1,2\big\}.$$

Definition 2.3. An L^1-Carathéodory function $f : E \to \mathbb{R}$ is said to satisfy the one-sided Nagumo-type growth condition in E if it satisfies either

$$f(t,x,y,z,w) \le \psi(t)h(|w|), \quad \forall (t,x,y,z,w) \in E, \tag{2.3}$$

or

$$f(t,x,y,z,w) \ge -\psi(t)h(|w|), \quad \forall (t,x,y,z,w) \in E, \tag{2.4}$$

for some positive continuous functions ψ, h, and some $\nu > 1$, such that

$$\int_0^{+\infty} \psi(s)ds < +\infty, \sup_{0 \le t < +\infty} \psi(t)(1+t)^\nu < +\infty, \quad \int_0^{+\infty} \frac{s}{h(s)}ds = +\infty. \tag{2.5}$$

Next lemma provides an *a priori* bound.

Lemma 2.4. *Let $f : [0,+\infty) \times \mathbb{R}^4 \to \mathbb{R}$ be an L^1-Carathéodory function satisfying* (2.3)*, or* (2.4)*, and* (2.5) *in E. Then for every $r > 0$ there exists $R > 0$ (not depending on u) such that every u solution of* (1.1)*,* (1.2) *satisfying*

$$\gamma_0(t) \le u(t) \le \Gamma_0(t), \gamma_1(t) \le u'(t) \le \Gamma_1(t), \gamma_2(t) \le u''(t) \le \Gamma_2(t), \tag{2.6}$$

for $t \in [0,+\infty)$, satisfies $\|u'''\|_0 < R$.

Proof. Let u be a solution of (1.1), (1.2) such that (2.6) holds. Consider $r > 0$ such that

$$r > \max\Big\{|\frac{C - \Gamma_2(0)}{a}|, |\frac{C - \gamma_2(0)}{a}|, |D|\Big\}. \tag{2.7}$$

By this inequality we cannot have $|u'''(t)| > r$ for all $t \in [0,+\infty)$, because

$$|u'''(0)| = |\frac{C - u''(0)}{a}| \le \max\Big\{|\frac{C - \Gamma_2(0)}{a}|, |\frac{C - \gamma_2(0)}{a}|\Big\} < r \tag{2.8}$$

and $|u'''(+\infty)| = |D| < r$.

If $|u'''(t)| \le r$ for all $t \in [0,+\infty)$, taking $R > r/2$ the proof is complete as

$$\|u'''\|_0 = \sup_{0\le t<+\infty} |\frac{u'''(t)}{2}| \le \frac{r}{2} < R.$$

If there exists $t \in (0,+\infty)$ such that $|u'''(t)| > r$, then by (2.5) we can take $R > r$ such that

$$\int_r^R \frac{s}{h(s)} ds > M \max\Big\{M_1 + \sup_{0\le t<+\infty} \frac{\Gamma_2(t)}{1+t}\frac{\nu}{\nu-1}, M_1 - \inf_{0\le t<+\infty} \frac{\gamma_2(t)}{1+t}\frac{\nu}{\nu-1}\Big\}$$

with $M := \sup_{0\le t<+\infty} \psi(t)(1+t)^\nu$ and $M_1 := \sup_{0\le t<+\infty} \frac{\Gamma_2(t)}{(1+t)^\nu} - \inf_{0\le t<+\infty} \frac{\gamma_2(t)}{(1+t)^\nu}$.

Assume that growth condition (2.3) holds. By (2.7), suppose that there are t_*, $t_+ \in (0,+\infty)$ such that $u'''(t_*) = r$, $u'''(t) > r$ for all $t \in (t_*, t_+]$. Then

$$\begin{aligned}
\int_{u'''(t_*)}^{u'''(t_+)} \frac{s}{h(s)} ds &= \int_{t_*}^{t_+} \frac{u'''(s)}{h(u'''(s))} u^{(4)}(s) ds \\
&= \int_{t_*}^{t_+} \frac{f(s,u(s),u'(s),u''(s),u'''(s))}{h(u'''(s))} u'''(s) ds \\
&\le \int_{t_*}^{t_+} \psi(s)\ u'''(s)\ ds \le M \int_{t_*}^{t_+} \frac{u'''(s)}{(1+s)^\nu}\ ds \\
&= M \int_{t_*}^{t_+} \Big(\frac{u''(s)}{(1+s)^\nu}\Big)' + \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds \\
&= M\Big(\frac{u''(t_+)}{(1+t_+)^\nu} - \frac{u''(t_*)}{(1+t_*)^\nu} + \int_{t_*}^{t_+} \frac{\nu u''(s)}{(1+s)^{1+\nu}} ds\Big) \\
&\le M\Big(M_1 + \sup_{0\le t<+\infty} \frac{\Gamma_2(t)}{1+t} \int_0^{+\infty} \frac{\nu}{(1+s)^\nu} ds\Big) \\
&< \int_r^R \frac{s}{h(s)} ds.
\end{aligned}$$

So $u'''(t_+) < R$ and as t_*, t_+ are arbitrary in $(0,+\infty)$, we have $u'''(t) < R$ for all $t \in [0,+\infty)$.

By the same technique as in (2.8), and considering t_- and t_* such that $u'''(t_*) = -r$, $u'''(t) < -r$ for all $t \in [t_-, t_*]$, it can be proved that $u'''(t) > -R$ for all $t \in [0,+\infty)$ and, therefore $\|u'''\| < R/2 < R$ for all $t \in [0,+\infty)$.

If f satisfies (2.4), following similar arguments, the same conclusion is achieved. □

Next result will play a key role to apply a fixed-point theorem.

Lemma 2.5 ([1, Theorem 6.2.2]). *A set $M \subset X$ is relatively compact if the following three conditions hold:*

1. *all functions from M are uniformly bounded;*
2. *all functions from M are equicontinuous on any compact interval of $[0,+\infty)$;*
3. *all functions from M are equiconvergent at infinity, that is, for any given $\epsilon > 0$, there exists a $t_\epsilon > 0$ such that*

$$|\frac{u^{(i)}(t)}{1+t^{3-i}} - \lim_{t\to+\infty} \frac{u^{(i)}(t)}{1+t^{3-i}}| < \epsilon, \text{ for all } t > t_\epsilon, x \in M \text{ and } i = 0,1,2,3.$$

The functions considered as lower and upper solutions for the initial problem are defined as it follows.

Definition 2.6. Given $a < 0$ and $A, B, C, D \in \mathbb{R}$, a function $\alpha \in C^4[0, +\infty) \cap X$ is said to be a lower solution of problem (1.1),(1.2) if

$$\alpha^{(4)}(t) \geq f(t, \overline{\alpha}(t), \alpha'(t), \alpha''(t), \alpha'''(t)), \quad t \in [0, +\infty),$$

and

$$\alpha'(0) \leq B, \quad \alpha''(0) + a\ \alpha'''(0) \leq C, \quad \alpha'''(+\infty) < D, \tag{2.9}$$

where $\overline{\alpha}(t) := \alpha(t) - \alpha(0) + A$.

A function β is an upper solution if it satisfies the reversed inequalities with $\overline{\beta}(t) := \beta(t) - \beta(0) + A$.

We point out that α and β need not to be well ordered or even ordered.

3. Main result

In this section we prove the existence of at least one solution for problem (1.1), (1.2).

Theorem 3.1. *Let $f : [0, +\infty) \times \mathbb{R}^4 \to \mathbb{R}$ be an L^1-Carathéodory function, and α, β lower and upper solutions of* (1.1), (1.2), *respectively, such that*

$$\alpha''(t) \leq \beta''(t), \quad \forall t \in [0, +\infty). \tag{3.1}$$

If f satisfies the one-sided Nagumo condition (2.3), *or* (2.4), *in the set*

$$E_* = \Big\{(t, x, y, z, w) \in [0, +\infty) \times \mathbb{R}^4 : \overline{\alpha}(t) \leq x \leq \overline{\beta}(t),\ \alpha'(t) \leq y \leq \beta'(t), \\ \alpha''(t) \leq z \leq \beta''(t)\Big\},$$

and

$$f(t, \overline{\alpha}(t), \alpha'(t), z, w) \geq f(t, x, y, z, w) \geq f(t, \overline{\beta}(t), \beta'(t), z, w), \tag{3.2}$$

for (t, z, w) fixed and $\overline{\alpha}(t) \leq x \leq \overline{\beta}(t)$, $\alpha'(t) \leq y \leq \beta'(t)$, then problem (1.1), (1.2) *has at least a solution $u \in C^4(0, +\infty) \cap X$ and there exists $R > 0$ such that*

$$\overline{\alpha}(t) \leq u(t) \leq \overline{\beta}(t), \alpha'(t) \leq u'(t) \leq \beta'(t), \\ \alpha''(t) \leq u''(t) \leq \beta''(t), -R < u'''(t) < R, \quad \forall t \in [0, +\infty).$$

Proof. Integrating (3.1) and (2.9), we have $\alpha'(t) \leq \beta'(t)$ and $\overline{\alpha}(t) \leq \overline{\beta}(t)$, for $t \in [0, +\infty)$. Therefore we can consider the modified and perturbed equation

$$\begin{aligned} u^{(4)}(t) &= f(t, \delta_0(t, u), \delta_1(t, u'), \delta_2(t, u''), \delta_3(t, u''')) \\ &\quad + \frac{1}{1+t^2} \frac{u''(t) - \delta_2(t, u'')}{1 + |u''(t) - \delta_2(t, u'')|}, \quad t \in [0, +\infty), \end{aligned} \tag{3.3}$$

where the functions $\delta_j : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, j = 0, 1, 2, 3$ are given by

$$\delta_0(t, x) = \begin{cases} \overline{\beta}(t), & x > \overline{\beta}(t) \\ x, & \overline{\alpha}(t) \leq x \leq \overline{\beta}(t) \\ \overline{\alpha}(t), & x < \overline{\alpha}(t), \end{cases}$$

$$\delta_i(t, y_i) = \begin{cases} \beta^{(i)}(t), & y_i > \beta^{(i)}(t) \\ y_i, & \alpha^{(i)}(t) \leq y_i \leq \beta^{(i)}(t) \\ \alpha^{(i)}(t), & y_i < \alpha^{(i)}(t), \end{cases} \qquad i = 1, 2,$$

$$\delta_3(t,w)=\begin{cases} R, & w>R\\ w, & -R\le w\le R\\ -R, & w<-R.\end{cases}$$

For clearness, we do the proof in several steps.

Step 1: Every solution of (3.3), (1.2) satisfies $\alpha''(t)\le u''(t)\le \beta''(t)$ for all $t\in[0,+\infty)$. Let u be a solution of the modified problem (3.3), (1.2) and suppose, by contradiction, that there exists $t\in(0,+\infty)$ such that $\alpha''(t)>u''(t)$. Therefore

$$\inf_{0\le t<+\infty}(u''(t)-\alpha''(t))<0.$$

By (2.9) this infimum can not be attained at $+\infty$. If

$$\inf_{0\le t<+\infty}(u''(t)-\alpha''(t)):=u''(0^+)-\alpha''(0^+)<0,$$

then the following contradiction is achieved

$$\begin{aligned} 0\le u'''(0^+)-\alpha'''(0^+)&=\frac{C-u''(0)}{a}+\frac{\alpha''(0)-C}{a}\\ &=-\frac{1}{a}(u''(0)-\alpha''(0))<0.\end{aligned}$$

If there is $t_*\in(0,+\infty)$ then, we can define

$$\min_{0\le t<+\infty}(u''(t)-\alpha''(t)):=u''(t_*)-\alpha''(t_*)<0,$$

with $u'''(t_*)=\alpha'''(t_*)$ and $u^{(4)}(t_*)-\alpha^{(4)}(t_*)\ge 0$. Therefore by (3.2) and Definition 2.6 we get the contradiction

$$\begin{aligned} 0&\le u^{(4)}(t_*)-\alpha^{(4)}(t_*)\\ &=f(t_*,\delta_0(t_*,u(t_*)),\delta_1(t_*,u'(t_*)),\delta_2(t_*,u''(t_*)),\delta_3(t_*,u'''(t_*)))\\ &\quad+\frac{1}{1+t_*^2}\frac{u''(t_*)-\delta_2(t_*,u''(t_*))}{1+|u''(t_*)-\delta_2(t_*,u''(t_*))|}-\alpha^{(4)}(t_*)\\ &=f(t_*,\delta_0(t_*,u(t_*)),\delta_1(t_*,u'(t_*)),\alpha''(t_*),\alpha'''(t_*))\\ &\quad+\frac{1}{1+t_*^2}\frac{u''(t_*)-\alpha''(t_*)}{1+|u''(t_*)-\alpha''(t_*)|}-\alpha^{(4)}(t_*)\\ &\le\frac{1}{1+t_*^2}\frac{u''(t_*)-\alpha''(t_*)}{1+|u''(t_*)-\alpha''(t_*)|}<0.\end{aligned}$$

So $u''(t)\ge\alpha''(t),\forall t\in[0,+\infty)$. Analogously it can be shown that $u''(t)\le\beta''(t),\forall t\in[0,+\infty)$.

As $\alpha'(0)\le B\le\beta'(0)$ and $u'(0)=B$, integrating on $[0,+\infty)$,

$$\begin{aligned}\alpha'(t)-\alpha'(0)=\int_0^t\alpha''(s)ds&\le\int_0^t u''(s)ds=u'(t)-B\\ &\le\int_0^t\beta''(s)ds=\beta'(t)-\beta'(0),\end{aligned}$$

$$\alpha'(t)\le\alpha'(t)-\alpha'(0)+B\le u'(t)\le\beta'(t)-\beta'(0)+B\le\beta'(t),$$

$$\alpha(t)-\alpha(0)=\int_0^t\alpha'(s)ds\le\int_0^t u'(s)ds=u(t)-A\le\int_0^t\beta'(s)ds=\beta(t)-\beta(0),$$

$$\overline{\alpha}(t)\le u(t)\le\overline{\beta}(t).$$

Step 2: Problem (3.3), (1.2) has at least one solution. Let us define the operator $T : X \to X$ by

$$Tu(t) = A + Bt + \frac{C - aD}{2}t^2 + \frac{D}{6}t^3 - \int_0^{+\infty} G(t,s)F(u(s))ds,$$

with

$$\begin{aligned} F(u(s)) &:= f(s, \delta_0(s,u), \delta_1(s,u'), \delta_2(s,u''), \delta_3(s,u''')) \\ &\quad + \frac{1}{1+s^2}\frac{u''(s) - \delta_2(s,u'')}{1+|u''(s) - \delta_2(s,u'')|}. \end{aligned}$$

By Lemma 2.2, the fixed points of T are solutions of problem (3.3), (1.2). So it is sufficient to prove that T has a fixed point.

(i) $T : X \to X$ is well defined. Let $u \in X$. As f is an L^1-Carathéodory function, so, for

$$\rho > \max\{\|u\|_X, \|\alpha\|_X, \|\beta\|_X\},$$

we obtain

$$\begin{aligned} \int_0^{+\infty} |F(u(s))|ds &\le \int_0^{+\infty} \phi_\rho(s) + \frac{1}{1+s^2}\frac{u''(s) - \delta_2(s,u'')}{1+|u''(s) - \delta_2(s,u'')|}ds \\ &\le \int_0^{+\infty} \phi_\rho(s) + \frac{1}{1+s^2}ds < +\infty, \end{aligned}$$

that is, F is also an L^1-Carathéodory function.

By the Lebesgue Dominated Convergence Theorem,

$$\begin{aligned} \lim_{t\to+\infty} \frac{(Tu)(t)}{1+t^3} &= \frac{D}{6} - \int_0^{+\infty} \lim_{t\to+\infty} \frac{G(t,s)}{1+t^3}F(u(s))ds \\ &= \frac{D}{6} - \frac{1}{6}\int_0^{+\infty} F(u(s))ds < +\infty, \end{aligned}$$

and analogously for

$$\lim_{t\to+\infty} \frac{(Tu)'(t)}{1+t^2}, \quad \lim_{t\to+\infty} \frac{(Tu)''(t)}{1+t} \text{ and } \lim_{t\to+\infty} \frac{(Tu)'''(t)}{2}.$$

Therefore $Tu \in X$.

(ii) T is continuous. For any convergent sequence $u_n \to u$ in X, there exists $\rho > 0$ such that $\sup_n \|u_n\|_X < \rho$, we have

$$\begin{aligned} \|Tu_n - Tu\|_X &= \max\Big\{\|Tu_n - Tu\|, \|(Tu_n)' - (Tu)'\|, \\ &\quad \|(Tu_n)'' - (Tu)''\|, \|(Tu_n)''' - (Tu)'''\|\Big\} \\ &\le \int_0^{+\infty} M|F(u_n(s)) - F(u(s))|ds \to 0, \\ &quadn \to +\infty, \end{aligned}$$

where

$$\begin{aligned} M = \max\Big\{&\sup_{0\le t<+\infty} |\frac{G(t,s)}{1+t^3}|, \sup_{0\le t<+\infty} |\frac{1}{1+t^2}\frac{\partial G(t,s)}{\partial t}|, \\ &\sup_{0\le t<+\infty} |\frac{1}{1+t}\frac{\partial^2 G(t,s)}{\partial t^2}|, 1\Big\}. \end{aligned}$$

(iii) T is compact. Let $B \subset X$ be any bounded subset, therefore there is $R > 0$ such that $\|u\|_X < R$ for all $u \in B$. For each $u \in B$, one has

$$\begin{aligned}\|Tu\| &= \sup_{0\le t<+\infty} \frac{|Tu(t)|}{1+t^3} \le |A|+|B|+|C-aD|+|D| \\ &\quad + \int_0^{+\infty} \sup_{0\le t<+\infty} \frac{|G(t,s)|}{1+t^3}|F(u(s))|ds \\ &\le |A|+|B|+|C-aD|+|D| + \int_0^{+\infty} M(\phi_R(s)+\frac{1}{1+s^2}) < +\infty.\end{aligned}$$

By the same arguments it can be proved that $\|(Tu)^{(i)}\| < +\infty$, for $i = 1, 2, 3$, and, therefore,

$$\|Tu\|_X = \max\{\|Tu\|, \|(Tu)'\|, \|(Tu)''\|, \|(Tu)'''\|\} < +\infty,$$

that is, TB is uniformly bounded.

TB is equicontinuous, because, for $L > 0$ and $t_1, t_2 \in [0, L]$, we have

$$\begin{aligned}|\frac{Tu(t_1)}{1+t_1^3} - \frac{Tu(t_2)}{1+t_2^3}| &\le |\frac{A+Bt_1+\frac{C-aD}{2}t_1^2+\frac{D}{6}t_1^3}{1+t_1^3} - \frac{A+Bt_2+\frac{C-aD}{2}t_2^2+\frac{D}{6}t_2^3}{1+t_2^3}| \\ &\quad + \int_0^{+\infty} |\frac{G(t_1,s)}{1+t_1^3} - \frac{G(t_2,s)}{1+t_2^3}||F(u(s))|ds \\ &\le |\frac{A+Bt_1+\frac{C-aD}{2}t_1^2+\frac{D}{6}t_1^3}{1+t_1^3} - \frac{A+Bt_2+\frac{C-aD}{2}t_2^2+\frac{D}{6}t_2^3}{1+t_2^3}| \\ &\quad + \int_0^{+\infty} |\frac{G(t_1,s)}{1+t_1^3} - \frac{G(t_2,s)}{1+t_2^3}|(\phi_R(s)+\frac{1}{1+s^2})ds \to 0,\end{aligned}$$

as $t_1 \to t_2$. By the same technique one shows that

$$|\frac{(Tu)^{(i)}(t_1)}{1+t_1^{3-i}} - \frac{(Tu)^{(i)}(t_2)}{1+t_2^{3-i}}| \to 0,$$

uniformly, for $i = 1, 2, 3$, as $t_1 \to t_2$.

Moreover TB is equiconvergent at infinity, because

$$\begin{aligned}&|\frac{Tu(t)}{1+t^3} - \lim_{t\to+\infty} \frac{Tu(t)}{1+t^3}| \\ &\le |\frac{A+Bt+\frac{C-aD}{2}t^2+\frac{D}{6}t^3}{1+t^3} - \frac{D}{6}| + \int_0^{+\infty} |\frac{G(t,s)}{1+t^3} - \frac{1}{6}||F(u(s))|ds \\ &\le |\frac{A+Bt+\frac{C-aD}{2}t^2+\frac{D}{6}t^3}{1+t^3} - \frac{D}{6}| + \int_0^{+\infty} |\frac{G(t,s)}{1+t^3} - \frac{1}{6}|(\phi_{\rho_1}+\frac{1}{1+s^2})ds \to 0,\end{aligned}$$

as $t \to +\infty$, and analogously for

$$|\frac{(Tu)^{(i)}(t)}{1+t^{3-i}} - \lim_{t\to+\infty} \frac{(Tu)^{(i)}(t)}{1+t^{3-i}}| \to 0, \quad \text{as } t \to +\infty.$$

So, by Lemma 2.5, the set TB is relatively compact.

As T is completely continuous then by Schauder Fixed Point Theorem, T has at least one fixed point $u \in X$. □

4. Example

Consider the fourth-order differential equation

$$(1+t^2)u^{(4)}(t) = -u(t)|u'''(t)-6|e^{u'''(t)} - e^{-t}(6t+2-u''(t)), \quad t>0, \tag{4.1}$$

with the boundary conditions

$$u(0)=A,\ u'(0)=0, \quad u''(0)+au'''(0)=0, \quad u'''(+\infty)=D, \tag{4.2}$$

where $A\geq 0$, $-\frac{1}{3}\leq a<0$ and $0<D<6$.

We remark that the above problem is a particular case of (1.1), (1.2) with $B=C=0$ and

$$f(t,x,y,z,w) = \frac{-x|w-6|e^{w} - e^{-t}(6t+2-z)}{1+t^2}. \tag{4.3}$$

Moreover the functions $\alpha(t)\equiv 0$ and $\beta(t)=t^3+t^2-1$ are, respectively, non ordered lower and upper solutions for (4.1), (4.2)), with $\overline{\alpha}(t)=A$ and $\overline{\beta}(t)=t^3+t^2+A$, f satisfies the Nagumo condition (2.3) with

$$\psi(t)=\frac{1}{1+t^2}, 1<\nu<2, h(|w|)\equiv 1,$$

on

$$\begin{aligned} E_0 = \Big\{&(t,x,y,z,w)\in[0,+\infty)\times\mathbb{R}^4 : A\leq x\leq t^3+t^2+A,\ 0\leq y\leq 3t^2+2t,\\ &0\leq z\leq 6t+2\Big\},\end{aligned}$$

and satisfies the assumptions of Theorem 3.1.

Therefore, there is at least a non trivial solution u of (4.1), (4.2), and $R>0$, such that

$$\begin{gathered} A\leq u(t)\leq t^3+t^2+A, \quad 0\leq u'(t)\leq 3t^2+2t,\\ 0\leq u''(t)\leq 6t+2, \quad \|u'''\|_0\leq R, \quad \forall t\in[0,+\infty). \end{gathered}$$

We remark that, this solution is unbounded and, from the location part, we notice that u is nondecreasing and convex. It is important to stress that the nonlinearity (4.3) does not satisfy the usual two-sided Nagumo-type condition. In fact, if there exist $\psi_0, h_0\in C(\mathbb{R}_0^+,\mathbb{R}^+)$ satisfying

$$|f(t,x,y,z,w)|\leq \psi_0(t)h_0(|w|), \quad \forall (t,x,y,z,w)\in E_0,$$

with $\int_0^{+\infty}\frac{s}{h_0(s)}ds=+\infty$, then, in particular,

$$-f(t,x,y,z,w)\leq \psi_0(t)h_0(|w|),$$

and, for $t\in[0,+\infty)$, $x=1$, $0\leq y\leq 3t^2+2t$, $z=6t+2$, and $w\in\mathbb{R}$,

$$-f(t,1,y,6t+2,w)=\frac{|w-6|e^{w}}{1+t^2}\leq \psi_0(t)h_0(|w|),$$

For $\psi_0(t)=1/(1+t^2)$ we have $|w-6|e^{w}\leq h_0(|w|)$ and the following contradiction holds

$$+\infty > \int_0^{+\infty}\frac{s}{(s-6)e^s}ds \geq \int_0^{+\infty}\frac{s}{h_0(s)}ds = +\infty.$$

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