Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 193, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

QUALITATIVE PROPERTIES OF A THIRD-ORDER DIFFERENTIAL EQUATION WITH A PIECEWISE CONSTANT ARGUMENT

HUSEYIN BEREKETOGLU, MEHTAP LAFCI, GIZEM S. OZTEPE

Abstract. We consider a third order differential equation with piecewise constant argument and investigate oscillation, nonoscillation and periodicity properties of its solutions.

1. Introduction

For many years, oscillation, non-oscillation and periodicity of third order differential equations have been investigated. Kim [15] studied oscillation properties of the equation

$$y''' + py'' + qy' + ry = 0,$$

where p, q and r are continuous on an interval. Tryhuk [24] established sufficient conditions for the existence of two linearly independent oscillatory solutions of the third order differential equation

$$y''' + p(t)y' + q(t)y = 0.$$

Cecchi [6] investigated the oscillatory behavior of the linear third-order differential equation of the form

$$y''' + p(x)y' + q(x)y = 0,$$

where the function $p(x)$ changes sign on the positive x-axis. Parhi and Das [18] considered the equation

$$(r(t)y'')' + q(t)y' + p(t)y = F(t),$$

and gave necessary and sufficient conditions for the existence of nonoscillatory or oscillatory solutions of this equation. In [19], they also investigated oscillatory and asymptotic properties of solutions of the equation

$$y''' + a(t)y'' + b(t)y' + c(t)y = 0,$$

where $a \in C^2$, $b \in C^1$, $c \in C^0$, $a(t), b(t), c(t) \le 0$ eventually and $b(t) \ne 0, c(t) \ne 0$ on any interval of positive measure. The oscillation of the solutions of

$$(b(t)(a(t)y'(t))')' + (q_1(t)y(t))' + q_2(t)y'(t) = 0$$

2010 *Mathematics Subject Classification.* 34K11.
Key words and phrases. Third order differential equation; piecewise constant argument; oscillation; periodicity.
©2017 Texas State University.
Submitted July 7, 2016. Published August 4, 2017.

and
$$(b(t)(a(t)y'(t))')' + q_1(t)y(t) + q_2(t))y(\tau(t)) = 0$$
was studied in [8] by Dahiya. Adamets and Lomtatidze [1] analyzed oscillatory properties of solutions of the third-order differential equation $u''' + p(t)u = 0$, where p is a locally integrable function on $[0, \infty)$ which is eventually of one sign. Han, Sun and Zhang [13] deduced new sufficient conditions which guarantee that every solution x of the delayed third order differential equation
$$(x(t) - a(t)x(\tau(t)))''' + p(t)x(\delta(t)) = 0$$
is either oscillatory or tends to zero. In [9], the authors stated necessary and sufficient conditions for the oscillation of the third-order nonhomogenous differential equation
$$y''' + a(t)y'' + b(t)y' + c(t)y = f(t),$$
under certain conditions given in terms of differentiability, continuity and signs of the coefficient functions and their derivatives. [10] was dedicated to studying the nonoscillatory solutions of the equation with mixed arguments
$$(a(t)(x'(t))^{\gamma})'' = q(t)f(x(\tau(t))) + p(t)g(x(\sigma(t))),$$
where $\tau(t) < t$, $\sigma(t) > t$. In 2015, Bartušek and Došlá [2] gave conditions under which every solution of the equation
$$x'''(t) + q(t)x'(t) + r(t)|x|^{\lambda}(t)\operatorname{sgn} x(t) = 0, \quad t \geq 0,$$
is either oscillatory or tends to zero. They also studied Kneser solutions vanishing at infinity and the existence of oscillatory solutions. Shoukaku [21] considered
$$y'''(t) - a(t)y''(t) - b(t)y'(t) - \sum_{i=1}^{m} c_i(t)y(\sigma_i(t)) = 0$$
using Riccati inequality. Ezeilo [11] studied the equation
$$x''' + ax'' + bx' + h(x) = p(t),$$
where a, b are constants, $p(t)$ is continuous and periodic with least period w. Using the Leray-Schauder technique, under certain conditions on a, b, h, p, the author guaranteed the existence of one solution of this equation with least period w. Tabueva In [22] studied the existence of a periodic solution of
$$x''' + \alpha x'' + \beta x' + \sin x = e(t).$$
Ezeilo [12] showed that the equation
$$x''' + \psi(x')x'' + \phi(x)x' + \theta(x) = p(t) + q(t, x, x')$$
has an w-periodic solution, where ψ, ϕ, θ, p and q are continuous in their respective arguments and p, q have a given period w, $w > 0$, in t. In 1979, Tejumola [23] proved the existence of at least one w periodic solution of the third order differential equation
$$x''' + f(x')x'' + g(x)x' + h(x) = p(t, x, x', x''),$$
where p is w-periodic in its first argument. In [25], the author gave a theorem on the existence of 2π-periodic solutions of the nonlinear third order differential equation with multiple deviating arguments
$$c(t)x'''(t) + \sum_{i=0}^{2}[a_i(x^{(i)})^{2k-1} + b_i(x^{(i)})^{2k-1}(t - \tau_i)] + g(t, x(t - \tau(t)), x'(t - \tau_3)) = p(t),$$

where $a_i, b_i (i = 0, 1, 2)$ and $\tau_i (i = 0, 1, 2, 3)$ are constants, k is a positive integer. Chen and Pan [7] proved sufficient conditions for the existence of periodic solutions of third order differential equations with deviating arguments of the type

$$x'''(t) + ax''(t - \tau_2(t)) + bx'(t - \tau_1(t)) + cx(t) + f(t, x(t - \tau(t)) = p(t).$$

As far as we know, there are some papers on the third-order differential equations with piecewise constant arguments. The oldest one was published in 1994 by Papaschinopoulos and Schinas [17]. They considered the equation

$$(y(t) + py(t - 1))''' = -qy(2[\frac{t + 1}{2}])$$

and proved existence, uniqueness and asymptotic stability of the solutions. Here $t \in [0, \infty)$, p, q are real constants and $[\cdot]$ denotes the greatest integer function. Liang and Wang [16] stated several sufficient conditions which insure that any solution of the equation

$$(r_2(t)(r_1(t)x'(t))')' + p(t)x'(t) + f(t, x([t])) = 0, \quad t \ge 0$$

oscillates or converges to zero. Shao and Liang [20] established sufficient conditions for the oscillation and asymptotic behaviour of the equation

$$(r(t)x''(t))' + f(t, x([t])) = 0.$$

In [5], the authors showed that every solution $x(t)$ of a third-order nonlinear differential equation with piecewise constant arguments of the type

$$(r_2(t)(r_1(t)x'(t))')' + p(t)x'(t) + f(t, x([t - 1])) + g(t, x([t])) = 0$$

oscillates or converges to zero, where $t \ge 0$, $r_1(t)$, $r_2(t)$ are continuous on $[0, \infty)$ with $r_1(t)$, $r_2(t) > 0$ and $r_1'(t) \ge 0$, $p(t)$ is continuously differentiable on $[0, \infty)$ with $p(t) \ge 0$.

On the other hand, the first and third authors considered an impulsive first order delay differential equation with piecewise constant argument in [14]. They investigated its oscillatory and periodic solutions. Then in [3], the same authors studied the oscillation, nonoscillation, periodicity and global asymptotic stability of an advanced type impulsive first order nonhomogeneous differential equation with piecewise constant arguments. Also, in 2011, oscillation, nonoscillation and periodicity of a second order

$$x''(t) - a^2x(t) = bx([t - 1]) + cx([t] + dx([t + 1]$$

differential equation with mixed type piecewise constant arguments were investigated [4].

In this paper, we extend our results on oscillation, nonoscillation and periodicity of solutions to first and second order linear differential equations with piecewise constant arguments to a third order linear differential equation with piecewise constant argument. For this purpose, we consider the following third order linear differential equation with a piecewise constant argument

$$x'''(t) - a^2x'(t) = bx([t - 1]) \tag{1.1}$$

with the initial conditions

$$x(-1) = \alpha_{-1}, x(0) = \alpha_0, x'(0) = \alpha_1, x''(0) = \alpha_2, \tag{1.2}$$

where $a \ne 0$ and $a, b, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2 \in \mathbb{R}$.

2. Existence and uniqueness

First we give the definition of a solution to (1.1). Then we use the technique in [26] to investigate the solution of this equation.

A function $x(t)$ defined on $[0,\infty)$ is said to be a solution of the initial value problem (1.1)–(1.2) if it satisfies the following conditions:

- (i) x is continuous on $[0,\infty)$,
- (ii) x'' exists and continuous on $[0,\infty)$,
- (iii) x''' exists on $[0,\infty)$ with the possible exception of the points $[t]\in[0,\infty)$, where one-sided derivatives exist,
- (iv) x satisfies (1.1) on each interval $[n,n+1)$ with $n\in N$.

Theorem 2.1. *Equation* (1.1) *has a solution on* $[0,\infty)$.

Proof. Let $x_n(t)$ be a solution of (1.1) on the interval $[n,n+1)$ with the conditions

$$x(n)=c_n,\ x(n-1)=c_{n-1},\ x'(n)=d_n,\ x''(n)=e_n.$$

Then (1.1) reduces to

$$x'''(t)-a^2x'(t)=bx(n-1).$$

The solution of the above equation is found as

$$x_n(t)=K_n+L_n\cosh a(t-n)+M_n\sinh a(t-n)-\frac{b}{a^2}tx_n(n-1) \tag{2.1}$$

with arbitrary constants $K_n,\ L_n$ and M_n. Writing $t=n$ in (2.1), we obtain

$$c_n=K_n+L_n-\frac{b}{a^2}nc_{n-1}. \tag{2.2}$$

If we take $t=n$ in the first and second derivatives of (2.1), respectively, we find

$$M_n=\frac{d_n}{a}+\frac{b}{a^3}c_{n-1},\quad L_n=\frac{e_n}{a^2}. \tag{2.3}$$

From (2.2) and (2.3),

$$K_n=c_n-\frac{e_n}{a^2}+\frac{b}{a^2}nc_{n-1} \tag{2.4}$$

is obtained. Substituting (2.3) and (2.4) in (2.1), we have

$$\begin{aligned}x_n(t)=&\frac{-1+\cosh a(t-n)}{a^2}e_n+\frac{\sinh a(t-n)}{a}d_n+c_n\\&+[\frac{b}{a^2}n-\frac{b}{a^2}t+\frac{b}{a^3}\sinh a(t-n)]c_{n-1}.\end{aligned} \tag{2.5}$$

First and second derivatives of (2.5) are found as

$$x_n'(t)=\frac{\sinh a(t-n)}{a}e_n+\cosh a(t-n)d_n+[\frac{b}{a^2}\cosh a(t-n)-\frac{b}{a^2}]c_{n-1}, \tag{2.6}$$

$$x_n''(t)=\cosh a(t-n)e_n+a\sinh a(t-n)d_n+\frac{b}{a}\sinh a(t-n)c_{n-1}. \tag{2.7}$$

Writing $t=n+1$ in (2.5), (2.6) and (2.7), it follows that

$$c_{n+1}=c_n+\frac{\sinh a}{a}d_n+\big(\frac{\cosh a}{a^2}-\frac{1}{a^2}\big)e_n+\big(\frac{b\sinh a}{a^3}-\frac{b}{a^2}\big)c_{n-1}, \tag{2.8}$$

$$d_{n+1}=(\cosh a)d_n+\frac{\sinh a}{a}e_n+(\Big(\frac{b\cosh a}{a^2}-\frac{b}{a^2}\Big)c_{n-1}, \tag{2.9}$$

$$e_{n+1}=a(\sinh a)d_n+(\cosh a)e_n+\frac{b\sinh a}{a}c_{n-1}. \tag{2.10}$$

Now, let us introduce the vector $v_n = \mathrm{col}(c_n, d_n, e_n)$ and the matrices

$$A = \begin{pmatrix} 1 & \frac{\sinh a}{a} & \frac{\cosh a}{a^2} - \frac{1}{a^2} \\ 0 & \cosh a & \frac{\sinh a}{a} \\ 0 & a\sinh a & \cosh a \end{pmatrix}, \quad B = \begin{pmatrix} \frac{b\sinh a}{a^3} - \frac{b}{a^2} & 0 & 0 \\ \frac{b\cosh a}{a^2} - \frac{b}{a^2} & 0 & 0 \\ \frac{b\sinh a}{a} & 0 & 0 \end{pmatrix}$$

so we can rewrite the system (2.8)-(2.10) as

$$v_{n+1} = Av_n + Bv_{n-1}. \tag{2.11}$$

Looking for a nonzero solution of this difference equation system in the form of $v_n = k\lambda^n$, with a constant vector k, leads us to

$$\det(\lambda^2 I - \lambda A - B) = 0,$$

and characteristic equation

$$\begin{aligned} &\lambda^4 + (-1 - 2\cosh a)\lambda^3 + (1 + \frac{b}{a^2} - \frac{b}{a^3}\sinh a + 2\cosh a)\lambda^2 \\ &+ (-1 - \frac{2b}{a^2}\cosh a + \frac{2b}{a^3}\sinh a)\lambda + (\frac{b}{a^2} - \frac{b}{a^3}\sinh a) = 0. \end{aligned} \tag{2.12}$$

Assuming that these roots are simple, we write the general solution of (2.12),

$$v_n = \lambda_1^n k_1 + \lambda_2^n k_2 + \lambda_3^n k_3 + \lambda_4^n k_4, \tag{2.13}$$

where $v_n = col(c_n, d_n, e_n)$ and $k_j = col(k_{ij})$, $i = 1, 2, 3, 4$ which can be found from adequate initial or boundary conditions. If some λ_j is a multiple zero of (2.12), then the expression for v_n also includes products of λ_j^n by n, n^2 or n^3. Finally, the solution $x_n(t)$ is obtained by substituting the appropriate components of the vectors v_n and v_{n-1} in (2.5). □

Remark 2.2. From (2.8), (2.9) and (2.10), we obtain

$$\begin{aligned} d_n = &\frac{-a}{2\sinh a} c_{n+2} + \frac{a(1+\cosh a)}{\sinh a} c_{n+1} \\ &+ \frac{(-a^3 - 2a^3\cosh a - ab + b\sinh a)}{2a^2\sinh a} c_n \\ &+ \frac{ab + 2ab\cosh a - 3b\sinh a}{2a^2\sinh a} c_{n-1}, \end{aligned} \tag{2.14}$$

$$\begin{aligned} e_n = &\frac{-a^2}{2(1-\cosh a)} c_{n+2} + \frac{a^2\cosh a}{1-\cosh a} c_{n+1} \\ &- \frac{(-a^3 + 2a^3\cosh a + ab - b\sinh a)}{2a(1-\cosh a)} c_n \\ &- \frac{ab - 2ab\cosh a + b\sinh a}{2a(1-\cosh a)} c_{n-1}. \end{aligned} \tag{2.15}$$

Substituting (2.14) and (2.15) in (2.8), gives us the difference equation

$$\begin{aligned} &c_{n+3} + (-1 - 2\cosh a)c_{n+2} + (1 + \frac{b}{a^2} - \frac{b}{a^3}\sinh a + 2\cosh a)c_{n+1} \\ &+ (-1 - \frac{2b}{a^2}\cosh a + \frac{2b}{a^3}\sinh a)c_n + (\frac{b}{a^2} - \frac{b}{a^3}\sinh a)c_{n-1} = 0 \end{aligned} \tag{2.16}$$

whose characteristic equation is the same as (2.12).

Theorem 2.3. *The boundary-value problem for* (1.1) *with the conditions*

$$x(-1) = c_{-1}, x(0) = c_0, x(1) = c_1, x(N-1) = c_{N-1} \tag{2.17}$$

has a unique solution on $0 \le t < \infty$ *if* $N > 2$ *is an integer and both of the following hypotheses are satisfied:*

(i) *The roots of* (2.12) λ_j *(characteristic roots) are nontrivial and distinct,*

(ii) $\lambda_1^N \lambda_2\lambda_3\lambda_4(-\lambda_4^2\lambda_1\lambda_2 + \lambda_4\lambda_1\lambda_2^2 + \lambda_4^2\lambda_1\lambda_3 - \lambda_1\lambda_2^2\lambda_3 - \lambda_4\lambda_1\lambda_3^2 + \lambda_1\lambda_2\lambda_3^2)$
$+ \lambda_1\lambda_2\lambda_3^N\lambda_4(-\lambda_4^2\lambda_1\lambda_3 + \lambda_4\lambda_1^2\lambda_3 + \lambda_4^2\lambda_2\lambda_3 - \lambda_1^2\lambda_2\lambda_3 - \lambda_4\lambda_2^2\lambda_3 + \lambda_1\lambda_2^2\lambda_3)$
$\neq \lambda_1\lambda_2\lambda_3\lambda_4^N(-\lambda_4\lambda_1^2\lambda_2 + \lambda_4\lambda_1\lambda_2^2 + \lambda_4\lambda_1^2\lambda_3 - \lambda_4\lambda_2^2\lambda_3 - \lambda_4\lambda_1\lambda_3^2 + \lambda_4\lambda_2\lambda_3^2)$
$+ \lambda_1\lambda_2^N\lambda_3\lambda_4(-\lambda_4^2\lambda_1\lambda_2 + \lambda_4\lambda_1^2\lambda_3 + \lambda_4^2\lambda_2\lambda_3 - \lambda_1^2\lambda_2\lambda_3 - \lambda_4\lambda_2\lambda_3^2 + \lambda_1\lambda_2\lambda_3^2)$.

Proof. The first row of the vector equation (2.13) gives us

$$c_n = \lambda_1^n k_{11} + \lambda_2^n k_{21} + \lambda_3^n k_{31} + \lambda_4^n k_{41}. \tag{2.18}$$

We get following system by applying the boundary conditions (2.17) to (2.18), respectively.

$$\lambda_1^{-1}k_{11} + \lambda_2^{-1}k_{21} + \lambda_3^{-1}k_{31} + \lambda_4^{-1}k_{41} = c_{-1}, \tag{2.19}$$

$$k_{11} + k_{21} + k_{31} + k_{41} = c_0, \tag{2.20}$$

$$\lambda_1 k_{11} + \lambda_2 k_{21} + \lambda_3 k_{31} + \lambda_4 k_{41} = c_1, \tag{2.21}$$

$$\lambda_1^{N-1}k_{11} + \lambda_2^{N-1}k_{21} + \lambda_3^{N-1}k_{31} + \lambda_4^{N-1}k_{41} = c_{N-1}. \tag{2.22}$$

From hypothesis (ii), the determination of the coefficients of this system is different from zero. Hence, we can find k_{ij} and also c_n uniquely. Furthermore, once the values c_n have been found, we calculate d_n and e_n from (2.14) and (2.15), respectively. Substituting c_n, d_n, e_n in (2.5), the unique solution $x_n(t)$ is obtained. □

The following four theorems depend on the characteristic roots. Their proofs are omitted because they are very similar to the proof of Theorem 2.3.

Theorem 2.4. *Let us assume that all characteristic roots are nontrivial and two of them are equal* ($\lambda_1 = \lambda_2$)*, others are different from each other* ($\lambda_3 \neq \lambda_4$)*. If*

$$\begin{aligned}
&\lambda_1^N \Big[(1-N)\lambda_1\lambda_3\lambda_4^3 + (N-2)\lambda_1^2\lambda_3\lambda_4^2 + N\lambda_4^3\lambda_3^2 + (2-N)\lambda_1^2\lambda_3^2\lambda_4 \\
&- N\lambda_4^2\lambda_3^3 + (N-1)\lambda_1\lambda_3^3\lambda_4\Big] \\
&\neq \lambda_3^N\big[\lambda_1\lambda_3\lambda_4^3 - 2\lambda_1^2\lambda_3\lambda_4^2 + \lambda_1^3\lambda_3\lambda_4\big] + \lambda_4^N\big[2\lambda_1^2\lambda_3^2\lambda_4 - \lambda_1\lambda_3^3\lambda_4 - \lambda_1^3\lambda_3\lambda_4\big],
\end{aligned}$$

then the boundary-value problem for (1.1) *with the conditions* (2.17) *has a unique solution on* $[0, \infty)$.

Theorem 2.5. *If the characteristic roots* λ_j *are nontrivial,* $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4$ *and*

$$\begin{aligned}
&\lambda_1^N\Big[(N-2)\lambda_1^2\lambda_2 + 2(1-N)\lambda_1\lambda_2^2 + N\lambda_2^3\Big] \\
&\neq \lambda_2^N\Big[(2-N)\lambda_1\lambda_2^2 + 2(N-1)\lambda_1^2\lambda_2 + N\lambda_1^3\Big],
\end{aligned}$$

then the boundary-value problem for (1.1) *with the conditions* (2.17) *has a unique solution on* $[0, \infty)$.

Theorem 2.6. *If the characteristic roots λ_j are nontrivial, $\lambda_1 = \lambda_2 = \lambda_3$ and*

$$\begin{aligned}\lambda_1^N\Big[&(-N^2+3N-2)\lambda_1^3\lambda_2 + (2N^2-5N+2)\lambda_1^2\lambda_2^2 + (2-N)\lambda_1^4\lambda_2^2 \\ &+ (-N^2+2N-1)\lambda_1\lambda_2^3 + (N-1)\lambda_1^3\lambda_2^3\Big] \neq \lambda_2^N\Big[\lambda_1^5\lambda_2 - \lambda_1^3\lambda_2\Big],\end{aligned}$$

then the boundary-value problem for (1.1) *with the conditions* (2.17) *has a unique solution on* $[0,\infty)$.

Theorem 2.7. *If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda$ and*

$$2N - 3N^2 + N^3 + (-2N + 3N^2 - N^3)\lambda^2 \neq 0,$$

then the boundary-value problem for (1.1) *with the conditions* (2.17) *has a unique solution on* $[0,\infty)$.

3. Main results

This section deals with the oscillation, nonoscillation and the periodicity of the solutions of (1.1). Also, we give an example to illustrate our results.

Theorem 3.1. *If*

$$0 < \frac{b}{a^2} < \frac{1 + 4\cosh a}{2\cosh a - 2},$$

then there exist oscillatory solutions of (1.1).

Proof. Equation (2.12) can be written as a polynomial of λ,

$$f(\lambda) = \lambda^4 + \beta_1\lambda^3 + \beta_2\lambda^2 + \beta_3\lambda + \beta_4, \tag{3.1}$$

where

$$\begin{aligned}\beta_1 &= -1 - 2\cosh a, \\ \beta_2 &= 1 + \frac{b}{a^2} - \frac{b}{a^3}\sinh a + 2\cosh a, \\ \beta_3 &= -1 - \frac{2b}{a^2}\cosh a + \frac{2b}{a^3}\sinh a, \\ \beta_4 &= \frac{b}{a^2} - \frac{b}{a^3}\sinh a.\end{aligned} \tag{3.2}$$

To prove the oscillation of solutions, we need to show that there exists a unique negative root of the characteristic equation (2.12). For this reason let us take the polynomial

$$f(-\lambda) = \lambda^4 - \beta_1\lambda^3 + \beta_2\lambda^2 - \beta_3\lambda + \beta_4.$$

Now, if hypothesis is true, then we find that

$$\beta_1 < 0, \beta_2 > 0, \beta_3 < 0, \beta_4 < 0.$$

By using Descartes' rule of signs, we conclude that there exists a unique negative root of (2.12). Let us take λ_1 as this root. Now, consider the following boundary conditions

$$x(0) = c_0, x(-1) = c_{-1} = c_0\lambda_1^{-1}, x(1) = c_1 = c_0\lambda_1, x(2) = c_2 = c_0\lambda_1^2.$$

Applying these conditions to (2.18), the coefficients k_{i1}, $i = 1,2,3,4$ are found as

$$k_{11} = c_0, \quad k_{21} = k_{31} = k_{41} = 0$$

and therefore, (2.18) becomes $c_n = x(n) = c_0\lambda_1^n$. Since $\lambda_1 < 0$, we see that

$$x(n)x(n+1) = \lambda_1 c_0^2\lambda_1^{2n} < 0, c_0 \neq 0$$

and so the solution $x(t)$ of (1.1) has a zero in each interval $(n, n+1)$. So there exist oscillatory solutions. □

Theorem 3.2. *If*

$$0 < b < \frac{a^3(1 + 2\cosh a)}{\sinh a - a} \tag{3.3}$$

or

$$b < \frac{a^3}{2(\sinh a - a\cosh a)} \tag{3.4}$$

is satisfied, then there exist nonoscillatory solutions of (1.1).

Proof. From (3.3), we have $\beta_1 < 0, \beta_2 > 0, \beta_3 < 0, \beta_4 < 0$. Also, we obtain $\beta_1 < 0, \beta_2 > 0, \beta_3 > 0, \beta_4 < 0$ by using (3.4) where $\beta_1, \beta_2, \beta_3, \beta_4$ are given by (3.2). So, from Descartes' rule of sign, if any of the above conditions is satisfied, then we obtain that the characteristic equation (2.12) has at least one positive root. Therefore, there are nonoscillatory solutions of (1.1). □

Theorem 3.3. *If*

$$b > \frac{a^3(1 + 2\cosh a)}{\sinh a - a}, \tag{3.5}$$

then there exist both oscillatory and nonoscillatory solutions of (1.1).

Proof. Condition (3.5) implies that $\beta_1 < 0$, $\beta_2 < 0$, $\beta_3 < 0$, $\beta_4 < 0$, where β_1, β_2, β_3, β_4 are given by (3.2). Hence, from Descartes' rule of sign, we conclude that there exists a single positive root of (2.12). So, the other roots are negative or complex. Positive root generates nonoscillatory solutions, and others give us the oscillatory solutions of (1.1). □

Theorem 3.4. *A necessary and sufficient condition for the solution of problem* (1.1)-(1.2) *to be k periodic, $k \in N - \{0\}$, is*

$$c(k) = c(0), \quad c(k-1) = c(-1), \quad d(k) = d(0), \quad e(k) = e(0). \tag{3.6}$$

Here $\{c(n)\}_{n\geq -1}$ is the solution of (2.16) *with the initial conditions*

$$c(-1) = \alpha_{-1}, \quad c(0) = \alpha_0, \quad d(0) = \alpha_1, \quad e(0) = \alpha_2.$$

Proof. In this proof, we use technique in [14]. If $x(t)$ is periodic with period k, then $x(t+k) = x(t)$ for $t \in [0, \infty)$. This implies that the equalities (3.6) is true.

For the proof of sufficiency case, suppose that (3.6) is satisfied. From (2.5),

$$\begin{aligned} x_k(t) &= \frac{-1 + \cosh a(t-k)}{a^2} e_k + \frac{\sinh a(t-k)}{a} d_k + c_k \\ &\quad + \Big(\frac{b}{a^2} k - \frac{b}{a^2} t + \frac{b}{a^3} \sinh a(t-k)\Big) c_{k-1}, \quad k \leq t < k+1, \end{aligned} \tag{3.7}$$

$$\begin{aligned} x_0(t) &= \frac{-1 + \cosh at}{a^2} e_0 + \frac{\sinh at}{a} d_0 + c_0 \\ &\quad + \Big(-\frac{b}{a^2} t + \frac{b}{a^3} \sinh at\Big) c_{-1}, \quad 0 \leq t < 1. \end{aligned} \tag{3.8}$$

So $x_k(t) = x_0(t-k)$, $k \le t < k+1$. Moreover

$$\begin{aligned} &x_{k+1}(t)\\ &= \frac{-1+\cosh a(t-(k+1))}{a^2} e_{k+1} + \frac{\sinh a(t-(k+1))}{a} d_{k+1} + c_{k+1} \\ &\quad + \Big(\frac{b}{a^2}(k+1) - \frac{b}{a^2}t + \frac{b}{a^3}\sinh a(t-(k+1))\Big) c_k, \quad k+1 \le t < k+2, \end{aligned} \tag{3.9}$$

$$\begin{aligned} x_1(t) = \frac{-1+\cosh a(t-1)}{a^2} e_1 + \frac{\sinh a(t-1)}{a} d_1 + c_1 \\ + (\frac{b}{a^2} - \frac{b}{a^2}t + \frac{b}{a^3}\sinh a(t-1)) c_0, \quad 1 \le t < 2. \end{aligned} \tag{3.10}$$

To show that

$$x_{k+1}(t) = x_1(t-k), \tag{3.11}$$

we need to show that

$$c(k+1) = c(1), \quad d(k+1) = d(1), \quad e(k+1) = e(1). \tag{3.12}$$

For this purpose, by using the continuity at $t = k+1$, we obtain

$$x_k(k+1) = x_{k+1}(k+1), \quad k \le t < k+1.$$

Here, $x_k(k+1)$ and $x_{k+1}(k+1)$ are obtained by taking $t = k+1$ in (3.7) and (3.9), respectively. Hence

$$x(k+1) = \frac{-1+\cosh a}{a^2} e_k + \frac{\sinh a}{a} d_k + c_k + \Big(\frac{-b}{a^2} + \frac{b}{a^3}\sinh a\Big) c_{k-1} \tag{3.13}$$

where $x(k+1) = x_{k+1}(k+1)$.

By using the same procedure at $t = 1$, we find

$$x(1) = \frac{-1+\cosh a}{a^2} e_0 + \frac{\sinh a}{a} d_0 + c_0 \Big(\frac{-b}{a^2} + \frac{b}{a^3}\sinh a\Big) c_{-1}. \tag{3.14}$$

Considering (3.6) in (3.13) and (3.14), we obtain $x(k+1) = x(1)$, that is

$$c(k+1) = c(k).$$

Taking the derivatives of (3.7) and (3.8) gives us

$$\begin{aligned} x_k'(t) = \frac{\sinh a(t-k)}{a} e_k + \cosh a(t-k) d_k \\ + \Big(\frac{b}{a^2}\cosh a(t-k) - \frac{b}{a^2}\Big) c_{k-1}, \quad k \le t < k+1, \end{aligned} \tag{3.15}$$

$$x_0'(t) = \frac{\sinh at}{a} e_0 + (\cosh at) d_0 + \Big(\frac{b}{a^2}\cosh at - \frac{b}{a^2}\Big) c_{-1}, \quad 0 < t < 1. \tag{3.16}$$

Using continuity at $t = k+1$ and $t = 1$ in (3.15) and (3.16), we find

$$x'(k+1) = \frac{\sinh a}{a} e_k + (\cosh a) d_k + (\frac{b}{a^2}\cosh a - \frac{b}{a^2}) c_{k-1}, \quad k \le t < k+1, \tag{3.17}$$

$$x'(1) = \frac{\sinh a}{a} e_0 + (\cosh a) d_0 + (\frac{b}{a^2}\cosh a - \frac{b}{a^2}) c_{-1}, \quad 0 < t < 1. \tag{3.18}$$

Considering (3.6) in (3.17) and (3.18), we have $x'(k+1) = x'(1)$ i.e.

$$d(k+1) = d(1).$$

Similarly, from the continuity at $t = k+1$ and $t = 1$ in the following derivatives

$$x_k''(t) = (\cosh a(t-k))e_k + a(\sinh a(t-k))d_k + \frac{b}{a}(\sinh a(t-k))c_{k-1}, \quad k \le t < k+1,$$

$$x_0''(t) = (\cosh at)e_0 + (a \sinh at)d_0 + \frac{b}{a}(\sinh at)c_{-1}, 0 < t < 1,$$

of (3.15) and (3.16), we obtain $e(k+1) = e(1)$. Therefore, we find (3.12). By induction, $x_{k+n}(t) = x_n(t-k)$. □

As an example we consider the differential equation

$$x'''(t) - x'(t) = (0.1)x([t-1]), \tag{3.19}$$

which is a special case of (1.1) with $a = 1$, $b = 0.1$. It is easily checked that (3.19) satisfies the condition of Theorem 3.1. Thus there are oscillatory solutions of (3.19). The solution $x_n(t)$ of (3.19) with the initial conditions

$$x(-1) = -64.91, \quad x(0) = 1, \quad x'(0) = -1.76448, \quad x''(0) = 1.94854$$

for $n = 0, 1, \dots, 13$ is shown in Figure 1

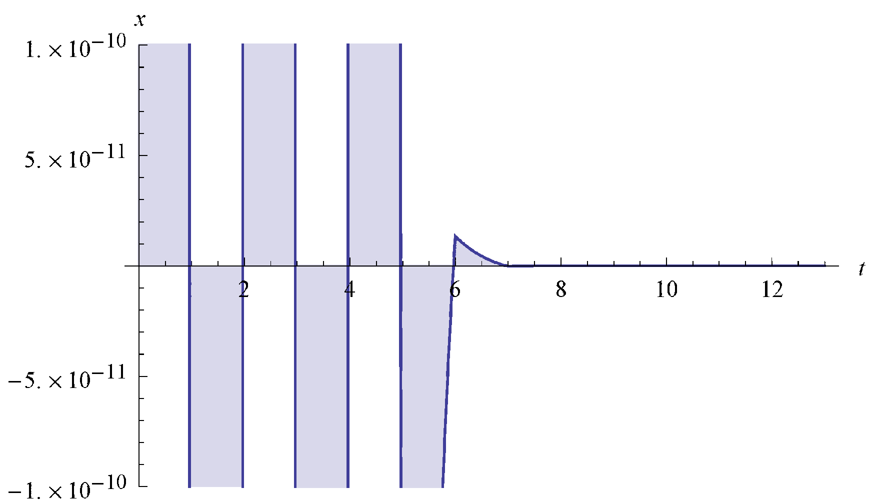

FIGURE 1. Solution of of (3.19)

REFERENCES

[1] Adamets, L.; Lomtatidze, A.; *Oscillation conditions for a third-order linear equation.* (Russian) Differ. Uravn. 37(6), 723-729, 2001; translation in Differ. Equ. 37(6), 755-762, 2001.

[2] Bartušek, M.; Došlá, Z.; *Oscillation of third order differential equation with damping term.* Czechoslovak Math. J., 65 (140), 2, 301-316, 2015.

[3] Bereketoglu, H.; Seyhan, G.; Ogun, A.; *Advanced impulsive differential equations with piecewise constant arguments.* Math. Model. Anal. 15(2), 175–187, 2010.

[4] Bereketoglu, H.; Seyhan, G.; Karakoc, F.; *On a second order differential equation with piecewise constant mixed arguments.* Carpathian J. Math. 27(1), 1–12, 2011.

[5] Bereketoglu, H.; Lafci, M.; S. Oztepe, G.; *On the oscillation of a third order nonlinear differential equation with piecewise constant arguments.* Mediterr. J. Math., 14 (3), Art. 123, 19 pp., 2017.

[6] Cecchi, M.; *Oscillation criteria for a class of third order linear differential equations.* Boll. Un. Mat. Ital. C (6) 2(1), 297-306, 1983.
[7] Chen, X. R.; Pan, L. J.; *Existence of periodic solutions for third order differential equations with deviating argument.* Far East J. Math. Sci. (FJMS), 32(3), 295-310, 2009.
[8] Dahiya, R. S.; *Oscillation of the third order differential equations with and without delay.* Differential equations and nonlinear mechanics (Orlando, FL, 1999), 75-87, Math. Appl. 528, Kluwer Acad. Publ., Dordrecht, 2001.
[9] Das, P.; Pati, J. K.; *Necessary and sufficient conditions for the oscillation a third-order differential equation.* Electron. J. Differential Equations, no. 174, 10 pp., 2010.
[10] Džurina, J.; Baculíková, B.; *Property (B) and oscillation of third-order differential equations with mixed arguments.* J. Appl. Anal. 19(1), 55-68, 2013.
[11] Ezeilo, J. O. C.; *On the existence of periodic solutions of a certain third-order differential equation.* Proc. Cambridge Philos. Soc. 56, 381-389, 1960.
[12] Ezeilo, J. O. C.; *Periodic solutions of certain third order differential equations.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., (8) 57 (1974), no. 1-2, 54-60 (1975).
[13] Han, Z.; Li, T.; Sun, S.; Zhang, C.; *An oscillation criteria for third order neutral delay differential equations.* J. Appl. Anal. 16(2), 295-303, 2010.
[14] Karakoc, F.; Bereketoglu, H.; Seyhan, G.; *Oscillatory and periodic solutions of impulsive differential equations with piecewise constant argument.* Acta Appl. Math. 110(1), 499–510, 2010.
[15] Kim, W. J.; *Oscillatory properties of linear third-order differential equations.* Proc. Amer. Math. Soc. 26, 286-293, 1970.
[16] Liang, H.; Wang, G.; *Oscillation criteria of certain third-order differential equation with piecewise constant argument.* J. Appl. Math. Art. ID 498073, 18 pp., 2012.
[17] Papaschinopoulos, G.; Schinas J. ; *Some results concerning second and third order neutral delay differential equations with piecewise constant argument.* Czechoslovak Math. J. 44(3), 501-512, 1994.
[18] Parhi, N.; Das, P.; *Oscillation and nonoscillation of nonhomogeneous third order differential equations.* Czechoslovak Math. J. 44, 119(3), 443-459, 1994.
[19] Parhi, N.; Das, P.; *On the oscillation of a class of linear homogeneous third order differential equations.* Arch. Math. (Brno) 34(4), 435-443, 1998.
[20] Shao, Y.; Liang, H.; *Oscillatory and asymptotic behavior for third order differential equations with piecewise constant argument.* Far East J. Dyn. Syst. 21(1), 45, 2013.
[21] Shoukaku, Y.; *Oscillation criteria for third order differential equations with functional arguments.* Math. Slovaca 65 (2015), no. 5, 1035-1048.
[22] Tabueva, V. A.; *Conditions for the existence of a periodic solution of a third-order differential equation.* Prikl. Mat. Meh. 25 961–962 (Russian); translated as J. Appl. Math. Mech. 25 1961 1445-1448.
[23] Tejumola, H. O.; *Periodic solutions of certain third order differential equations.* Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8) 6, no. 4, 243-249, 1979.
[24] Tryhuk, V.; *An oscillation criterion for third order linear differential equations.* Arch. Math. (Brno) 11(1975) no:2, 99-104 (1976).
[25] Wang, N.; *Sufficient conditions for the existence of periodic solutions to some third order differential equations with deviating argument.* Math. Appl. (Wuhan) 21(4), 661-670, 2008.
[26] Wiener, J.; Lakshmikantham, V.; *Excitability of a second-order delay differential equation.* Nonlinear Anal. Ser. B: Real World Appl. 38(1), 1–11, 1999.

Huseyin Bereketoglu
Department of Mathematics, Faculty of Sciences, Ankara University, 06100, Ankara, Turkey
E-mail address: bereket@science.ankara.edu.tr

Mehtap Lafci
Department of Mathematics, Faculty of Sciences, Ankara University, 06100, Ankara, Turkey
E-mail address: mlafci@ankara.edu.tr

Gizem S. Oztepe (corresponding author)
Department of Mathematics, Faculty of Sciences, Ankara University, 06100, Ankara, Turkey
E-mail address: gseyhan@ankara.edu.tr