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SCHRÖDINGER EQUATIONS WITH MAGNETIC FIELDS AND HARDY-SOBOLEV CRITICAL EXPONENTS

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Abstract. This article is motivated by problems in astrophysics. We consider nonlinear Schrödinger equations and related systems with magnetic fields and Hardy-Sobolev critical exponents. Under proper conditions, existence of ground state solutions to these equations and systems are established.

1. Introduction

Astrophysics pose a rich class of nonlinear problems, in particular,

$$(-i\nabla + A)^2 u = \frac{|u|^{2^*(s)-2}u}{|x|^s}, \quad u \in D_A^{1,2}(\mathbb{R}^N), \tag{1.1}$$

with the Hardy-Sobolev term models the dynamics of galaxies; we refer to [3, 4] and the references therein. In the present paper we consider the semilinear stationary Schrödinger equation (1.1) with a magnetic field and a Hardy-Sobolev critical exponents, but also

$$\begin{gathered}(-i\nabla + A)^2 u - \lambda u = \frac{|u|^{2^*(s)-2}u}{|x|^s}, \quad u \in H_A^1(\Omega),\\ u = 0, \quad \text{on } \partial\Omega,\end{gathered} \tag{1.2}$$

and related systems thereof, viz.

$$\begin{gathered}(-i\nabla + A)^2 u = \mu_1 \frac{|u|^{2^*(s)-2}u}{|x|^s} + \frac{\alpha\gamma}{2^*(s)} \frac{|u|^{\alpha-2}u|v|^{\beta}}{|x|^s},\\ (-i\nabla + B)^2 v = \mu_2 \frac{|v|^{2^*(s)-2}v}{|x|^s} + \frac{\beta\gamma}{2^*(s)} \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s},\\ u \in D_A^{1,2}(\mathbb{R}^N), \quad v \in D_B^{1,2}(\mathbb{R}^N),\end{gathered} \tag{1.3}$$

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and

$$
\begin{gathered}
(-i\nabla + A)^2 u - \lambda_1 u = \mu_1 \frac{|u|^{2^*(s)-2}u}{|x|^s} + \frac{\alpha\gamma}{2^*(s)} \frac{|u|^{\alpha-2}u|v|^\beta}{|x|^s}, \\
(-i\nabla + B)^2 v - \lambda_2 v = \mu_2 \frac{|v|^{2^*(s)-2}v}{|x|^s} + \frac{\beta\gamma}{2^*(s)} \frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s}, \\
u \in H_A^1(\Omega), \quad v \in H_B^1(\Omega), \quad u = v = 0, \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.4}
$$

where $u, v : \mathbb{R}^N \to \mathbb{C}, N \geq 3$, $A = (A_1, \dots, A_N), B = (B_1, \dots, B_N) : \mathbb{R}^N \to \mathbb{R}^N$ are magnetic vector potentials, $0 \leq s < 2, \lambda, \lambda_1, \lambda_2, \mu_1, \mu_2, \gamma > 0$, $\alpha, \beta > 1$ with $\alpha + \beta = 2^*(s) := \frac{2(N-s)}{N-2}$, and Ω is a smooth bounded domain containing the origin as an interior point. Set $-\Delta_A := (-i\nabla + A)^2, \nabla_A := \nabla + iA$, and

$$
\begin{gathered}
D_A^{1,2}(\mathbb{R}^N) := \big\{u \in L^{2^*}(\mathbb{R}^N) : |\nabla_A u| \in L^2(\mathbb{R}^N)\big\}, \\
H_A^1(\Omega) := \big\{u \in L^2(\Omega) : |\nabla_A u| \in L^2(\Omega)\big\}.
\end{gathered}
$$

Then $-\Delta_A u = -\Delta u - iu \operatorname{div} A - 2iA \cdot \nabla u + |A|^2 u$, $D_A^{1,2}(\mathbb{R}^N)$ and $H_A^1(\Omega)$ are Hilbert spaces obtained by the closures of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ and $C_c^\infty(\Omega, \mathbb{C})$ with respect to scaler products

$$
\operatorname{Re}\Big(\int_{\mathbb{R}^N} \nabla_A u \cdot \overline{\nabla_A v}\Big) \quad \text{and} \quad \operatorname{Re}\Big(\int_{\Omega} \nabla_A u \cdot \overline{\nabla_A v}\Big)
$$

respectively, where the bar denotes complex conjugation. Here and in the following, $\int \cdot$ means $\int \cdot \, dx$. We regard the range of function as $\mathbb{C}$, except the places where we emphasize that the range is $\mathbb{R}$. $L^p\big(\Omega, \frac{dx}{|x|^s}\big)$ denotes the space of L^p-integrable functions with respect to the measure $\frac{dx}{|x|^s}$, endowed with norm

$$
|u|_{p,s} := \Big(\int_\Omega \frac{|u|^p}{|x|^s}\Big)^{1/p}.
$$

For $\Omega = \mathbb{R}^N$, denote the L^p norm by

$$
|u|_{p,s,\mathbb{R}^N} := \Big(\int_{\mathbb{R}^N} \frac{|u|^p}{|x|^s}\Big)^{1/p}.
$$

Write $|u|_p := |u|_{p,0}$ and $|u|_{p,\mathbb{R}^N} := |u|_{p,0,\mathbb{R}^N}$ for simplicity. Define

$$
\begin{gathered}
\mu_s^A(\mathbb{R}^N) := \inf_{u \in D_A^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{|\nabla_A u|_{2,\mathbb{R}^N}^2}{|u|_{2^*(s),s,\mathbb{R}^N}^2}, \\
\mu_s^{A,\lambda}(\Omega) := \inf_{u \in H_A^1(\Omega)\setminus\{0\}} \frac{|\nabla_A u|_2^2 - \lambda|u|_2^2}{|u|_{2^*(s),s}^2}.
\end{gathered}
$$

The first existence results for this kind of problems with a magnetic potential (i.e., $A \in L^2_{\rm loc}$) were established in the seminal work [11]. Leaving aside periodic and singular magnetic fields, a number of papers dealt with nonlinear Schrödinger equations with regular fields, for example, [5, 10, 16, 20, 21, 22], including [1, 2, 6, 15, 18, 23] for the critical Sobolev exponent and [9] for the critical Hardy exponent.

As far as we know, there are no results for problems of this type with Hardy-Sobolev critical exponents, in particular for the system case. The Hardy-Sobolev term has the same homogeneity as the Laplacian but it does not belong to the Kato class and, therefore, the resulting functional lacks compactness.

The present paper is mainly motivated by [2]; we apply existence results of ground state solutions obtained in [8, 13, 14, 25] to extend [2, Theorems 1.1 and

1.2] to the case of Hardy-Sobolev critical exponent and also systems; it is worth to emphasize that systems are not considered in [2]. First, we establish results for single equations.

Theorem 1.1. *If $A \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$, then $\mu_s^A(\mathbb{R}^N)$ is attained by a $u \in D^{1,2}_A(\mathbb{R}^N) \setminus \{0\}$ if and only if $\operatorname{curl} A \equiv 0$, where $\operatorname{curl} A$ is the usual curl operator for $N = 3$ and the $N \times N$ skew-symmetric matrix with entries $a_{jk} = \partial_j A_k - \partial_k A_j$ for $N \geq 4$.*

Theorem 1.2. *Assume that*

(A1) $A \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N), \operatorname{curl} A \equiv 0$ *or*
(A2) $A \in L^2_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N), A$ *is continuous at* 0

holds. Let $N \geq 4$ and $\sigma(-\Delta_A - \lambda) \subset (0, +\infty)$, where $\sigma(\cdot)$ is the spectrum in $L^2(\mathbb{R}^N)$. Then $\mu_s^{A,\lambda}(\Omega)$ is attained by some $u \in H^1_A(\Omega) \setminus \{0\}$.

Second, we establish results for systems. For this purpose we define

$$\bar{\mu}_s^{A,B}(\mathbb{R}^N) := \inf_{(u,v)\in D_{A,B}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{D_{A,B}}}{\left(\int_{\mathbb{R}^N}\left(\mu_1\frac{|u|^{2^*(s)}}{|x|^s} + \mu_2\frac{|v|^{2^*(s)}}{|x|^s} + \gamma\frac{|u|^\alpha|v|^\beta}{|x|^s}\right)\right)^{\frac{2}{2^*(s)}}},$$

$$\bar{\mu}_s^{A,B}(\Omega) := \inf_{(u,v)\in H_{A,B}\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_{H_{A,B}}}{\left(\int_{\Omega}\left(\mu_1\frac{|u|^{2^*(s)}}{|x|^s} + \mu_2\frac{|v|^{2^*(s)}}{|x|^s} + \gamma\frac{|u|^\alpha|v|^\beta}{|x|^s}\right)\right)^{\frac{2}{2^*(s)}}},$$

where $D_{A,B} := D^{1,2}_A(\mathbb{R}^N) \times D^{1,2}_B(\mathbb{R}^N)$, endowed with norm

$$\|(u,v)\|^2_{D_{A,B}} := |\nabla_A u|^2_{2,\mathbb{R}^N} + |\nabla_B v|^2_{2,\mathbb{R}^N},$$

and $H_{A,B} := H^1_A(\Omega) \times H^1_B(\Omega)$, endowed with norm

$$\|(u,v)\|^2_{H_{A,B}} := |\nabla_A u|_2^2 - \lambda_1|u|_2^2 + |\nabla_B v|_2^2 - \lambda_2|v|_2^2.$$

Then we have the following result.

Theorem 1.3. *Assume that $A, B \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$ and*

(A3) $N \geq 3, 1 < \alpha, \beta < 2, \gamma > 0$ *holds.*

Then $\bar{\mu}_s^{A,B}(\mathbb{R}^N)$ is attained by some $(u,v) \in D_{A,B}$ such that $u \not\equiv 0, v \not\equiv 0$ if and only if $\operatorname{curl} A \equiv 0 \equiv \operatorname{curl} B$.

Theorem 1.4. *Assume that* (A3) *is satisfied and*

(A4) $A, B \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N), \operatorname{curl} A \equiv 0 \equiv \operatorname{curl} B$, *or*
(A5) $A, B \in L^2_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N), A$ *and* B *are continuous at* 0

holds. If $\sigma(-\Delta_A - \lambda_1), \sigma(-\Delta_B - \lambda_2) \subset (0, +\infty)$ and $N \geq 4$, then $\bar{\mu}_s^{A,B}(\Omega)$ is attained by some $(u,v) \in H_{A,B}$ such that $u \not\equiv 0$, $v \not\equiv 0$.

The corresponding energy functionals $I : D_{A,B} \to \mathbb{R}$ and $E : H_{A,B} \to \mathbb{R}$ of (1.3) and (1.4) are

$$\begin{aligned} &I(u,v) \\ &= \frac{1}{2}\|(u,v)\|^2_{D_{A,B}} - \frac{1}{2^*(s)}\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \mu_2|v|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \gamma\int_{\mathbb{R}^N}\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big), \end{aligned}$$

and

$$E(u,v) = \frac{1}{2}\|(u,v)\|^2_{H_{A,B}} - \frac{1}{2^*(s)}\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s} + \mu_2|v|^{2^*(s)}_{2^*(s),s} + \gamma\int_{\Omega}\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big),$$

respectively. Define

$$\begin{aligned}\mathcal{N} := \Big\{(u,v) \in D_{A,B} \setminus \{(0,0)\} : &\|(u,v)\|^2_{D_{A,B}} \\ &= \mu_1 |u|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \mu_2 |v|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \gamma \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^s} \Big\}, \\ \mathcal{M} := \Big\{(u,v) \in H_{A,B} \setminus \{(0,0)\} : &\|(u,v)\|^2_{H_{A,B}} \\ &= \mu_1 |u|^{2^*(s)}_{2^*(s),s} + \mu_2 |v|^{2^*(s)}_{2^*(s),s} + \gamma \int_{\Omega} \frac{|u|^\alpha |v|^\beta}{|x|^s} \Big\},\end{aligned}$$

$M_0 := \inf_{(u,v)\in\mathcal{N}} I(u,v)$ and $M := \inf_{(u,v)\in\mathcal{M}} E(u,v)$. By nontrivial solutions $(u,v) \in D_{A,B}$ of (1.3), we mean $u \neq 0, v \neq 0$. A solution of (1.3) is called a ground state solution if $(u,v) \in \mathcal{N}$ and $I(u,v) = M_0$. A ground state solution is semi-trivial if it is of type $(u,0)$ or $(0,v)$. Similar definitions applies to (1.4) and single equations (1.1) and (1.2). For ground states, we obtain

Theorem 1.5. *If* (A1) *holds, then* (1.1) *has a nontrivial ground state solution with energy* $M_1 := \frac{2-s}{2(N-s)}\big(\mu_s^A(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}$.

Theorem 1.6. *Assume that* (A1) *or* (A2) *holds. If* $N \geq 4$ *and* $\sigma(-\Delta_A - \lambda) \subset (0,+\infty)$*, then* (1.2) *has a nontrivial ground state solution with energy given by* $M_2 := \frac{2-s}{2(N-s)}\big(\mu_s^{A,\lambda}(\Omega)\big)^{\frac{N-s}{2-s}}$.

Theorem 1.7. *If* (A3) *and* (A4) *hold, then* (1.3) *has a nontrivial ground state solution with energy given by* $M_0 := \frac{2-s}{2(N-s)}\big(\bar{\mu}_s^{A,B}(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}$.

Theorem 1.8. *Assume that* (A3) *and one of* (A4) *and* (A5) *hold. If*

$$\sigma(-\Delta_A - \lambda_1), \sigma(-\Delta_B - \lambda_2) \subset (0,+\infty)$$

and $N \geq 4$*, then* (1.4) *has a nontrivial ground state solution with energy* $M := \frac{2-s}{2(N-s)}\big(\bar{\mu}_s^{A,B}(\Omega)\big)^{\frac{N-s}{2-s}}$.

Remark 1.9. Although the symmetric and decaying information about ground state solutions of equations with magnetic fields is not known, the existence of ground state solutions is heavily dependent on that of equations without magnetic fields, under proper conditions, such as (A1)–(A5).

Consider the nonlinear system

$$\begin{gathered}\mu_1 k^{\frac{2^*(s)-2}{2}} + \frac{\alpha\gamma}{2^*(s)} k^{\frac{\alpha-2}{2}} l^{\beta/2} = 1, \\ \mu_2 l^{\frac{2^*(s)-2}{2}} + \frac{\beta\gamma}{2^*(s)} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} = 1, \\ k>0, \quad l>0.\end{gathered} \tag{1.5}$$

Theorem 1.10. *Assume that* (A4) *and*

(A6) $N \geq 4$*,* $1 < \alpha, \beta < 2$*, and*

$$\gamma \geq \frac{2(N-s)(2-s)}{(N-2)^2} \max\Big\{\frac{\mu_1}{\alpha}\Big(\frac{2-\beta}{2-\alpha}\Big)^{\frac{2-\beta}{2}}, \frac{\mu_2}{\beta}\Big(\frac{2-\alpha}{2-\beta}\Big)^{\frac{2-\alpha}{2}}\Big\}.$$

If $A = B$, then (1.3) *has a nontrivial ground state solution* $(\sqrt{k_0}U, \sqrt{l_0}U)$ *with energy* $M_0 = \frac{2-s}{2(N-s)}(k_0 + l_0)\big(\mu_s(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}$*, where U is a nontrivial ground state solution of* (1.1)*, obtained in Theorem 1.5,*

$$(k_0, l_0) \text{ satisfies } (1.5) \text{ and } k_0 = \min\{k : (k,l) \text{ is a solution of } (1.5)\}. \tag{1.6}$$

That is, M_0 is attained at $(\sqrt{k_0}U, \sqrt{l_0}U)$.

Theorem 1.11. *Assume that* (A6) *and either* (A4) *or* (A5) *holds. If $A = B$, $\lambda_1 = \lambda_2 = \lambda$, and $\sigma(-\Delta_A - \lambda) \subset (0, +\infty)$, then* (1.4) *has a nontrivial ground state solution $(\sqrt{k_0}\omega, \sqrt{l_0}\omega)$ with energy* $M = \frac{2-s}{2(N-s)}(k_0 + l_0)\big(\mu_s^{A,\lambda}(\Omega)\big)^{\frac{N-s}{2-s}}$*, where (k_0, l_0) satisfies* (1.6) *and ω is a nontrivial ground state solution of* (1.2)*, obtained in Theorem 1.6. That is, M is attained at $(\sqrt{k_0}\omega, \sqrt{l_0}\omega)$.*

Remark 1.12. By [17, Lemma 1.1], we see that the above theorems also hold when conditions (A1) and (A4) are replaced with (A1') and (A4') respectively:

- (A1') $A \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$, there exists $\varphi \in W^{1,N}_{\rm loc}(\mathbb{R}^N, \mathbb{R})$ such that $\nabla\varphi = A$,
- (A4') $A, B \in L^N_{\rm loc}(\mathbb{R}^N, \mathbb{R}^N)$, there exist $\varphi, \psi \in W^{1,N}_{\rm loc}(\mathbb{R}^N, \mathbb{R})$ such that $\nabla\varphi = A, \nabla\psi = B$.

For more details, we refer to [11, Theorem 3.7] and the proof of Theorem 1.3 in this paper.

The paper is organized as follows. In Section 2, we establish several auxiliary results for the proof of our main results; key ingredients are Lemma 2.1 and Lemma 2.4, not found elsewhere. The latter is proven by using Ekeland's variational principle. In Section 3, we discuss the attainability of the infimum defined above by applying the method of concentration-compactness. The existence of ground state solution to the Schrödinger problems is studied in Section 4. Finally, in Section 5 we consider a magnetic field in three dimensions as an application of some of the above theorems.

2. Preliminaries

Define

$$\mu_s(\mathbb{R}^N) := \inf_{u \in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{|\nabla u|^2_{2,\mathbb{R}^N}}{|u|^2_{2^*(s),s,\mathbb{R}^N}},$$

where $D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*(\mathbb{R}^N)} : |\nabla u| \in L^2(\mathbb{R}^N)\}$. Then, by [13], $\mu_s(\mathbb{R}^N)$ is attained by functions of form

$$y_\varepsilon(x) := \big((N-s)(N-2)\big)^{\frac{N-2}{2(2-s)}} \varepsilon^{\frac{N-2}{2}} \big(\varepsilon^{2-s} + |x|^{2-s}\big)^{-\frac{N-2}{2-s}},$$

where $\varepsilon > 0$. The function y_ε is a positive solution of $-\Delta u = \frac{|u|^{2^*(s)-2}u}{|x|^s}$, and moreover,

$$|\nabla y_\varepsilon|^2_{2,\mathbb{R}^N} = |y_\varepsilon|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} = \big(\mu_s(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}.$$

Define

$$\bar\mu_s(\mathbb{R}^N) := \inf_{(u,v)\in D\setminus\{(0,0)\}} \frac{\|(u,v)\|^2_D}{\Big(\int_{\mathbb{R}^N} \Big(\mu_1 \frac{|u|^{2^*(s)}}{|x|^s} + \mu_2 \frac{|v|^{2^*(s)}}{|x|^s} + \gamma \frac{|u|^\alpha |v|^\beta}{|x|^s}\Big)\Big)^{\frac{2}{2^*(s)}}},$$

where $D := D^{1,2}(\mathbb{R}^N) \times D^{1,2}(\mathbb{R}^N)$, endowed with norm

$$\|(u,v)\|^2_D := |\nabla u|^2_{2,\mathbb{R}^N} + |\nabla v|^2_{2,\mathbb{R}^N}.$$

Then, by Lemma 2.4, [8, 14] ($s = 0$) and [25] ($0 < s < 2$), we see that, under condition (A3), $\bar{\mu}_s(\mathbb{R}^N)$ is attained by (U, V), where U and V are positive, radially symmetric functions which decay as follows:

$$U(x) + V(x) \le C(1 + |x|)^{2-N}, \quad |\nabla U(x)| + |\nabla V(x)| \le C(1 + |x|)^{1-N}. \tag{2.1}$$

As proved in [11, 19], for any $u \in D_A^{1,2}(\mathbb{R}^N)$ or $H_A^1(\Omega)$, the following (weak) *diamagnetic inequality* holds pointwise for almost every $x \in \mathbb{R}^N$ or Ω,

$$\big|\nabla|u|\big| = \Big|\operatorname{Re}\Big(\nabla u \frac{\overline{u}}{|u|}\Big)\Big| = \Big|\operatorname{Re}\Big((\nabla u + iAu)\frac{\overline{u}}{|u|}\Big)\Big| \le |\nabla_A u|.$$

Then, for $u \in D_A^{1,2}(\mathbb{R}^N)$ or $H_A^1(\Omega)$, we see that $|u|$ belongs to the usual Sobolev space $D^{1,2}(\mathbb{R}^N)$ or $H_0^1(\Omega)$. Moreover, we have the following lemma.

Lemma 2.1. *The embedding $H_A^1(\Omega) \hookrightarrow L^p\big(\Omega, \frac{dx}{|x|^s}\big)$ is continuous for $1 \le p \le 2^*(s)$, and it is compact for $1 \le p < 2^*(s)$, where $0 \le s < 2$. The embedding $D_A^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}\big(\mathbb{R}^N, \frac{dx}{|x|^s}\big)$ is continuous for $0 \le s < 2$.*

Proof. By the diamagnetic inequality and the Hardy-Sobolev inequality, it is easy to see that the embeddings $H_A^1(\Omega) \hookrightarrow L^p\big(\Omega, \frac{\mathrm{d}x}{|x|^s}\big)$ and $D_A^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}\big(\mathbb{R}^N, \frac{\mathrm{d}x}{|x|^s}\big)$ are continuous, where $1 \le p \le 2^*(s)$ and $0 \le s < 2$.

Let $\{u_n\}$ be a bounded sequence in $H_A^1(\Omega)$. For compactness of the embedding, it remains to show that there exists a subsequence of $\{u_n\}$, strongly converging in $L^p\big(\Omega, \frac{\mathrm{d}x}{|x|^s}\big)$, where $1 \le p < 2^*(s)$.

For the case $s = 0$, since $\{|u_n|\}$ is bounded in $H_0^1(\Omega)$, we can consider the real parts R_n and imaginary parts I_n of u_n separately, and follow the arguments of Rellich-Kondrachov Compactness Theorem (cf. [12]), passing to a subsequence, we may prove that $R_n \to R$ and $I_n \to I$ strongly in $L^p(\Omega)$, where $1 \le p < 2^* := 2^*(0)$. That is, $u_n \to u$ strongly in $L^p(\Omega)$, where $u = R + iI$.

For the case $0 < s < 2$, applying the ideas of [2, Lemma 2.6] and [7, Lemma 2.1], we may extract a subsequence, still denoted by u_n, such that $u_n \rightharpoonup u$ weakly in $H_A^1(\Omega)$. Then, $u_n \rightharpoonup u$ weakly in $L^{2^*}(\Omega)$, and $|u_n - u|$ is bounded in $H_0^1(\Omega)$. Hence, up to a subsequence, $|u_n - u| \rightharpoonup 0$ weakly in $H_0^1(\Omega)$ and $u_n \to u$ a.e. on Ω. By Rellich-Kondrachov Theorem, we see that $u_n \to u$ strongly in $L^q(\Omega)$, where $1 \le q < 2^*$. Since $H_A^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$, there exists a constant C such that $|u_n - u|_{2^*}^{2^*} \le C$. For any $\varepsilon > 0$, let $\Omega_\varepsilon := \Omega \cap B_\varepsilon$ and $\Omega_\varepsilon^c := \Omega \setminus \Omega_\varepsilon$, where B_ε is the ball centered at 0 with radius ε. Noting $N - \frac{2^* s}{2^* - p} > 0$, we have

$$\begin{aligned}
\int_{\Omega_\varepsilon} \frac{|u_n - u|^p}{|x|^s} &\le \Big(\int_{\Omega_\varepsilon} |u_n - u|^{2^*}\Big)^{\frac{p}{2^*}} \Big(\int_{\Omega_\varepsilon} |x|^{-\frac{2^* s}{2^* - p}}\Big)^{\frac{2^* - p}{2^*}} \\
&\le C\Big(\int_0^\varepsilon r^{-\frac{2^* s}{2^* - p}} r^{N-1} \mathrm{d}r\Big)^{\frac{2^* - p}{2^*}} \\
&= O\big(\varepsilon^{\frac{(N-2)(2^*(s) - p)}{2}}\big).
\end{aligned} \tag{2.2}$$

On the other hand, for any $x \in \Omega_\varepsilon^c$, there exists a constant $C_\varepsilon > 0$ such that $\frac{1}{|x|^s} \le C_\varepsilon$. It follows from Rellich-Kondrachov Compactness Theorem that $\int_{\Omega_\varepsilon^c} \frac{|u_n - u|^p}{|x|^s} = o(1)$. Combining this and (2.2), we get that $\lim_{n\to\infty} |u_n - u|_{p,s}^p = 0$. $\square$

Remark 2.2. If $\sigma(-\Delta_A-\lambda_1),\sigma(-\Delta_B-\lambda_2)\subset(0,+\infty)$, then by Lemma 2.1, it is standard to see that the quantities

$$\mu_s^A(\mathbb{R}^N),\mu_s^{A,\lambda_1}(\Omega),\mu_s^{B,\lambda_2}(\Omega),\bar\mu_s^{A,B}(\mathbb{R}^N),\ \text{and}\ \bar\mu_s^{A,B}(\Omega)$$

are strictly positive.

Lemma 2.3. *If $A,B\in L^N_{\rm loc}(\mathbb{R}^N,\mathbb{R}^N)$, then $\mu_s^A(\mathbb{R}^N)=\mu_s(\mathbb{R}^N)$ and $\bar\mu_s^{A,B}(\mathbb{R}^N)=\bar\mu_s(\mathbb{R}^N)$.*

Proof. We only prove the latter equality. For any $(u,v)\in D_{A,B}\setminus\{(0,0)\}$, by the diamagnetic inequality, we have

$$\begin{aligned}
&\frac{|\nabla_A u|^2_{2,\mathbb{R}^N}+|\nabla_B v|^2_{2,\mathbb{R}^N}}{\big(\int_{\mathbb{R}^N}\big(\mu_1\frac{|u|^{2^*(s)}}{|x|^s}+\mu_2\frac{|v|^{2^*(s)}}{|x|^s}+\gamma\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)\big)^{\frac{2}{2^*(s)}}}\\
&\geq\frac{\big|\nabla|u|\big|^2_{2,\mathbb{R}^N}+\big|\nabla|v|\big|^2_{2,\mathbb{R}^N}}{\big(\int_{\mathbb{R}^N}\big(\mu_1\frac{|u|^{2^*(s)}}{|x|^s}+\mu_2\frac{|v|^{2^*(s)}}{|x|^s}+\gamma\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)\big)^{\frac{2}{2^*(s)}}}\\
&\geq\bar\mu_s(\mathbb{R}^N),
\end{aligned}$$

which implies that $\bar\mu_s^{A,B}(\mathbb{R}^N)\geq\bar\mu_s(\mathbb{R}^N)$. Define

$$\big(U_\varepsilon(x),V_\varepsilon(x)\big):=\Big(\varepsilon^{-\frac{N-2}{2}}U\big(\frac{x}{\varepsilon}\big),\varepsilon^{-\frac{N-2}{2}}V\big(\frac{x}{\varepsilon}\big)\Big) \tag{2.3}$$

and

$$\big(u_\varepsilon(x),v_\varepsilon(x)\big):=\big(\phi(x)U_\varepsilon(x),\phi(x)V_\varepsilon(x)\big),$$

where (U,V) achieves $\bar\mu_s(\mathbb{R}^N)$ with (2.1), and $\phi\in C_0^1(B_2)$ is a cut-off function satisfying $\phi\equiv1$ on B_1. Then, a direct computation yields

$$\int_\Omega|\nabla u_\varepsilon|^2\leq\int_{\mathbb{R}^N}|\nabla U|^2+O(\varepsilon^{N-2}), \tag{2.4}$$

$$\int_\Omega|\nabla v_\varepsilon|^2\leq\int_{\mathbb{R}^N}|\nabla V|^2+O(\varepsilon^{N-2}), \tag{2.5}$$

$$\int_\Omega\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}\geq\int_{\mathbb{R}^N}\frac{|U|^{2^*(s)}}{|x|^s}+O(\varepsilon^{N-s}), \tag{2.6}$$

$$\int_\Omega\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}\geq\int_{\mathbb{R}^N}\frac{|V|^{2^*(s)}}{|x|^s}+O(\varepsilon^{N-s}), \tag{2.7}$$

$$\int_\Omega\frac{|u_\varepsilon|^\alpha|v_\varepsilon|^\beta}{|x|^s}\geq\int_{\mathbb{R}^N}\frac{|U|^\alpha|V|^\beta}{|x|^s}+O(\varepsilon^{N-s}). \tag{2.8}$$

It follows from $\{u_\varepsilon\}$ that it is bounded in $L^{2^*}(\mathbb{R}^N)$ and $u_\varepsilon\to0$ a.e. in $\mathbb{R}^N$ as $\varepsilon\to0$ that for any $\varphi\in L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N)$,

$$\Big|\int_{\mathbb{R}^N}u_\varepsilon\varphi\Big|\leq\Big(\int_{\mathbb{R}^N}u_\varepsilon^{2^*}\Big)^{\frac{1}{2^*}}\Big(\int_{\mathbb{R}^N}|\varphi|^{\frac{2^*}{2^*-1}}\Big)^{\frac{2^*-1}{2^*}}\to0,$$

i.e., $u_\varepsilon\rightharpoonup0$ weakly in $L^{2^*}(\mathbb{R}^N)$. Hence, $u_\varepsilon^2\rightharpoonup0$ weakly in $L^{\frac{2^*}{2}}(\mathbb{R}^N)$. Since $|A|^2\in L^{\frac{N}{2}}_{\rm loc}(\mathbb{R}^N)=\big(L^{\frac{2^*}{2}}_{\rm loc}(\mathbb{R}^N)\big)'$, the dual space of $L^{\frac{2^*}{2}}_{\rm loc}(\mathbb{R}^N)$, we obtain

$$\int_{\mathbb{R}^N}|Au_\varepsilon|^2=\langle|A|^2,u_\varepsilon^2\rangle\to0,$$

where the duality product is taken with respect to $L^{\frac{N}{2}}(\mathbb{R}^N)$ and $L^{\frac{2^*}{2}}(\mathbb{R}^N)$. Similarly, we have $\int_{\mathbb{R}^N}|Bv_\varepsilon|^2\to 0$ as $\varepsilon\to 0$. Let $\delta>0$. For ε small enough, noting that u_ε and v_ε are real-valued, by (2.4)–(2.8), we have

$$\begin{aligned}
\bar\mu_s^{A,B}(\mathbb{R}^N) &\le \frac{|\nabla_A u_\varepsilon|^2_{2,\mathbb{R}^N}+|\nabla_B v_\varepsilon|^2_{2,\mathbb{R}^N}}{\big(\int_{\mathbb{R}^N}\big(\mu_1\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}+\mu_2\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}+\gamma\frac{|u_\varepsilon|^\alpha|v_\varepsilon|^\beta}{|x|^s}\big)\big)^{\frac{2}{2^*(s)}}}\\
&=\frac{\int_{\mathbb{R}^N}\big(|\nabla u_\varepsilon|^2+|Au_\varepsilon|^2+|\nabla v_\varepsilon|^2+|Bv_\varepsilon|^2\big)}{\big(\int_{\mathbb{R}^N}\big(\mu_1\frac{|u_\varepsilon|^{2^*(s)}}{|x|^s}+\mu_2\frac{|v_\varepsilon|^{2^*(s)}}{|x|^s}+\gamma\frac{|u_\varepsilon|^\alpha|v_\varepsilon|^\beta}{|x|^s}\big)\big)^{\frac{2}{2^*(s)}}}\\
&\le\frac{\int_{\mathbb{R}^N}\big(|\nabla U|^2+|\nabla V|^2+|Au_\varepsilon|^2+|Bv_\varepsilon|^2\big)+O(\varepsilon^{N-2})}{\big(\int_{\mathbb{R}^N}\big(\mu_1\frac{|U|^{2^*(s)}}{|x|^s}+\mu_2\frac{|V|^{2^*(s)}}{|x|^s}+\gamma\frac{|U|^\alpha|V|^\beta}{|x|^s}\big)+O(\varepsilon^{N-s})\big)^{\frac{2}{2^*(s)}}}\\
&\le\bar\mu_s(\mathbb{R}^N)+\delta,
\end{aligned}$$

which implies that $\bar\mu_s^{A,B}(\mathbb{R}^N)\le\bar\mu_s(\mathbb{R}^N)$. Therefore, $\bar\mu_s^{A,B}(\mathbb{R}^N)=\bar\mu_s(\mathbb{R}^N)$. □

Lemma 2.4. *The following conclusions hold.*

(i) $\mu_s^A(\mathbb{R}^N)$ *is attained if and only if* (1.1) *has a nontrivial ground state solution;*
(ii) $\mu_s^{A,\lambda}(\Omega)$ *is attained if and only if* (1.2) *has a nontrivial ground state solution;*
(iii) $\bar\mu_s^{A,B}(\mathbb{R}^N)$ *is attained by* $(u,v)\in D_{A,B}$ *with* $u\not\equiv 0, v\not\equiv 0$ *if and only if* (1.3) *has a nontrivial ground state solution;*
(iv) $\bar\mu_s^{A,B}(\Omega)$ *is attained by* $(u,v)\in H_{A,B}$ *with* $u\not\equiv 0, v\not\equiv 0$ *if and only if* (1.4) *has a nontrivial ground state solution.*

Proof. We only prove (iv). Setting

$$F(u,v):=\frac{\|(u,v)\|^2_{H_{A,B}}}{\big(\int_\Omega\big(\mu_1\frac{|u|^{2^*(s)}}{|x|^s}+\mu_2\frac{|v|^{2^*(s)}}{|x|^s}+\gamma\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)\big)^{\frac{2}{2^*(s)}}},$$

then $\bar\mu_s^{A,B}(\Omega)=\inf_{(u,v)\in H_{A,B}\setminus\{(0,0)\}}F(u,v)$ and $F(tu,tv)=F(u,v)$ for any $t\in\mathbb{R}$. Obviously, for any $(u,v)\in H_{A,B}\setminus\{(0,0)\}$, there exists an unique

$$t_{u,v}=\Big(\frac{\|(u,v)\|^2_{H_{A,B}}}{\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}}\Big)^{\frac{1}{2^*(s)-2}}$$

such that $(t_{u,v}u,t_{u,v}v)\in\mathcal{M}$. Therefore,

$$\begin{aligned}
\bar\mu_s^{A,B}(\Omega)&=\inf_{(u,v)\in H_{A,B}\setminus\{(0,0)\}}F(t_{u,v}u,t_{u,v}v)\\
&=\inf_{(u,v)\in\mathcal{M}}F(u,v)=\inf_{(u,v)\in\mathcal{M}}\|(u,v)\|_{H_{A,B}}^{\frac{2^*(s)-2}{2^*(s)}}.
\end{aligned}$$

Noting that $M=\frac{2-s}{2(N-s)}\inf_{(u,v)\in\mathcal{M}}\|(u,v)\|^2_{H_{A,B}}$, we see that $\bar\mu_s^{A,B}(\Omega)$ is attained if and only if M is attained. Assume that (1.4) has a nontrivial ground state solution, i.e., M is attained by a nontrivial element in $H_{A,B}$. Then, $\bar\mu_s^{A,B}(\Omega)$ is attained by some $(u,v)\in H_{A,B}$ with $u\not\equiv 0$ and $v\not\equiv 0$. On the other hand, assume that $\bar\mu_s^{A,B}(\Omega)$ is achieved by a nontrivial element in $H_{A,B}$. Then, there exists $(u,v)\in H_{A,B}$ with $u\not\equiv 0$ and $v\not\equiv 0$ such that $M=\inf_{\mathcal{M}}E=E(u,v)$. It remains to show that (u,v) is a solution of (1.4). It is easy to see that $E|_{\mathcal{M}}\in C^1(\mathcal{M},\mathbb{R})$ is bounded below.

By Ekeland's variational principle (e.g. [24]), for $\varepsilon, \delta > 0$, there exists $(u', v') \in \mathcal{M}$ such that

$$E(u', v') \le E(u, v) + 2\varepsilon, \ \|E'(u', v')\|_{H'_{A,B}} < \frac{8\varepsilon}{\delta}, \ \|(u', v') - (u, v)\|_{H_{A,B}} \le 2\delta. \tag{2.9}$$

Choosing $\varepsilon_n = \frac{1}{n}$ and $\delta_n = \frac{1}{\sqrt{n}}$ in (2.9), there exists $\{(u_n, v_n)\}$ such that $(u_n, v_n) \to (u, v)$ in $H_{A,B}$, $E'(u_n, v_n) \to 0$ in $H'_{A,B}$, and $E(u_n, v_n) \to E(u, v)$, as $n \to \infty$. Hence, $E'(u, v) = 0$ in $H'_{A,B}$, that is (u, v) is a solution of (1.4). $\square$

3. Attainability of the infimum

Since the proofs of Theorems 1.1 and 1.2 are similar to that of [2, Theorems 1.1 and 1.2] and easier than that of Theorems 1.3 and 1.4 in the present paper, we only prove Theorems 1.3 and 1.4 here. Note that systems are not treated in [2] and the concentration-compactness arguments therein, going back to Willem [24], has to be combined with new arguments in order to treat these systems.

Proof of Theorem 1.3. (Necessary condition) Let (u, v) be a minimizer of $\bar{\mu}_s^{A,B}(\mathbb{R}^N)$ normalized by $\mu_1 |u|_{2^*(s),s,\mathbb{R}^N} + \mu_2 |v|_{2^*(s),s,\mathbb{R}^N} + \gamma \int_{\mathbb{R}^N} \frac{|u|^\alpha |v|^\beta}{|x|^s} = 1$. By the diamagnetic inequality and Lemma 2.3, we have

$$\begin{aligned} \bar{\mu}_s^{A,B}(\mathbb{R}^N) &= \int_{\mathbb{R}^N} \left(|\nabla_A u|^2 + |\nabla_B v|^2 \right) \\ &\ge \int_{\mathbb{R}^N} \left(\big|\nabla |u|\big|^2 + \big|\nabla |v|\big|^2 \right) \\ &\ge \bar{\mu}_s(\mathbb{R}^N) = \bar{\mu}_s^{A,B}(\mathbb{R}^N), \end{aligned}$$

which means the above inequality must be equality and

$$\begin{aligned} |\nabla_A u| &= \big|\nabla |u|\big| = \Big| \operatorname{Re}\Big(\nabla u \frac{\overline{u}}{|u|} \Big) \Big| = \Big| \operatorname{Re}\Big((\nabla u + iAu) \frac{\overline{u}}{|u|} \Big) \Big|, \\ |\nabla_B v| &= \big|\nabla |v|\big| = \Big| \operatorname{Re}\Big(\nabla v \frac{\overline{v}}{|v|} \Big) \Big| = \Big| \operatorname{Re}\Big((\nabla v + iBv) \frac{\overline{v}}{|v|} \Big) \Big|. \end{aligned}$$

Then, we deduce that $\operatorname{Im}\big(\nabla u \frac{\overline{u}}{|u|}\big) = 0$ and $\operatorname{Im}\big(\nabla v \frac{\overline{v}}{|v|}\big) = 0$, which are equivalent to $A = -\operatorname{Im}\big(\frac{\nabla u}{u}\big)$ and $B = -\operatorname{Im}\big(\frac{\nabla v}{v}\big)$. Since $\operatorname{curl}\big(\frac{\nabla u}{u}\big) = 0$ and $\operatorname{curl}\big(\frac{\nabla v}{v}\big) = 0$, we infer that $\operatorname{curl} A = 0$ and $\operatorname{curl} B = 0$.

(Sufficient condition) Assume that $\operatorname{curl} A = 0$ and $\operatorname{curl} B = 0$. By [17, Lemma 1.1], there exist $\varphi, \psi \in W^{1,N}_{\rm loc}(\mathbb{R}^N, \mathbb{R})$ such that $\nabla\varphi = A, \nabla\psi = B$. Let

$$\big(u_\varepsilon(x), v_\varepsilon(x)\big) = \big(U_\varepsilon(x) e^{-i\varphi(x)}, V_\varepsilon(x) e^{-i\psi(x)}\big),$$

where $\varepsilon > 0$ and $(U_\varepsilon, V_\varepsilon)$ is defined in (2.3). It follows from Lemma 2.3 that $(u_\varepsilon, v_\varepsilon)$ is a minimizer for $\bar{\mu}_s^{A,B}(\mathbb{R}^N)$. $\square$

Lemma 3.1. *If* (A1) *or* (A2) *holds,* $N \ge 4$, *and* $\sigma(-\Delta_A - \lambda_1), \sigma(-\Delta_B - \lambda_2) \subset (0, +\infty)$, *then* $\bar{\mu}_s^{A,B}(\Omega) < \min\Big\{ \mu_1^{-\frac{2}{2^*(s)}} \mu_s^{A,\lambda_1}(\Omega),\ \mu_2^{-\frac{2}{2^*(s)}} \mu_s^{B,\lambda_2}(\Omega) \Big\}$.

Proof. By Theorem 1.2, we assume that u_{μ_1} achieves $\mu_s^{A,\lambda_1}(\Omega)$ with $|u_{\mu_1}|_{2^*(s),s} = \big(\frac{\mu_s^{A,\lambda_1}(\Omega)}{\mu_1}\big)^{\frac{1}{2^*(s)-2}}$. Define $t(\epsilon) := t_{u_{\mu_1}, \epsilon u_{\mu_1}}$, i.e.,

$$t(\epsilon) = \Big(\frac{\|u_{\mu_1}\|^2_{H_A} + \epsilon^2 \|u_{\mu_1}\|^2_{H_B}}{\big(\mu_1 + \mu_2 |\epsilon|^{2^*(s)} + \gamma |\epsilon|^\beta\big) |u_{\mu_1}|^{2^*(s)}_{2^*(s),s}} \Big)^{\frac{1}{2^*(s)-2}},$$

where $\|u\|_{H_A}^2 := |\nabla_A u|_2^2 - \lambda_1 |u|_2^2$ and $\|u\|_{H_B}^2 := |\nabla_B u|_2^2 - \lambda_2 |u|_2^2$. It is easy to see that $\big(t(\epsilon)u_{\mu_1}, t(\epsilon)\epsilon u_{\mu_1}\big) \in \mathcal{M}$. Noting that $\|u_{\mu_1}\|_{H_A}^2 = \mu_1 |u_{\mu_1}|_{2^*(s),s}^{2^*(s)}$ and $t(0) = 1$, we deduce that

$$\lim_{\epsilon\to 0} \frac{t'(\epsilon)}{|\epsilon|^{\beta-2}\epsilon} = -\frac{\gamma\beta}{\big(2^*(s)-2\big)\mu_1},$$

that is,

$$t'(\epsilon) = -\frac{\gamma\beta|\epsilon|^{\beta-2}\epsilon}{\big(2^*(s)-2\big)\mu_1}\big(1+o(1)\big), \quad \text{as } \epsilon \to 0.$$

Then

$$t(\epsilon) = 1 - \frac{\gamma|\epsilon|^{\beta}}{\big(2^*(s)-2\big)\mu_1}\big(1+o(1)\big), \quad \text{as } \epsilon \to 0,$$

and hence,

$$t(\epsilon)^{2^*(s)} = 1 - \frac{2^*(s)\gamma|\epsilon|^{\beta}}{\big(2^*(s)-2\big)\mu_1}\big(1+o(1)\big), \quad \text{as } \epsilon \to 0.$$

Thus, we have

$$\begin{aligned}
& t(\epsilon)^{2^*(s)}\big(\mu_1 + \mu_2|\epsilon|^{2^*(s)} + \gamma|\epsilon|^{\beta}\big)|u_{\mu_1}|_{2^*(s),s}^{2^*(s)} \\
&= \Big(1 - \frac{2^*(s)\gamma|\epsilon|^{\beta}}{\big(2^*(s)-2\big)\mu_1}\big(1+o(1)\big)\Big)\big(\mu_1 + \mu_2|\epsilon|^{2^*(s)} + \gamma|\epsilon|^{\beta}\big)|u_{\mu_1}|_{2^*(s),s}^{2^*(s)} \\
&= \mu_1 |u_{\mu_1}|_{2^*(s),s}^{2^*(s)} - \frac{\gamma|\epsilon|^{\beta}}{2^*(s)}|u_{\mu_1}|_{2^*(s),s}^{2^*(s)} + o(|\epsilon|^{\beta}) \\
&< \mu_1 |u_{\mu_1}|_{2^*(s),s}^{2^*(s)} \quad \text{for } |\epsilon| \text{ small enough.}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\bar{\mu}_s^{A,B}(\Omega) &= \inf_{(u,v)\in\mathcal{M}} \Big(\mu_1 |u|_{2^*(s),s}^{2^*(s)} + \mu_2 |v|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u|^{\alpha}|v|^{\beta}}{|x|^s}\Big)^{\frac{2^*(s)-2}{2^*(s)}} \\
&\le \Big(t(\epsilon)^{2^*(s)}\big(\mu_1 + \mu_2|\epsilon|^{2^*(s)} + \gamma|\epsilon|^{\beta}\big)|u_{\mu_1}|_{2^*(s),s}^{2^*(s)}\Big)^{\frac{2^*(s)-2}{2^*(s)}} \\
&< \mu_1^{\frac{2^*(s)-2}{2^*(s)}} |u_{\mu_1}|_{2^*(s),s}^{2^*(s)-2} \\
&= \mu_1^{-\frac{2}{2^*(s)}} \mu_s^{A,\lambda_1}(\Omega).
\end{aligned}$$

Similarly, $\bar{\mu}_s^{A,B}(\Omega) < \mu_2^{-\frac{2}{2^*(s)}} \mu_s^{B,\lambda_2}(\Omega)$. □

Proof of Theorem 1.4. Since the proof under the assumption (A4) is similar to that of Theorem 1.3, we only prove it under the assumption (A5). Setting

$$\theta(x) := -\sum_{j=1}^{N} A_j(0)x_j, \quad \vartheta(x) := -\sum_{j=1}^{N} B_j(0)x_j,$$

we have

$$\begin{aligned}
\nabla\theta(x) &= \big(-A_1(0), \dots, -A_N(0)\big) = -A(0), \\
\nabla\vartheta(x) &= \big(-B_1(0), \dots, -B_N(0)\big) = -B(0),
\end{aligned}$$

which imply that $(\nabla\theta + A)(0) = 0$ and $(\nabla\vartheta + B)(0) = 0$. Then, continuity ensures that there exists $\delta > 0$ satisfying

$$\big|(\nabla\theta + A)(x)\big|^2 \le \frac{\lambda_3}{2}, \quad \big|(\nabla\vartheta + B)(x)\big|^2 \le \frac{\lambda_3}{2}, \qquad \forall |x| < \delta, \tag{3.1}$$

where $\lambda_3 = \min\{\lambda_1, \lambda_2\}$. There exists $\rho > 0$ such that $B_\rho \subset \Omega$. Let $2r := \min\{\delta, \rho\}$ and

$$\big(u_\varepsilon(x), v_\varepsilon(x)\big) = \Big(\phi(x)U_\varepsilon(x)e^{i\theta(x)}, \phi(x)V_\varepsilon(x)e^{i\vartheta(x)}\Big),$$

where $\phi \in C_0^1(B_{2r})$ is a cut-off function such that $\phi(x) = 1$ in B_r and $(U_\varepsilon, V_\varepsilon)$ is defined by (2.3). By (2.4), (2.5) and (3.1), we deduce that

$$\begin{aligned}
&\int_\Omega \big(|\nabla_A u_\varepsilon|^2 - \lambda_1|u_\varepsilon|^2 + |\nabla_B v_\varepsilon|^2 - \lambda_2|v_\varepsilon|^2\big)\\
&= \int_\Omega \big(|\nabla(\phi U_\varepsilon)|^2 + \phi^2U_\varepsilon^2|\nabla\theta + A|^2 - \lambda_1\phi^2U_\varepsilon^2\big)\\
&\quad + \int_\Omega \big(|\nabla(\phi V_\varepsilon)|^2 + \phi^2V_\varepsilon^2|\nabla\vartheta + B|^2 - \lambda_2\phi^2V_\varepsilon^2\big)\\
&\le \int_{\mathbb{R}^N} \big(|\nabla U|^2 + |\nabla V|^2\big) + O(\varepsilon^{N-2}) + \frac{\lambda_3}{2}\int_{B_{2r}} \phi^2U_\varepsilon^2\\
&\quad - \lambda_1\int_{B_{2r}} \phi^2U_\varepsilon^2 + \frac{\lambda_3}{2}\int_{B_{2r}} \phi^2V_\varepsilon^2 - \lambda_2\int_{B_{2r}} \phi^2V_\varepsilon^2\\
&\le \int_{\mathbb{R}^N} \big(|\nabla U|^2 + |\nabla V|^2\big) + O(\varepsilon^{N-2}) - \frac{\lambda_3}{2}\int_{B_r} (U_\varepsilon^2 + V_\varepsilon^2).
\end{aligned}$$

Since

$$\begin{aligned}
\int_{B_r} |U_\varepsilon|^2 &\ge \int_{|x|\le r} \varepsilon^{2-N}\big|U\big(\frac{x}{\varepsilon}\big)\big|^2 \mathrm{d}x\\
&= \varepsilon^2\int_{\mathbb{R}^N} |U(y)|^2\mathrm{d}y - \varepsilon^2\int_{|y|\ge\frac{r}{\varepsilon}} |U(y)|^2\mathrm{d}y\\
&\ge C\varepsilon^2 - C\varepsilon^2\int_{|y|\ge\frac{r}{\varepsilon}} |y|^{4-2N}\mathrm{d}y\\
&= C\varepsilon^2 + O(\varepsilon^{N-2}),
\end{aligned} \tag{3.2}$$

and

$$\int_{B_r} |V_\varepsilon|^2 \ge C\varepsilon^2 + O(\varepsilon^{N-2}),$$

by (2.6)–(2.8), we have

$$\begin{aligned}
\bar\mu_s^{A,B}(\Omega) &\le \frac{\int_\Omega \big(|\nabla_A u_\varepsilon|^2 - \lambda_1|u_\varepsilon|^2 + |\nabla_B v_\varepsilon|^2 - \lambda_2|v_\varepsilon|^2\big)}{\big(\mu_1|u_\varepsilon|_{2^*(s),s}^{2^*(s)} + \mu_2|v_\varepsilon|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u_\varepsilon|^\alpha|v_\varepsilon|^\beta}{|x|^s}\big)^{\frac{2}{2^*(s)}}}\\
&\le \frac{\int_{\mathbb{R}^N} \big(|\nabla U|^2 + |\nabla V|^2\big) - C\varepsilon^2 + O(\varepsilon^{N-2})}{\big(\int_{\mathbb{R}^N} \big(\mu_1\frac{|U|^{2^*(s)}}{|x|^s} + \mu_2\frac{|V|^{2^*(s)}}{|x|^s} + \gamma\frac{|U|^\alpha|V|^\beta}{|x|^s}\big) + O(\varepsilon^{N-s})\big)^{\frac{N-2}{N-s}}}\\
&< \bar\mu_s(\mathbb{R}^N).
\end{aligned} \tag{3.3}$$

Let $\{(u_n, v_n)\}$ be a minimizing sequence for $\bar\mu_s^{A,B}(\Omega)$ normalized as

$$\mu_1|u_n|_{2^*(s),s}^{2^*(s)} + \mu_2|v_n|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u_n|^\alpha|v_n|^\beta}{|x|^s} = 1;$$

that is,

$$|\nabla_A u_n|_2^2 - \lambda_1|u_n|_2^2 + |\nabla_B v_n|_2^2 - \lambda_2|v_n|_2^2 = \bar{\mu}_s^{A,B}(\Omega) + o(1). \tag{3.4}$$

Noting that $\{u_n\}$ is bounded in $H_A^1(\Omega)$ and $\{v_n\}$ is bounded in $H_B^1(\Omega)$, by Lemma 2.1, we may extract two subsequences-still denoted by $\{u_n\}$ and $\{v_n\}$-such that

$$\begin{gathered}
u_n \rightharpoonup u \quad \text{weakly in } H_A^1(\Omega),\\
v_n \rightharpoonup v \quad \text{weakly in } H_B^1(\Omega)\\
u_n \to u, \quad v_n \to v \quad \text{strongly in } L^2(\Omega),\\
u_n \to u, \quad v_n \to v \quad \text{a.e. on } \Omega,
\end{gathered}$$

with

$$\mu_1|u|_{2^*(s),s}^{2^*(s)} + \mu_2|v|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^s} \le 1.$$

Setting $w_n := u_n - u$ and $z_n := v_n - v$, then $w_n \rightharpoonup 0$ weakly in $H_A^1(\Omega)$, $z_n \rightharpoonup 0$ weakly in $H_B^1(\Omega)$ and $w_n \to 0, z_n \to 0$ a.e. on Ω. It follows from diamagnetic inequality and (3.4) that

$$\begin{gathered}
|\nabla_A u_n|_2^2 + |\nabla_B v_n|_2^2 \ge \big|\nabla|u_n|\big|_2^2 + \big|\nabla|v_n|\big|_2^2 \ge \bar{\mu}_s(\mathbb{R}^N),\\
\bar{\mu}_s^{A,B}(\Omega) + \lambda_1|u_n|_2^2 + \lambda_2|v_n|_2^2 + o(1) \ge \bar{\mu}_s(\mathbb{R}^N).
\end{gathered}$$

By (3.3), we see that $\lambda_1|u|_2^2 + \lambda_2|v|_2^2 \ge \bar{\mu}_s(\mathbb{R}^N) - \bar{\mu}_s^{A,B}(\Omega) > 0$, which means that $(u,v) \not\equiv (0,0)$. Since $w_n \rightharpoonup 0$ weakly in $H_A^1(\Omega)$ and $z_n \rightharpoonup 0$ weakly in $H_B^1(\Omega)$, we have

$$\begin{aligned}
|\nabla_A u_n|_2^2 &= \int_\Omega |\nabla_A w_n|^2 + \int_\Omega |\nabla_A u|^2 + 2\operatorname{Re}\Big(\int_\Omega \nabla_A w_n \cdot \overline{\nabla_A u}\Big)\\
&= |\nabla_A w_n|_2^2 + |\nabla_A u|_2^2 + o(1),\\
|\nabla_B v_n|_2^2 &= |\nabla_B z_n|_2^2 + |\nabla_B v|_2^2 + o(1).
\end{aligned}$$

Then, (3.4) yields

$$\bar{\mu}_s^{A,B}(\Omega) = |\nabla_A w_n|_2^2 + |\nabla_A u|_2^2 - \lambda_1|u|_2^2 + |\nabla_B z_n|_2^2 + |\nabla_B v|_2^2 - \lambda_2|v|_2^2 + o(1). \tag{3.5}$$

The Brezis-Lieb Lemma guarantees that

$$\begin{aligned}
1 &= \mu_1|u+w_n|_{2^*(s),s}^{2^*(s)} + \mu_2|v+z_n|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u+w_n|^\alpha|v+z_n|^\beta}{|x|^s}\\
&= \mu_1|u|_{2^*(s),s}^{2^*(s)} + \mu_2|v|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^s}\\
&\quad + \mu_1|w_n|_{2^*(s),s}^{2^*(s)} + \mu_2|z_n|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|w_n|^\alpha|z_n|^\beta}{|x|^s} + o(1).
\end{aligned}$$

Noting

$$\begin{gathered}
\mu_1|u|_{2^*(s),s}^{2^*(s)} + \mu_2|v|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^s} \le 1,\\
\mu_1|w_n|_{2^*(s),s}^{2^*(s)} + \mu_2|z_n|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|w_n|^\alpha|z_n|^\beta}{|x|^s} \le 1,
\end{gathered}$$

we have

$$1 \le \Big(\mu_1|u|_{2^*(s),s}^{2^*(s)} + \mu_2|v|_{2^*(s),s}^{2^*(s)} + \gamma\int_\Omega \frac{|u|^\alpha|v|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}$$

$$
\begin{aligned}
&+\Big(\mu_1|w_n|^{2^*(s)}_{2^*(s),s}+\mu_2|z_n|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|w_n|^\alpha|z_n|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}+o(1)\\
&\le\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}\\
&+\frac{1}{\bar\mu_s(\mathbb{R}^N)}\big(\big|\nabla|w_n|\big|_2^2+\big|\nabla|z_n|\big|_2^2\big)+o(1)\\
&\le\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}\\
&\quad+\frac{1}{\bar\mu_s(\mathbb{R}^N)}\big(|\nabla_A w_n|_2^2+|\nabla_B z_n|_2^2\big)+o(1).
\end{aligned}
$$

It follows from (3.3), (3.5) and $\bar\mu_s^{A,B}(\Omega)>0$ that

$$
\begin{aligned}
&|\nabla_A u|_2^2-\lambda_1|u|_2^2+|\nabla_B v|_2^2-\lambda_2|v|_2^2\\
&\le\bar\mu_s^{A,B}(\Omega)\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}\\
&\quad+\Big(\frac{\bar\mu_s^{A,B}(\Omega)}{\bar\mu_s(\mathbb{R}^N)}-1\Big)\big(|\nabla_A w_n|_2^2+|\nabla_B z_n|_2^2\big)+o(1)\\
&<\bar\mu_s^{A,B}(\Omega)\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big)^{\frac{2}{2^*(s)}}+o(1),
\end{aligned}
$$

which, combining with $(u,v)\not\equiv(0,0)$, implies

$$
\frac{|\nabla_A u|_2^2-\lambda_1|u|_2^2+|\nabla_B v|_2^2-\lambda_2|v|_2^2}{\big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\big)^{\frac{2}{2^*(s)}}}\le\bar\mu_s^{A,B}(\Omega).
$$

Then, $\bar\mu_s^{A,B}(\Omega)$ is attained by (u,v). It remains to show that (u,v) can not be the type of $(u,0)$ or $(0,v)$. Suppose by contradiction that $\bar\mu_s^{A,B}(\Omega)$ is attained by $(u,0)$. Then

$$
\bar\mu_s^{A,B}(\Omega)=\frac{|\nabla_A u|_2^2-\lambda_1|u|_2^2}{\mu_1^{\frac{2}{2^*(s)}}|u|^2_{2^*(s),s}}\ge\mu_1^{-\frac{2}{2^*(s)}}\mu_s^{A,\lambda_1}(\Omega),
$$

which contradicts to Lemma 3.1. Hence, (u,v) can not be the type of $(u,0)$. Similarly, it can not be $(0,v)$, which completes the proof. □

Remark 3.2. Even if $\bar\mu_s^{A,B}(\Omega)\le 0$, it is also attained. Indeed, by (3.5) and $\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\le 1$, we obtain

$$
\begin{aligned}
&|\nabla_A u|_2^2-\lambda_1|u|_2^2+|\nabla_B v|_2^2-\lambda_2|v|_2^2\\
&\le\mu_s^{A,B}(\Omega)\\
&\le\mu_s^{A,B}(\Omega)\Big(\mu_1|u|^{2^*(s)}_{2^*(s),s}+\mu_2|v|^{2^*(s)}_{2^*(s),s}+\gamma\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big).
\end{aligned}
$$

4. Ground states for the equations

By Lemma 2.4, Theorems 1.5–1.8, follow from Theorems 1.1–1.4 respectively.

Considering (1.3), by Theorem 1.5, we assume that u_{μ_1} and v_{μ_2} are ground state solutions of $-\Delta_A u=\mu_1\frac{|u|^{2^*(s)-2}u}{|x|^s}$ and $-\Delta_B v=\mu_1\frac{|v|^{2^*(s)-2}v}{|x|^s}$, respectively. It follows from Lemma 2.3 that the ground state energies are

$$M_{\mu_1} := \frac{2-s}{2(N-s)}\mu_1^{-\frac{N-2}{2-s}}\big(\mu_s(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}, \quad M_{\mu_2} := \frac{2-s}{2(N-s)}\mu_2^{-\frac{N-2}{2-s}}\big(\mu_s(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}.$$

We claim that if $\gamma < 0$, then (1.3) has no nontrivial ground state solution, which is the reason that we only consider the case $\gamma > 0$ in this paper. In fact, if $\gamma < 0$, then

$$|\nabla_A u|^2_{2,\mathbb{R}^N} - \mu_1 |u|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} = \frac{\alpha\gamma}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u|^\alpha |v|^\beta}{|x|^s} \le 0,$$

which implies

$$\mu_1 |u|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} \ge |\nabla_A u|^2_{2,\mathbb{R}^N} \ge \mu_s^A(\mathbb{R}^N)|u|^2_{2^*(s),s,\mathbb{R}^N}.$$

If $u \in D_A^{1,2}(\mathbb{R}^N)\setminus\{0\}$, then $|u|_{2^*(s),s,\mathbb{R}^N} \ge \big(\frac{\mu_s^A(\mathbb{R}^N)}{\mu_1}\big)^{\frac{1}{2^*(s)-2}}$, which yields that $|\nabla_A u|^2_{2,\mathbb{R}^N} \ge \mu_s^A(\mathbb{R}^N)\big(\frac{\mu_s^A(\mathbb{R}^N)}{\mu_1}\big)^{\frac{2}{2^*(s)-2}}$. Therefore,

$$M_{\mu_1} = \frac{2-s}{2(N-s)}\mu_1^{-\frac{N-2}{2-s}}\big(\mu_s^A(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}} \le \frac{2-s}{2(N-s)}|\nabla_A u|^2_{2,\mathbb{R}^N}.$$

Similarly, $M_{\mu_2} \le \frac{2-s}{2(N-s)}|\nabla_B v|^2_{2,\mathbb{R}^N}$ for any $v \in D_B^{1,2}(\mathbb{R}^N)\setminus\{0\}$. Suppose that (u,v) is a ground state solution of (1.3). Then

$$\begin{aligned} M_0 = I(u,v) &= \frac{2-s}{2(N-s)}\big(|\nabla_A u|^2_{2,\mathbb{R}^N} + |\nabla_B v|^2_{2,\mathbb{R}^N}\big) \\ &\ge \begin{cases} M_{\mu_2}, & \text{if } u = 0, v \ne 0, \\ M_{\mu_1} + M_{\mu_2}, & \text{if } u \ne 0, v \ne 0, \\ M_{\mu_1}, & \text{if } u \ne 0, v = 0. \end{cases} \end{aligned}$$

It can be seen that $M_0 \le \min\{M_{\mu_1},\ M_{\mu_2}\}$, which means that (1.3) has no nontrivial ground state solution. Define

$$\begin{aligned} \mathcal{N}' := \Big\{&(u,v) \in D_{A,A} : u \not\equiv 0, v \not\equiv 0, \\ &|\nabla_A u|^2_{2,\mathbb{R}^N} = \mu_1 |u|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \frac{\alpha\gamma}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u|^\alpha|v|^\beta}{|x|^s}, \\ &|\nabla_A v|^2_{2,\mathbb{R}^N} = \mu_2 |v|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \frac{\beta\gamma}{2^*(s)}\int_{\mathbb{R}^N}\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big\}, \end{aligned}$$

$$\begin{aligned} \mathcal{M}' := \Big\{&(u,v) \in H_{A,A} : u \not\equiv 0, v \not\equiv 0, \\ &|\nabla_A u|^2_2 - \lambda|u|^2_2 = \mu_1 |u|^{2^*(s)}_{2^*(s),s} + \frac{\alpha\gamma}{2^*(s)}\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}, \\ &|\nabla_A v|^2_2 - \lambda|v|^2_2 = \mu_2 |v|^{2^*(s)}_{2^*(s),s} + \frac{\beta\gamma}{2^*(s)}\int_\Omega\frac{|u|^\alpha|v|^\beta}{|x|^s}\Big\}, \end{aligned}$$

$M_0' := \inf_{(u,v)\in\mathcal{N}'} I(u,v)$ and $M' := \inf_{(u,v)\in\mathcal{M}'} E(u,v)$. It can be seen from Theorems 1.5 and 1.6 that

$$|\nabla_A u|^2_{2,\mathbb{R}^N} \ge \Big(\frac{2(N-s)}{2-s}M_1\Big)^{\frac{2-s}{N-s}}|u|^2_{2^*(s),s,\mathbb{R}^N}, \quad \forall u \in D_A^{1,2}(\mathbb{R}^N) \tag{4.1}$$

and

$$|\nabla_A u|_2^2 - \lambda |u|_2^2 \geq \Big(\frac{2(N-s)}{2-s}M_2\Big)^{\frac{2-s}{N-s}} |u|^2_{2^*(s),s}, \quad \forall u \in H^1_A(\Omega).$$

Define functions:

$$\begin{aligned} F_1(k,l) &:= \mu_1 k^{\frac{2^*(s)-2}{2}} + \frac{\alpha\gamma}{2^*(s)} k^{\frac{\alpha-2}{2}} l^{\beta/2} - 1, \quad k>0, l\geq 0;\\ F_2(k,l) &:= \mu_2 l^{\frac{2^*(s)-2}{2}} + \frac{\beta\gamma}{2^*(s)} k^{\frac{\alpha}{2}} l^{\frac{\beta-2}{2}} - 1, \quad k\geq 0, l>0. \end{aligned} \tag{4.2}$$

Following the arguments as in [8, Lemma 2.4] or [14, Proposition 2.2], we have the following result.

Proposition 4.1. *If* (A6) *holds, then*

$$\begin{gathered} k+l \leq k_0 + l_0,\\ F_1(k,l)\geq 0, \quad F_2(k,l)\geq 0,\\ k,l\geq 0, \quad (k,l)\neq (0,0) \end{gathered} \tag{4.3}$$

has a unique solution $(k,l) = (k_0,l_0)$*, where* (k_0,l_0) *is defined by* (1.6)*.*

Proof of Theorem 1.10. Recalling (1.5), we see that $(\sqrt{k_0}U, \sqrt{l_0}U) \in \mathcal{N}'$, that $(\sqrt{k_0}U, \sqrt{l_0}U)$ is a nontrivial solution of (1.3), and that

$$M_0' \leq I(\sqrt{k_0}U, \sqrt{l_0}U) = \Big(\frac12 - \frac{1}{2^*(s)}\Big)(k_0+l_0)|\nabla_A U|^2_{2,\mathbb{R}^N} = (k_0+l_0)M_1. \tag{4.4}$$

On the other hand, assume that $\{(u_n,v_n)\}\subset \mathcal{N}'$ is a minimizing sequence for M_0', that is, $I(u_n,v_n)\to M_0'$ as $n\to\infty$. Define

$$c_n = |u_n|^2_{2^*(s),s,\mathbb{R}^N}, \quad d_n = |v_n|^2_{2^*(s),s,\mathbb{R}^N},$$

and by (4.1), we obtain

$$\begin{aligned} \Big(\frac{2(N-s)M_1}{2-s}\Big)^{\frac{2-s}{N-s}} c_n &\leq |\nabla_A u_n|^2_{2,\mathbb{R}^N}\\ &= \mu_1 |u_n|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \frac{\alpha\gamma}{2^*(s)}\int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\\ &\leq \mu_1 c_n^{\frac{2^*(s)}{2}} + \frac{\alpha\gamma}{2^*(s)} c_n^{\frac{\alpha}{2}} d_n^{\beta/2},\\ \Big(\frac{2(N-s)M_1}{2-s}\Big)^{\frac{2-s}{N-s}} d_n &\leq |\nabla_A v_n|^2_{2,\mathbb{R}^N}\\ &= \mu_2 |v_n|^{2^*(s)}_{2^*(s),s,\mathbb{R}^N} + \frac{\beta\gamma}{2^*(s)}\int_{\mathbb{R}^N} \frac{|u_n|^\alpha |v_n|^\beta}{|x|^s}\\ &\leq \mu_2 d_n^{\frac{2^*(s)}{2}} + \frac{\beta\gamma}{2^*(s)} c_n^{\frac{\alpha}{2}} d_n^{\beta/2}. \end{aligned} \tag{4.5}$$

Dividing both sides of the inequalities by $\big(\frac{2(N-s)M_1}{2-s}\big)^{\frac{2-s}{N-s}} c_n$ and $\big(\frac{2(N-s)M_1}{2-s}\big)^{\frac{2-s}{N-s}} d_n$, respectively, and setting

$$\tilde c_n = \frac{c_n}{\big(\frac{2(N-s)M_1}{2-s}\big)^{\frac{N-2}{N-s}}}, \quad \tilde d_n = \frac{d_n}{\big(\frac{2(N-s)M_1}{2-s}\big)^{\frac{N-2}{N-s}}},$$

we have

$$\mu_1 \tilde{c}_n^{\frac{2^*(s)-2}{2}} + \frac{\alpha\gamma}{2^*(s)} \tilde{c}_n^{\frac{\alpha-2}{2}} \tilde{d}_n^{\beta/2} \geq 1,$$
$$\mu_2 \tilde{d}_n^{\frac{2^*(s)-2}{2}} + \frac{\beta\gamma}{2^*(s)} \tilde{c}_n^{\frac{\alpha}{2}} \tilde{d}_n^{\frac{\beta-2}{2}} \geq 1,$$

i.e., $F_1(\tilde{c}_n, \tilde{d}_n) \geq 0$ and $F_2(\tilde{c}_n, \tilde{d}_n) \geq 0$. Then, Proposition 4.1 ensures that $\tilde{c}_n + \tilde{d}_n \geq k_0 + l_0$, which means that

$$c_n + d_n \geq (k_0 + l_0)\Big(\frac{2(N-s)M_1}{2-s}\Big)^{\frac{N-2}{N-s}}. \tag{4.6}$$

It follows from (4.4), (4.5) and $I(u_n, v_n) = \frac{2-s}{2(N-s)} \|(u_n, v_n)\|^2_{D_{A,A}}$ that

$$\begin{aligned}
\Big(\frac{2(N-s)M_1}{2-s}\Big)^{\frac{2-s}{N-s}}(c_n + d_n) &\leq \frac{2(N-s)}{2-s} I(u_n, v_n) \\
&= \frac{2(N-s)}{2-s} M_0' + o(1) \\
&\leq \frac{2(N-s)}{2-s}(k_0 + l_0)M_1 + o(1).
\end{aligned}$$

Combining this with (4.6), we obtain

$$c_n + d_n \to (k_0 + l_0)\Big(\frac{2(N-s)M_1}{2-s}\Big)^{\frac{N-2}{N-s}}, \quad \text{as } n \to \infty.$$

Therefore,

$$\begin{aligned}
M_0' &= \lim_{n\to\infty} I(u_n, v_n) \\
&\geq \lim_{n\to\infty} \frac{2-s}{2(N-s)}\Big(\frac{2(N-s)B_1}{2-s}\Big)^{\frac{2-s}{N-s}}(c_n + d_n) = (k_0 + l_0)M_1.
\end{aligned}$$

By (4.4), we have

$$M_0' = (k_0 + l_0)M_1 = I(\sqrt{k_0}U, \sqrt{l_0}U). \tag{4.7}$$

Theorem 1.7 ensures that M_0 is attained by a nontrivial ground state solution $(u, v) \in \mathcal{N}$ of (1.3) with $B = A$. It is easy to see that $(u, v) \in \mathcal{N}'$, which implies that

$$M_0 = I(u, v) \geq \inf_{(\tilde{u},\tilde{v})\in\mathcal{N}'} I(\tilde{u}, \tilde{v}) = M_0'.$$

Obviously, $M_0 \leq M_0'$ follows from $\mathcal{N}' \subset \mathcal{N}$. Therefore, $M_0 = M_0'$, and combining this with (4.7), we see that $(\sqrt{k_0}U, \sqrt{l_0}U)$ is a ground state solution of (1.3). By (4.7), Theorem 1.5 and Lemma 2.3, we have

$$M_0 = \frac{2-s}{2(N-s)}(k_0 + l_0)\big(\mu_s(\mathbb{R}^N)\big)^{\frac{N-s}{2-s}}.$$

□

The proof Theorem 1.11 is similar to that of Theorem 1.10, it is omitted.

5. Application in three dimensions

In this section, we consider a constant magnetic field in dimension 3 as an application of Theorems 1.1 and 1.3.

Constant magnetic field. Let $\tilde{A}:\mathbb{R}^3\to\mathbb{R}^3$ by $\tilde{A}(x_1,x_2,x_3):=(-x_2,x_1,0)$, which is called constant magnetic potential as $\operatorname{curl}\tilde{A}=2\neq 0$. Theorem 1.1 guarantees that $\mu_s^{r\tilde{A}}(\mathbb{R}^3)$ is not achieved, and then

$$(-i\nabla+r\tilde{A})^2u=\frac{|u|^{2^*(s)-2}u}{|x|^s},\quad u\in D^{1,2}_{\tilde{A}}(\mathbb{R}^N)$$

has no ground state solution, where r is a nonzero real number. By Theorem 1.3, we obtain that under condition (A3), $\bar{\mu}_s^{r_1\tilde{A},r_2\tilde{A}}(\mathbb{R}^3)$ is not attained, and thus,

$$\begin{gathered}(-i\nabla+r_1\tilde{A})^2u=\mu_1\frac{|u|^{2^*(s)-2}u}{|x|^s}+\frac{\alpha\gamma}{2^*(s)}\frac{|u|^{\alpha-2}u|v|^\beta}{|x|^s},\\ (-i\nabla+r_2\tilde{A})^2v=\mu_2\frac{|v|^{2^*(s)-2}v}{|x|^s}+\frac{\beta\gamma}{2^*(s)}\frac{|u|^{\alpha}|v|^{\beta-2}v}{|x|^s},\\ u,\ v\in D^{1,2}_{\tilde{A}}(\mathbb{R}^3)\end{gathered}$$

has no ground state solution, where r_1 and r_2 are nonzero real numbers.

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