Electronic Journal of Differential Equations, Vol. 2017 (2017), No. 201, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

NONLOCAL INITIAL BOUNDARY VALUE PROBLEM FOR THE TIME-FRACTIONAL DIFFUSION EQUATION

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Communicated by Mokhtar Kirane

Abstract. In this article we discuss a method for constructing trace formulae for the heat-volume potential of the time-fractional diffusion equation to lateral surfaces of cylindrical domains and use these conditions to construct as well as to study a nonlocal initial boundary value problem for the time-fractional diffusion equation.

1. Introduction

Let us consider the one-dimensional potential

$$u(t) = \int_0^1 -\frac{1}{2}|t-\tau| f(\tau) d\tau \quad \text{in } \Omega = (0,1), \tag{1.1}$$

where f is an integrable function in Ω. The kernel of the one-dimensional potential is a fundamental solution of the second order differential equation; that is,

$$-\partial_t^2 E(t-\tau) = \delta(t-\tau), \tag{1.2}$$

where $E(t-\tau) = -\frac{1}{2}|t-\tau|$ and δ is the Dirac distribution. Hence the potential (1.1) satisfies the equation

$$-\partial_t^2 u(t) = f(t),\ t \in \Omega. \tag{1.3}$$

An interesting question having several important applications (in general) is what boundary condition can be put on u on the boundary of Ω so that equation (1.3) complemented by this boundary condition would have a unique solution in Ω still given by the same formula (1.1) (with the same kernel). This amounts to finding the trace of the one-dimensional Newton potential (1.1) to the boundary of Ω.

Simply, by using integration by parts, one obtains that boundary conditions for the potential (1.1) are

$$u'(0) + u'(1) = 0, -u'(1) + u(0) + u(1) = 0. \tag{1.4}$$

Hence if we solve equation (1.3) with the boundary conditions (1.4), then we find a unique solution of this boundary value problem in the form (1.1). This problem becomes more interesting for PDE. The trace of the Newton potential

2010 *Mathematics Subject Classification.* 26A33, 31A10.
Key words and phrases. Time-fractional diffusion equation; fundamental solution; time-fractional heat potential; layer potentials; nonlocal boundary condition.

Submitted August 22, 2016. Published August 31, 2017.

on a boundary surface appeared in Kac's work [4], where he called it and the subsequent spectral analysis as "the principle of not feeling the boundary". This was further expanded in Kac's book [5] with several further applications to the spectral theory and the asymptotics of the Weyl eigenvalue counting function. Some results towards answering these questions can be found in papers of Kac [4, 5], Saito [21], as well as in systematic studies of Kal'menov and Suragan [8, 9, 10, 11, 22], see also Kal'menov and Otelbaev [6] for the more general analysis. The analogues of the problem for the Kohn Laplacian and its powers on the Heisenberg group have been recently investigated by Ruzhanksy and Suragan in [19] as well as in [20] for general stratified Lie groups.

The main purpose of this paper is to construct trace formulae for the heat-volume potentials of the time-fractional diffusion equation to piecewise smooth lateral surfaces of cylindrical domains and use these conditions to construct as well as to study a nonlocal initial boundary value problem for the time-fractional diffusion equation. Consider

$$\lozenge_{\alpha,t} u = \partial_t^\alpha u - \Delta u = f \quad \text{in } \Omega \times (0,T), \tag{1.5}$$

$$u(0,x) = 0, \quad x \in \Omega, \tag{1.6}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with the boundary $\partial\Omega \in C^{1+\gamma}$, $0 < \gamma < 1$, $\Delta = \sum_{i=1}^n \partial_{x_i}^2$ is the Laplacian and

$$\partial_t^\alpha u(t,x) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\tau} u'_\tau(\tau,x) d\tau$$

is the fractional Caputo time derivative of order $0 < \alpha \le 1$. Here Γ is the gamma function. We shall note that for $\alpha = 1$ the fractional derivative coincides with the standard time derivative.

For the convenience of the reader let us now briefly recapture the main results of this paper:

We establish trace formulae for the time-fractional heat potential operator

$$\int_0^t d\tau \int_\Omega E(x-y, t-\tau) f(\tau, y) dy$$

to the surface $\partial\Omega \times (0,T)$, where $\partial\Omega$ is the boundary of the bounded domain $\Omega \subset \mathbb{R}^n$. Then we use this to introduce a version of Kac's boundary value problem, that is Kac's principle of "not feeling the boundary" for the time-fractional heat operator $\lozenge_{\alpha,t}$.

In Section 2 we very briefly review the main concepts of potential theory for the fractional diffusion equation and fix the notation. In Section 3 we derive trace formulae and give the analogues of Kac's boundary value problem for the time-fractional diffusion equation in Theorem 3.1.

2. Preliminaries

In this section we very briefly review some important concepts of the time-fractional diffusion equation and fix the notation. For the general background details on potential theory of the time-fractional diffusion equation we refer to [12, 15, 16, 1]. The fundamental solution of the time-fractional diffusion equation (1.5) is given by

$$E(x,t) = \theta(t)\pi^{-d/2} t^{\alpha-1} |x|^{-d} H_{12}^{20}\Big(\frac{1}{4}|x|^2 t^{-\alpha}\big|_{(-d/2,1),(1,1)}^{(\alpha,\alpha)}\Big), \tag{2.1}$$

where H is the Fox H-function (see e.g. [17]) and θ is the Heaviside step function. It is constructed by taking the Laplace-transform in the time and the Fourier-transform in the spatial variable of the time-fractional diffusion equation

$$\Diamond_{\alpha,t} E(x,t) := (\partial_t^\alpha - \Delta_x)E(x,t) = \delta(x,t),$$

where $\delta(x,t)$ is the Dirac distribution at the origin, and by using the inverse Fourier-transform of the Mittag-Leffler function. Heat volume potential, single and double layer potentials of the time-fractional diffusion equation, respectively, can be defined by

$$(\Diamond_{\alpha,t}^{-1}\rho)(x,t) = \int_0^t \int_\Omega E(x-y,t-\tau)\rho(y,\tau)dyd\tau, \tag{2.2}$$

$$(S\rho)(x,t) = \int_0^t \int_{\partial\Omega} E(x-y,t-\tau)\rho(y,\tau)dyd\tau, \tag{2.3}$$

$$(D\rho)(x,t) = \int_0^t \int_{\partial\Omega} \partial_n E(x-y,t-\tau)\rho(y,\tau)dyd\tau, \tag{2.4}$$

where ∂_n is the outer normal derivative on the boundary $\partial\Omega$ of the bounded domain Ω. Here we also recall Green's formula (see, for example, [15]) for the the time-fractional diffusion operator

$$\int_0^T \int_\Omega (\Diamond_{\alpha,\tau} u P_T v - P_T u \Diamond_{\alpha,\tau} v)\, dxd\tau = \int_0^T \int_{\partial\Omega} (u\partial_n P_T v - \partial_n u P_T v) dSd\tau, \tag{2.5}$$

where P_T is a time involution operator on the interval $(0,T)$ and is defined by setting

$$P(T)v(\tau) = v(T-\tau).$$

3. Trace formula and initial boundary value problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded domain with Lyapunov boundary $\partial\Omega \in C^{1+\lambda}$, $0 < \lambda < 1$, and $f \in C(\overline{(0,T) \times \Omega})$ such that $f(\cdot,t)$ is Hölder continuous uniformly in $t \in [0,T]$ and $\operatorname{supp} f(\cdot,t) \subset \Omega$, $t \in [0,T]$. Consider the following time-fractional generalization of the heat potential (time-fractional heat potential)

$$u(x,t) := \Diamond_{\alpha,t}^{-1} f = \int_0^t d\tau \int_\Omega E(x-y,t-\tau)f(\tau,y)dy,\ x \in \Omega,\ t \in (0,T), \tag{3.1}$$

where E is a fundamental solution of $\Diamond_{\alpha,t}$. Here our aim is to find a boundary condition for u on the boundary $\partial\Omega$ of a bounded domain Ω such that with this boundary condition the equation

$$\begin{gathered} \Diamond_{\alpha,t} u(x,t) = f(x,t), \quad \text{in} \quad \Omega \times (0,T), \\ u(x,0) = 0, \quad x \in \Omega, \end{gathered} \tag{3.2}$$

has a unique classical solution and this solution is the time-fractional heat potential (3.1). This amounts to finding the trace of the integral operator in (3.1) on $\partial\Omega$.

A starting point for us will be that if $f \in C(\overline{\Omega \times (0,T)})$ such that $f(\cdot,t)$ is Hölder continuous uniformly in $t \in [0,T]$ and $\operatorname{supp} f(\cdot,t) \subset \Omega$, $t \in [0,T]$, then u defined by (3.1) is well defined and satisfies the initial problem (3.2) (see [13, Theorem 2.4]).

Our main result for the time-fractional heat potential operator is the following variant of Kac's formula (see the discussion in the introduction of [18] and [19]) for a case of setting of the time-fractional diffusion equation.

Theorem 3.1. *For each $f \in C(\overline{\Omega \times (0,T)})$ such that $f(\cdot,t)$ is Hölder continuous uniformly in $t \in [0,T]$ and $\operatorname{supp} f(\cdot,t) \subset \Omega$, $t \in [0,T]$, the time-fractional heat potential $u = \Diamond_{\alpha,t}^{-1} f$ satisfies the following nonlocal boundary condition:*

$$\begin{aligned} &-\frac{u(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y,t-\tau) u(y,\tau) dS_y \\ &- \int_0^t d\tau \int_{\partial\Omega} E(x-y,t-\tau) \partial_n u(y,\tau) dS_y = 0, \end{aligned} \tag{3.3}$$

for all $x \in \partial\Omega$ and $t \in (0,T)$. Conversely, if u is a solution of the time-fractional diffusion equation

$$\Diamond_{\alpha,t} u = f, \tag{3.4}$$

satisfying the initial condition

$$u|_{t=0} = 0, \quad \text{on } \Omega, \tag{3.5}$$

and the boundary condition (3.3)*, then it is given as the time-fractional heat potential $u = \Diamond_{\alpha,t}^{-1} f$ by formula* (3.1) *and it is unique.*

Corollary 3.2. *It follows from Theorem 3.1 that the kernel E, which is a fundamental solution of the time-fractional diffusion equation, is Green's function of the nonlocal initial boundary value problem* (3.3)-(3.5) *in $\Omega \times (0,T)$. Therefore, the initial nonlocal boundary value problem* (3.3)-(3.5) *can serve as an example of an explicitly solvable initial boundary value problem for the time-fractional diffusion equation for any $0 < \alpha \le 1$ (and independent of the shape of the domain Ω).*

Proof of Theorem 3.1. By using Green's formula (2.5), for any $x \in \Omega$ and $t \in (0,T)$, we obtain

$$\begin{aligned} u(x,T-t) &= \int_0^{T-t} d\tau \int_\Omega E(x-y,T-t-\tau) f(y,\tau) dy \\ &= \int_0^{T-t} d\tau \int_\Omega E(x-y,T-t-\tau) \Diamond_{\alpha,\tau} u(y,\tau) dy \\ &= \int_0^{T} d\tau \int_\Omega E(x-y,T-t-\tau) \Diamond_{\alpha,\tau} u(y,\tau) dy \\ &= \int_0^{T} d\tau \int_\Omega \Diamond_{\alpha,\tau} E(x-y,\tau-t) u(y,T-\tau) dy \\ &\quad + \int_0^T d\tau \int_{\partial\Omega} \partial_n E(x-y,T-t-\tau) u(y,\tau) dS_y \\ &\quad - \int_0^T d\tau \int_{\partial\Omega} E(x-y,T-t-\tau) \partial_n u(y,\tau) dS_y \\ &= u(y,T-t) + \int_0^T d\tau \int_{\partial\Omega} \partial_n E(x-y,T-t-\tau) u(y,\tau) dS_y \\ &\quad - \int_0^T d\tau \int_{\partial\Omega} E(x-y,T-t-\tau) \partial_n u(y,\tau) dS_y, \end{aligned}$$

for any $x \in \Omega$ and $t \in (0,T)$. That is, we have

$$\begin{aligned} &\int_0^T d\tau \int_{\partial\Omega} \partial_n E(x-y, T-t-\tau) u(y,\tau) dS_y \\ &- \int_0^T d\tau \int_{\partial\Omega} E(x-y, T-t-\tau) \partial_n u(y,\tau) dS_y \equiv 0, \end{aligned} \tag{3.6}$$

for any $x \in \Omega$ and $t \in (0,T)$. Since $\theta(T-t-\tau) = 0$ for $T-t < \tau$, this means

$$\begin{aligned} &\int_0^{T-t} d\tau \int_{\partial\Omega} \partial_n E(x-y, T-t-\tau) u(y,\tau) dS_y \\ &- \int_0^{T-t} d\tau \int_{\partial\Omega} E(x-y, T-t-\tau) \partial_n u(y,\tau) dS_y \equiv 0, \end{aligned} \tag{3.7}$$

for any $x \in \Omega$ and $t \in (0,T)$. Therefore, denoting $T-t$ by t, we obtain

$$\begin{aligned} &\int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y, t-\tau) u(y,\tau) dS_y \\ &- \int_0^t d\tau \int_{\partial\Omega} E(x-y, t-\tau) \partial_n u(y,\tau) dS_y = 0, \end{aligned} \tag{3.8}$$

for all $t \in (0,T)$ and $x \in \Omega$. By using the properties of the (time-fractional) double and single layer potentials (see [12, Theorem 1] and [14, Theorem 2.1]) as x approaches the boundary $\partial\Omega$ from the interior, from (3.8), we obtain

$$\begin{aligned} &-\frac{u(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y, t-\tau) u(y,\tau) dS_y \\ &- \int_0^t d\tau \int_{\partial\Omega} E(x-y, t-\tau) \partial_n u(y,\tau) dS_y = 0, \end{aligned} \tag{3.9}$$

for all $t \in (0,T)$ and $x \in \partial\Omega$. This shows that (3.1) is a solution of the initial boundary value problem (3.4)-(3.5)-(3.3).

Now let us prove its uniqueness. If the initial boundary value problem has two solutions u and u_1, then the function $w = u - u_1$ satisfies

$$\begin{gathered} \Diamond_{\alpha,t} w(x,t) = 0, \quad \text{in } \Omega \times (0,T), \\ w(x,0) = 0, \quad x \in \Omega, \end{gathered} \tag{3.10}$$

and the boundary condition (3.3), i.e.

$$\begin{aligned} &-\frac{w(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y, t-\tau) w(y,\tau) dS_y \\ &- \int_0^t d\tau \int_{\partial\Omega} E(x-y, t-\tau) \partial_n w(y,\tau) dS_y = 0, \end{aligned} \tag{3.11}$$

for all $t \in (0,T)$ and $x \in \partial\Omega$.

Since $f = 0$ in this case, instead of (3.8) we have the representation formula

$$\begin{aligned} w(x,t) = &- \int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y, t-\tau) w(y,\tau) dS_y \\ &+ \int_0^t d\tau \int_{\partial\Omega} E(x-y, t-\tau) \partial_n w(y,\tau) dS_y, \end{aligned} \tag{3.12}$$

for all $t \in (0,T)$ and $x \in \Omega$. As above, by using the properties of the double and single layer potentials as $\Omega \ni x \to \partial\Omega$, we obtain

$$\begin{aligned} -w(x,t) = -\frac{w(x,t)}{2} + \int_0^t d\tau \int_{\partial\Omega} \partial_n E(x-y,t-\tau) w(y,\tau) dS_y \\ - \int_0^t d\tau \int_{\partial\Omega} E(x-y,t-\tau)\partial_n w(y,\tau) dS_y, \end{aligned} \tag{3.13}$$

for any $x \in \partial\Omega$ and $t \in (0,T)$. Comparing this with (3.11), we arrive at $w(t,x) = 0$, $x \in \partial\Omega$, $t \in (0,T)$, by uniqueness of the solution of the mixed Cauchy-Dirichlet problem (see [13], see also [2] for more general discussions) we get $w \equiv 0$, i.e. $u = \Diamond_{\alpha,t}^{-1} f$. So we obtain the desired result. □

Acknowledgements. The authors were supported in parts by the MES RK grants 0825/GF4 and 4075/GF4 as well as by the MES RK target grant 0085/PTSF-14.

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