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EXISTENCE OF WEAK SOLUTIONS TO A NONLINEAR REACTION-DIFFUSION SYSTEM WITH SINGULAR SOURCES

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Abstract. We discuss the existence of a class of weak solutions to a nonlinear parabolic system of reaction-diffusion type endowed with singular production terms by reaction. The singularity is due to a potential occurrence of quenching localized to the domain boundary. The kind of quenching we have in mind is due to a twofold contribution: (i) the choice of boundary conditions, modeling in our case the contact with an infinite reservoir filled with ready-to-react chemicals and (ii) the use of a particular nonlinear, non-Lipschitz structure of the reaction kinetics. Our working techniques use fine energy estimates for approximating non-singular problems and uniform control on the set where singularities are localizing.

1. Introduction

Our main interest lies in combining homogenization asymptotics together with either fast reactions (like in [18, 19]) or with singular reactions (like in [4] and [11]). In this article, we set the foundations for such investigations by exploring the effect of the choice of a particular type of singularity on the weak solvability of the model equations. The singularity is supposed here to appear due to the occurrence of a localized strong quenching behavior.

The quenching phenomenon is expected to be due to the kinetics of diffusion-limited reactions in random and/or confined geometries; see e.g. [12] and references cited therein. However, we are not aware of a multi-particle system derivation of the structure of the (macroscopic) singularity in the reaction rate in the case of quenching. Our approach here is simply ansatz-based. Traditionally, the mass action law of chemical kinetics usually requires integer partial reaction orders (cf. [26], e.g.). In our setting, we use instead a power-law reaction rate, sometimes referred to as being based on a pseudo-mass action kinetics. The reader can find in [16], e.g., a number of concrete examples of chemical reaction mechanisms of fractional order. We use this occasion to refer also, for instance, to [2] (and references cited therein) for classes of chemical reactions not respecting the classical mass action kinetics.

The goal in this paper is to study the existence of weak solutions to the following system of nonlinear equations of reaction-diffusion type endowed with singular production terms by reaction, mimicking the quenching feature:

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Let $Q_T := \Omega \times (0,T)$, where Ω is a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$ and $T > 0$,

$$\begin{gathered}
u_t - \operatorname{div}(a(x,t,u,\nabla u)) = f(u,v) \quad \text{in } \Omega \times (0,T) \\
v_t - \operatorname{div}(b(x,t,v,\nabla v)) = g(u,v) \quad \text{in } \Omega \times (0,T) \\
u(x,0) = u_0(x) \quad \text{in } \Omega \\
v(x,0) = v_0(x) \quad \text{in } \Omega \\
u(x,t) = 0 \quad \text{on } \Gamma_1 \times (0,T) \\
v(x,t) = 0 \quad \text{on } \Gamma_2 \times (0,T) \\
a(x,t,u,\nabla u) \cdot \nu = 0 \quad \text{on } \Gamma_2 \times (0,T) \\
b(x,t,v,\nabla v) \cdot \nu = 0 \quad \text{on } \Gamma_1 \times (0,T)
\end{gathered} \tag{1.1}$$

where Γ_1 and Γ_2 are such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The Haussdorff measure of Γ_1 and Γ_2 does not vanish, i.e. $\mathcal{H}(\Gamma_1) \neq 0$ and $\mathcal{H}(\Gamma_2) \neq 0$.

Here ν denotes the outer normal to $\partial\Omega$. The functions $f(r,s) : [0,+\infty) \to \mathbb{R}$ and $g(r,s) : [0,+\infty) \to \mathbb{R}$ are defined as

$$f(r,s) := k\frac{s}{r^\gamma}, \tag{1.2}$$

$$g(r,s) := -k\frac{s}{r^\gamma} \tag{1.3}$$

with $k > 0$ and $0 < \gamma \leq 1$ real parameters. Note that the functions f and g are singular at $r = 0$, i.e. they can take the value $+\infty$ when $r = 0$ and $s \neq 0$.

Problem (1.1) has a clear physical meaning. To fix ideas, just imagine the following scenario: let u, v denote the mass concentration of two distinct chemical species (reactant and product), being involved in the chemical reaction mechanism

$$U \to V, \tag{1.4}$$

where U denotes the chemical species associated to u and V denotes the chemical species associated to v. Such chemical mechanism can refer either to a gas-liquid reaction (cf. [21, sec. 2.4.3.3]) or to a gas-gas reaction (cf. [1]). These chemicals are provided (from infinite reservoirs) at Γ_1 and Γ_2, they travel a heterogeneous medium Ω (modeled here by the use of nonlinear diffusivities $a(\cdot)$ and $b(\cdot)$), and finally, they mix. It is worth noting that the mechanism (1.4) does not require *per se* that the species U and V coexist.

We restrict our attention by using the following assumptions: the functions $a(x,t,s,\xi)$ and $b(x,t,s,\xi) : \Omega \times (0,T) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and satisfy the following Leray-Lions conditions:

- (A1) $a(x,t,s,\xi) \cdot \xi \geq \alpha|\xi|^p$, $\alpha \in \mathbb{R}^+$, $p > 1$, a.e. in Q_T and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$;
- (A2) $(a(x,t,s,\xi) - a(x,t,s,\eta)) \cdot (\xi - \eta) > 0$, for every $s \in \mathbb{R}$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ such that $\xi \neq \eta$;
- (A3) $|a(x,t,s,\xi)| \leq \alpha_1|\xi|^{p-1}$ with $\alpha_1 \in \mathbb{R}^+$;
- (A4) $b(x,t,s,\xi) \cdot \xi \geq \beta|\xi|^p$, $\beta \in \mathbb{R}^+$, $p > 1$ a.e. in Q_T and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$;
- (A5) $(b(x,t,s,\xi) - b(x,t,s,\eta)) \cdot (\xi - \eta) > 0$, for every $s \in \mathbb{R}$ and for all $(s,\xi) \in \mathbb{R} \times \mathbb{R}^N$ such that $\xi \neq \eta$;
- (A6) $|b(x,t,s,\xi)| \leq \beta_1|\xi|^{p-1}$ with $\beta_1 \in \mathbb{R}^+$;
- (A7) the functions u_0 and v_0 are nonnegative functions that belong to $L^\infty(\Omega)$.

We set our problem in the following spaces:

$$V := \{\varphi \in W^{1,p}(\Omega) : \varphi = 0 \quad \text{on } \Gamma_1\},$$

$$W := \{\psi \in W^{1,p}(\Omega) : \psi = 0 \quad \text{on } \Gamma_2\},$$

with $p > 1$. The dual spaces of V and W, respectively, are denoted by $(V)^*$ and $(W)^*$.

We can now give our definition of weak solution to problem (1.1).

Definition 1.1. A weak solution to problem (1.1) is a nonnegative couple $(u, v) \in [L^p(0,T;V) \cap L^\infty(0,T;L^2(\Omega)] \times [L^p(0,T;W) \cap L^\infty(0,T;L^2(\Omega))]$ with
$(u_t, v_t) \in [L^{p'}(0,T;(V)^*) + L^1(0,T;L^1_{\rm loc}(\Omega))] \times [L^{p'}(0,T;(W)^*) + L^1(0,T;L^1_{\rm loc}(\Omega))]$
such that:

$$(u(x,0), v(x,0)) = (u_0(x), v_0(x)) \quad \text{a.e. } x \in \Omega, \tag{1.5}$$

$$\iint_{Q_T} \frac{v}{u^\gamma}\phi < +\infty, \tag{1.6}$$

$$-\int_\Omega u_0(x)\varphi(x,0) - \iint_{Q_T} u\frac{\partial \varphi}{\partial t} + \iint_{Q_T} a(x,t,u,\nabla u)\nabla\varphi = \iint_{Q_T} f(u,v)\varphi, \tag{1.7}$$

$$-\int_\Omega v_0(x)\psi(x,0) - \iint_{Q_T} v\frac{\partial \psi}{\partial t} + \iint_{Q_T} b(x,t,v,\nabla v)\nabla\psi = \iint_{Q_T} g(u,v)\psi, \tag{1.8}$$

for all $\varphi, \psi, \phi \in C_0^\infty(\Omega \times [0,T))$.

We give the following existence result for the solution of problem (1.1).

Theorem 1.2 (Existence)**.** *Assume* $0 < \gamma \le 1$, (A1)–(A7)*. Then there exists a solution* (u, v) *to problem* (1.1) *in the sense of Definition 1.1.*

Problem (1.1) consists of a system of two weakly coupled equations which present in the right hand side singular lower order term in the variable u. By *singular* we mean, in this context, that the terms $f(r,s)$ and/or $g(r,s)$ can become unbounded when $r = 0$. Scalar parabolic problems which present lower order terms of this type were studied previously in [8, 9, 10].

Essentially, problems of type (1.1) which exhibit equations with singular lower order term of type $f(u) = -\frac{1}{u^p}$, $p > 0$, have a global solution for which there exists a time T such that $inf_{x\in\Omega} f \to 0$ as $t \to T$. So, the reaction term tends to blow up when the solution goes towards extinction. This kind of phenomenon is called *quenching* (or in some case *extinction*, as in [6]). For example, if we solve the ordinary differential equation

$$u' = -\frac{1}{u^p}, \quad t > 0, \quad u(0) = 1$$

($p > 0$), we obtain

$$u(t) = [1 - (1+p)t]^{\frac{1}{p+1}}, \quad \text{for some } t > 0.$$

The main observation is here that the solution is smooth for $t \in \left(0, \frac{1}{p+1}\right)$ and $u(t) \to 0$ for $t \to \frac{1}{p+1}$, that is u quenches in finite time.

If we consider the partial differential equation

$$u_t - \Delta u = -\frac{1}{u^p}$$

the situation becomes somewhat more complicated. Now, the presence of the diffusion term Δu attempts to prevent the quenching phenomenon and an intrinsic reaction-diffusion competition appears.

In this paper, we search for local-in-time weak solutions to our problem (1.1). It is worth however mentioning that, under additional strong structural conditions on our Leray-Lions-like operators, working technical ideas from [22], which rely on sub- and super-solutions or at least on the existence of some global bounds, can be used in principle to extend our concept of local-in-time weak solution up to a global weak solution.

2. Nonsingular approximating problems

To deal with problem (1.1) we use a couple of approximations. In particular, we consider the following sequence of nonsingular approximating problems (2.1). Essentially, we are truncating in such a way as to eliminate the singularity. The approximating Problem reads: Find $(u_n, v_n) \in [L^p(0,T;V)\cap L^\infty(Q_T)]\times[L^p(0,T;W)\cap L^\infty(Q_T)]$ such that

$$\begin{gathered}
(u_n)_t - \operatorname{div}(a(x,t,u_n,\nabla u_n)) = f_n(u_n,v_n) \quad \text{in } \Omega\times(0,T)\\
(v_n)_t - \operatorname{div}(b(x,t,v_n,\nabla v_n)) = g_n(u_n,v_n) \quad \text{in } \Omega\times(0,T)\\
u(x,0) = u_{0,n}(x) \quad \text{in } \Omega\\
v(x,0) = v_{0,n}(x) \quad \text{in } \Omega\\
u_n(x,t) = 0 \quad \text{on } \Gamma_1\times(0,T)\\
v_n(x,t) = 0 \quad \text{on } \Gamma_2\times(0,T)\\
a(x,t,u_n,\nabla u_n)\cdot\nu = 0 \quad \text{on } \Gamma_2\times(0,T)\\
b(x,t,v_n,\nabla v_n)\cdot\nu = 0 \quad \text{on } \Gamma_1\times(0,T)
\end{gathered} \tag{2.1}$$

where

$$f_n(u_n,v_n) = \begin{cases} k\frac{v_n}{(u_n+\frac{1}{n})^\gamma}, & \text{if } u_n\ge 0 \text{ and } v_n\ge 0\\ 0, & \text{otherwise,}\end{cases}$$
$$g_n(u_n,v_n) = \begin{cases} -k\frac{v_n}{(u_n+\frac{1}{n})^\gamma}, & \text{if } u_n\ge 0 \text{ and } v_n\ge 0\\ 0, & \text{otherwise,}\end{cases}$$

while $u_{0,n}, v_{0,n} \in L^\infty(\Omega)\cap H_0^1(\Omega)$ are suitable regularizations of the initial data obtained by a standard convolution technique (see [5]) such that

$$\lim_{n\to\infty}\frac{1}{n}\|u_{0,n}\|_{H_0^1(\Omega)} = 0, \tag{2.2}$$
$$\lim_{n\to\infty}\frac{1}{n}\|v_{0,n}\|_{H_0^1(\Omega)} = 0. \tag{2.3}$$

Lemma 2.1. *Problem* (2.1) *admits a nonnegative couple of solutions*

$$(u_n,v_n)\in[L^p(0,T;V)\cap L^\infty(Q_T)]\times[L^p(0,T;W)\cap L^\infty(Q_T)]$$

such that

$$\begin{aligned}
&-\int_\Omega u_{0,n}(x)\varphi(x,0) - \iint_{Q_T} u_n\frac{\partial\varphi}{\partial t} + \iint_{Q_T} a(x,t,u_n,\nabla u_n)\nabla\varphi\\
&= k\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\varphi,
\end{aligned} \tag{2.4}$$

$$
\begin{aligned}
&-\int_\Omega v_{0,n}(x)\psi(x,0)-\iint_{Q_T} v_n\frac{\partial\psi}{\partial t}+\iint_{Q_T} b(x,t,v_n,\nabla v_n)\nabla\psi\\
&=-k\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\psi,
\end{aligned}
\tag{2.5}
$$

for every $\varphi,\psi\in C_0^\infty(\Omega\times[0,T))$.

Proof. The existence of a solution (u_n,v_n) can be proved following the line of standard results of [17]. For simplicity, we suppose $u_{0,n}=0$ and $v_{0,n}=0$. Then, using the method by Stampacchia [25], we can prove that $u_n\geq 0$ taking as test function in the first equation of the problem (2.1) the function $\varphi=-u_n^-$.

Since $u_n^+=0$ on the support of u_n^- (i.e. where $u_n\leq 0$) and remember that

$$
f_n(u_n,v_n)=\begin{cases} k\frac{v_n}{(u_n+\frac{1}{n})^\gamma}, & \text{if } u_n\geq 0 \text{ and } v_n\geq 0\\ 0, & \text{otherwise,}\end{cases}
$$

we have that the right hand side of (2.4) is zero, so we obtain

$$
\iint_{Q_T}(u_n)_t(-u_n^-)+\iint_{Q_T} a(x,t,u_n,\nabla u_n)\nabla(-u_n^-)=0.
$$

We rewrite the last equality as

$$
\iint_{Q_T}(u_n^+-u_n^-)_t(-u_n^-)+\iint_{Q_T} a(x,t,u_n^+-u_n,\nabla(u_n^+-u_n^-))\nabla(-u_n^-)=0
$$

from which, by (A1) we obtain

$$
\frac{1}{2}\int_\Omega(u_n^-)^2(t)+\alpha\iint_{Q_T}|\nabla u_n^-|^p\leq 0,
$$

and we deduce that

$$
u_n^-=0\quad \text{a.e. in } Q_T,
$$

i.e. that $u_n\geq 0$ a.e. in Ω and for all $t\in[0,T)$. In the same way, to obtain that $v_n\geq 0$, we can reason as before, by choosing as test function $\psi=-v_n^-$. □

From now on, we denote with C a generic constant. Its precise value changes depending on the context. Usually C is thought to be independent of n, if not otherwise mentioned. We recall here the definition of the usual truncation function T_k, defined as

$$
T_k(s)=\max\{-k,\min\{k,s\}\},\quad k\geq 0,\ s\in\mathbb{R}^+. \tag{2.6}
$$

In the following we will denote by $\langle\cdot,\cdot\rangle$ the duality product between $(V)^*$ and V (and also between $(W)^*$ and W).

3. A priori uniform estimates

3.1. Uniform estimate for (u_n,v_n) in $L^\infty(Q_T)$.

Proposition 3.1. *Assume* (A1)–(A7). *Then there exist positive constants* M_1 *and* M_2*, independent of* n*, such that:*

$$
\|u_n\|_{L^\infty(Q_T)}\leq M_1, \tag{3.1}
$$

$$
\|v_n\|_{L^\infty(Q_T)}\leq M_2. \tag{3.2}
$$

Proof. The uniform estimate (3.1) for the sequence $\{u_n\}$ follows by the [10, Proposition 2.13].

For simplicity we suppose $v_{0,n}(x) = 0$. To handle the equation solved by v_n we choose as test function $\psi = G_{M_2}(v_n) := (v_n - M_2)^+$, with $M_2 > 1$ fixed. By (A4), we obtain

$$\iint_{Q_t} (v_n)_t (v_n - M_2)^+ + \beta \iint_{Q_t} |\nabla G_{M_2}(v_n)|^p \le -k \iint_{Q_t} \frac{v_n G_{M_2}(v_n)}{(u_n + \frac{1}{n})^\gamma} \le 0,$$

where $Q_t := \Omega \times [0,t)$. Neglecting the nonnegative term on the left hand side, it follows that

$$\frac{1}{2}\int_\Omega [(v_n - M_2)^+]^2(t) = 0$$

from which $(v_n - M_2)^+ = 0$ a.e. in Q_T, i.e. (3.2) is proved. □

3.2. **Energy estimate for (u_n, v_n) in $L^p(0,T;V) \times L^p(0,T;W)$.**

Proposition 3.2. *Assume* (A1)–(A6). *Then there exists a positive constant C, independent of n, such that:*

$$\|u_n\|_{L^p(0,T;V)} \le C, \tag{3.3}$$

$$\|v_n\|_{L^p(0,T;W)} \le C. \tag{3.4}$$

Proof. Choosing as test function $\varphi = u_n \in L^p(0,T;V)$ in the first equation of problem (2.1) solved by u_n and integrating over $\Omega \times [0,t)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^T \|u_n\|^2_{L^2(\Omega)} dt + \iint_{Q_T} a(x,t,u_n,\nabla u_n)\nabla u_n = k \iint_{Q_T} \frac{v_n u_n}{(u_n + \frac{1}{n})^\gamma}.$$

By assumption (A1) and observing that $\frac{u_n}{(u_n+\frac{1}{n})^\gamma} \le u_n^{1-\gamma}$, the previous equality leads to

$$\frac{1}{2}\int_\Omega u_n^2(t) + \alpha \iint_{Q_T} |\nabla u_n|^p \le k \iint_{Q_T} v_n u_n^{1-\gamma} + C\|u_0\|^2_{L^2(\Omega)}.$$

Following the same steps as in the proof of [8, Lemma 2.4-(i)], we find that

$$\frac{1}{2}\int_\Omega u_n^2(t) + \alpha \iint_{Q_T} |\nabla u_n|^p \le C. \tag{3.5}$$

Now, from (3.5) we deduce also that

$$\|u_n\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{3.6}$$

To handle the second equation of problem (2.1), we choose as test function $\psi = v_n \in L^p(0,T;W)$. By (A4) we obtain the inequality

$$\begin{aligned}\frac{1}{2}\int_\Omega v_n^2(t) + \beta \iint_{Q_T} |\nabla v_n|^p &\le -k \iint_{Q_T} \frac{v_n^2}{(u_n + \frac{1}{n})^\gamma} + C\|v_{0,n}\|^2_{L^2(\Omega)} \\ &\le C\|v_{0,n}\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.7}$$

From (3.7) we deduce also that

$$\|v_n\|_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{3.8}$$

Summing (3.5) and (3.7) leads to the estimates (3.3) and (3.4). □

An important *a priori* estimate for controlling the singular lower order term is the following.

Proposition 3.3. *Assume* $\gamma > 0$, (A1)–(A7). *Then there exists a positive constant* C*, independent of* n*, such that*

$$k \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \le C \quad \textit{for all } n \in \mathbb{N}, \tag{3.9}$$

for every $\varphi \in C_0^\infty(\Omega), \varphi \ge 0$.

Proof. We multiply the first equation of problem (2.1) by the test function $\varphi^p(x)$ and get

$$\int_0^T \langle (u_n)_t, \varphi^p(x) \rangle + p \iint_{Q_T} a(x,t,u_n,\nabla u_n)\varphi^{p-1}\nabla\varphi = k \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x),$$

from which, using (A3), we obtain

$$\begin{aligned} k \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) &\le p \iint_{Q_T} |a(x,t,u_n,\nabla u_n)| \varphi^{p-1} |\nabla\varphi| + C \\ &\le \alpha_1 p \iint_{Q_T} |\nabla u_n|^{p-1} \varphi^{p-1} |\nabla \varphi| + C \le C. \end{aligned}$$

Here we have used once again Young's inequality with exponents $\frac{p}{p-1}$ and p together with the energy estimate (3.3). □

3.3. **Uniform estimate on the sets** $\{(x,t) \in Q_T : u_n(x,t) \le \delta\}$ **and** $\{(x,t) \in Q_T : v_n(x,t) \le \delta\}$**.** In this subsection, we focus our attention on the critical sets

$$\begin{gathered} \{(x,t) \in Q_T : u_n(x,t) \le \delta\}, \\ \{(x,t) \in Q_T : v_n(x,t) \le \delta\}. \end{gathered}$$

These sets are prone to hosting the locations of the singularity, i.e., where the lower order term is unbounded when $u_n = 0$, or when an indeterminate situation appears when $u_n = 0$ and $v_n = 0$. In fact, we wish to avoid a potential blow up of the solutions on these sets. This is ensured by the following key result.

Proposition 3.4. *Assume* $\gamma > 0$, (A1)–(A7). *Then*

$$k \iint_{Q_T \cap \{0 \le u_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \le C\delta, \tag{3.10}$$

$$k \iint_{Q_T \cap \{0 \le v_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \le \begin{cases} C\delta^{1-\gamma} & \textit{if } 0 < \gamma < 1 \\ C\sqrt{\delta} & \textit{if } \gamma = 1. \end{cases} \tag{3.11}$$

Proof. We begin by proving (3.10). Following the ideas in the proof of [10, Proposition 2.20], we choose as test function in the equation solved by u_n the function $\varphi_\sigma = \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^p(x)$, with $\varphi \in C_0^\infty(\Omega), \varphi \ge 0$. Consequently we obtain

$$\begin{aligned} &\int_0^T \langle (u_n)_t, \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma} \varphi^p(x) \rangle \\ &+ \frac{1}{\sigma} \iint_{Q_T} a(x,t,u_n,\nabla u_n) \nabla (T_\sigma(-(u_n-\delta)^-)) \varphi^p(x) \\ &+ p \iint_{Q_T} a(x,t,u_n,\nabla u_n) \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma} \varphi^{p-1} \nabla \varphi \\ &= +k \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma} \varphi^p(x). \end{aligned} \tag{3.12}$$

First, we want to show that

$$\int_0^T \langle (u_n)_t, \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^p(x)\rangle \geq -\delta|\Omega|, \tag{3.13}$$

where $|\Omega|$ is the Lebesgue measure of Ω. To this end, we introduce the function $v_{\sigma,\nu} = \frac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}$, where $u_{n,\nu}$ is, for any fixed $n \in \mathbb{N}$ and $\sigma \in \mathbb{N}$, the solution of the following ordinary differential equation problem

$$\begin{gathered} \frac{1}{\sigma}[u_{n,\nu}]_t + u_{n,\nu} = u_n \\ u_{n,\nu}(0) = u_{0,n}. \end{gathered} \tag{3.14}$$

The function $u_{n,\nu}$ satisfies the following properties (see [14, 15]):

$$\begin{gathered} u_{n,\nu} \in L^p(0,T;W_0^{1,p}(\Omega)), \quad (u_{n,\nu})_t \in L^p(0,T;W_0^{1,p}(\Omega)), \\ \|u_{n,\nu}\|_{L^\infty(Q_T)} \leq \|u_n\|_{L^\infty(Q_T)}, \\ u_{n,\nu} \to u_n \quad \text{in } L^p(0,T;W_0^{1,p}(\Omega)) \quad \text{as } \nu \to +\infty, \\ (u_{n,\nu})_t \to (u_n)_t \quad \text{in } L^{p'}(0,T;W^{-1,p'}(\Omega)) \text{ as } \nu \to +\infty. \end{gathered}$$

So, we have

$$\begin{aligned} &\int_0^T \langle (u_n)_t, \frac{T_\sigma(-(u_n-\delta)^-)}{\sigma}\varphi^p(x)\rangle \\ &= \lim_{\nu\to\infty} \iint_{Q_T} (u_{n,\nu}-\delta)_t^+ \frac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}\varphi^p(x) \\ &\quad - \lim_{\nu\to\infty} \iint_{Q_T} (u_{n,\nu}-\delta)_t^- \frac{T_\sigma(-(u_{n,\nu}-\delta)^-)}{\sigma}\varphi^p(x) \\ &= \lim_{\nu\to\infty} \iint_{Q_T} (u_{n,\nu}-\delta)_t^- \frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\sigma}\varphi^p(x). \end{aligned} \tag{3.15}$$

Introducing now the function $\Phi_\sigma(s) := \int_0^{(s-\delta)^-} \frac{T_\sigma(\rho)}{\sigma} d\rho$, from (3.15), we obtain

$$\begin{aligned} &\lim_{\nu\to\infty} \iint_{Q_T} (u_{n,\nu}-\delta)_t^- \frac{T_\sigma((u_{n,\nu}-\delta)^-)}{\delta}\varphi^p(x) \\ &= \lim_{\nu\to\infty} \iint_{Q_T} \frac{d}{dt}\Phi_\sigma(u_{n,\nu}) \\ &= \lim_{\nu\to\infty} \int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(T) - \lim_{\nu\to\infty} \int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(0) \\ &\geq -\lim_{\nu\to\infty} \int_\Omega \Phi_\sigma(u_{n,\nu}-\delta)^-(0) \\ &= -\int_\Omega \Phi_\sigma(u_n-\delta)^-(0) \geq -\delta|\Omega|, \end{aligned}$$

since $\int_\Omega \Phi_\sigma(u_n-\delta)^-(0) \leq \delta|\Omega|$. This proves (3.13). By (3.13), observing also that $\frac{T_\sigma(-(u_n-\delta)^-)}{\sigma} = 0$ on the set $\{(x,t) \in Q_T : u_n(x,t) \geq \delta\}$, the equality (3.12) becomes

$$\begin{aligned}
&\frac{1}{\sigma}\iint_{Q_T\cap\{\delta-\sigma\le u_n\le\delta\}} a(x,,t,u_n,\nabla u_n)\nabla u_n\varphi^p(x)\\
&+k\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\frac{T_\sigma((u_n-\delta)^-)}{\sigma}\varphi^p(x)\\
&\le p\iint_{Q_T\cap\{u_n\le\delta\}}|a(x,t,u_n,\nabla u_n)|\varphi^{p-1}|\nabla\varphi|+\delta|\Omega|.
\end{aligned}\tag{3.16}$$

Note that, in view of (A1), the first term in the left-hand side of (3.16) is nonnegative. By (A3) and using Hölder's inequality in the right hand side, we obtain

$$\begin{aligned}
&k\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\frac{T_\sigma((u_n-\delta)^-)}{\sigma}\varphi^p(x)\\
&\le p\alpha_1\iint_{Q_T\cap\{u_n\le\delta\}}|\nabla u_n|^{p-1}\varphi^{p-1}|\nabla\varphi|+\delta|\Omega|\\
&\le p\alpha\Big(\iint_{Q_T\cap\{u_n\le\delta\}}|\nabla u_n|^p\varphi^p\Big)^{\frac{p-1}{p}}\Big(\iint_{Q_T}|\nabla\varphi|^p\Big)^{1/p}+\delta|\Omega|.
\end{aligned}\tag{3.17}$$

We observe now that

$$\iint_{Q_T\cap\{u_n\le\delta\}}|\nabla u_n|^p\varphi^p(x)\le C\delta.\tag{3.18}$$

Indeed, multiplying problem (2.1) by the test function $-(u_n-\delta)^-\varphi^p(x)$, $\varphi\in C_0^\infty(\Omega)$, $\varphi\ge 0$, we obtain

$$\begin{aligned}
&\int_0^T\langle(u_n)_t,(-(u_n-\delta)^-)\varphi^p(x)\rangle+\iint_{Q_T\cap\{u_n\le\delta\}}a(x,t,u_n,\nabla u_n)\nabla u_n\varphi^p(x)\\
&-p\iint_{Q_T}a(x,t,u_n,\nabla u_n)(u_n-\delta)^-\varphi^{p-1}\nabla\varphi\le 0.
\end{aligned}\tag{3.19}$$

To deal with the term involving time derivative, we use the same argument as that used to achieve (3.13). Hence we obtain

$$\int_0^T\langle(u_n)_t,(-(u_n-\delta)^-)\varphi^p(x)\rangle\ge-\delta|\Omega|.\tag{3.20}$$

By (A1), (A3) and (3.20), the inequality (3.19) becomes

$$\alpha\iint_{Q_T\cap\{u_n\le\delta\}}|\nabla u_n|^p\varphi^p\le p\alpha_1\iint_{Q_T\cap\{u_n<\delta\}}|\nabla u_n|^{p-1}(\delta-u_n)\varphi^{p-1}|\nabla\varphi|+\delta|\Omega|,$$

which, by Hölder's inequality and (3.3), leads to

$$\begin{aligned}
\iint_{Q_T\cap\{u_n\le\delta\}}|\nabla u_n|^p\varphi^p&\le\frac{p\delta\alpha_1}{\alpha}\Big(\iint_{Q_T}|\nabla u_n|^p\varphi^p\Big)^{\frac{p-1}{p}}\Big(\iint_{Q_T}|\nabla\varphi|^p\Big)^{1/p}+\frac{\delta|\Omega|}{\alpha}\\
&\le C\delta.
\end{aligned}$$

Thus, (3.18) holds. Finally, we have obtained that

$$k\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\frac{T_\sigma((u_n-\delta)^-)}{\sigma}\varphi^p(x)\le C\delta.\tag{3.21}$$

Now, we can pass to the limit in (3.21) for $\sigma \to 0$ and n fixed, relying on Lebesgue dominate convergence Theorem since $\frac{T_\sigma((u_n-\delta)^-)}{\sigma}$ converges a.e. to 1 on the set $\{(x,t) \in Q_T : u_n(x,t) < \delta\}$. Therefore, we obtain:

$$\iint_{Q_T \cap \{0 \le u_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \le C\delta,$$

and hence, (3.10) holds.

We now focus the attention on the estimate (3.11). We distinguish two cases, depending on the value of parameter γ.

If $0 < \gamma < 1$, we consider the decomposition

$$\begin{aligned}
&\iint_{Q_T \cap \{0 \le v_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \\
&= \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{0 \le u_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \\
&\quad + \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{u_n > \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \\
&\le \iint_{Q_T \cap \{0 \le u_n \le \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) \\
&\quad + \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{u_n > \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x) = I + II.
\end{aligned} \tag{3.22}$$

By (3.10) we obtain

$$I \le C\delta. \tag{3.23}$$

To handle the term II, we proceed as follows:

$$II \le \delta \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{u_n > \delta\}} \frac{\varphi^p(x)}{\delta^\gamma} = \delta^{1-\gamma} \iint_{Q_T} \varphi^p(x) \le C\delta^{1-\gamma}. \tag{3.24}$$

If $\gamma = 1$, we consider the decomposition

$$\begin{aligned}
&\iint_{Q_T \cap \{0 \le v_n \le \delta\}} \frac{v_n}{u_n + \frac{1}{n}} \varphi^p(x) \\
&= \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{0 \le u_n \le \sqrt{\delta}\}} \frac{v_n}{u_n + \frac{1}{n}} \varphi^p(x) \\
&\quad + \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{u_n > \sqrt{\delta}\}} \frac{v_n}{u_n + \frac{1}{n}} \varphi^p(x) \\
&\le \iint_{Q_T \cap \{0 \le u_n \le \sqrt{\delta}\}} \frac{v_n}{u_n + \frac{1}{n}} \varphi^p(x) \\
&\quad + \iint_{Q_T \cap \{0 \le v_n \le \delta\} \cap \{u_n > \sqrt{\delta}\}} \frac{v_n}{u_n + \frac{1}{n}} \varphi^p(x) = \tilde{I} + \tilde{II}.
\end{aligned} \tag{3.25}$$

Choosing as test function in the equation solved by u_n the function

$$\phi_\sigma = \frac{T_\sigma\big(-(u_n - \sqrt{\delta})^-\big)}{\sigma} \varphi^p(x),$$

with $\varphi \in C_0^\infty(\Omega), \varphi \ge 0$, and repeating the same arguments of the proof of (3.10), we obtain

$$\tilde{I} \le C\sqrt{\delta}. \tag{3.26}$$

For the term $\tilde{II}$, we obtain:

$$\begin{aligned}
&\iint_{Q_T\cap\{0\le v_n\le\delta\}\cap\{u_n>\sqrt{\delta}\}} \frac{v_n}{u_n+\frac{1}{n}}\varphi^p(x) \\
&\le \delta \iint_{Q_T\cap\{0\le v_n\le\delta\}\cap\{u_n>\sqrt{\delta}\}} \frac{\varphi^p(x)}{\delta^{\frac12}} \\
&= \delta^{1-\frac12}\iint_{Q_T}\varphi^p(x)\le C\sqrt{\delta}.
\end{aligned} \tag{3.27}$$

Consequently, by (3.23), (3.24), (3.22), (3.26), (3.27), (3.25), we finally get (3.11). $\square$

4. Convergence and compactness results

To pass to the limit as $n\to\infty$ in the distributional formulations (2.4) and (2.5) we need strongly convergent subsequences. Their existence is ensured in the next result.

Proposition 4.1. *Assume $0<\gamma\le 1$, (A1)–(A7). Then there exists a couple $(u,v)\in[L^p(0,T,V)\cap L^\infty(Q_T)]\times[L^p(0,T,W)\cap L^\infty(Q_T)]$ such that, as $n\to\infty$, we have:*

$$u_n \rightharpoonup u \quad \text{weakly in } L^p(0,T;V), \tag{4.1}$$
$$v_n \rightharpoonup v \quad \text{weakly in } L^p(0,T;W), \tag{4.2}$$
$$u_n \rightharpoonup u \quad \text{weakly* in } L^\infty(Q_T), \tag{4.3}$$
$$v_n \rightharpoonup v \quad \text{weakly* in } L^\infty(Q_T), \tag{4.4}$$
$$u_n \to u \quad \text{strongly in } L^1(Q_T), \tag{4.5}$$
$$v_n \to v \quad \text{strongly in } L^1(Q_T), \tag{4.6}$$
$$u_n \to u \quad \text{a.e. in } Q_T, \tag{4.7}$$
$$v_n \to v \quad \text{a.e. in } Q_T, \tag{4.8}$$

up to a subsequence.

Proof. Convergences (4.1) and (4.2) are direct consequences of the *a priori* estimates (3.3) and (3.4) obtained respectively for the sequences $\{u_n\}$ and $\{v_n\}$. Convergences (4.3) and (4.4) are direct consequences of the *a priori* estimates (3.1) and (3.2) obtained respectively for the sequences $\{u_n\}$ and $\{v_n\}$.

To prove (4.5) and (4.7) we observe that thanks to the uniform estimate (3.9) we have

$$\frac{v_n\varphi^p}{(u_n+\frac1n)^\gamma}\in L^1(Q_T) \tag{4.9}$$

for every $\varphi\in C_0^\infty(\Omega)$, $\varphi\ge 0$. Moreover, observing that $a(x,t,s,\xi)$ is uniformly bounded in $L^{p'}(0,T;(V)^*)$ we have that

$$\frac{\partial(u_n\varphi^p)}{\partial t} \quad \text{is bounded in } L^{p'}(0,T;(V)^*)+L^1(Q_T). \tag{4.10}$$

By (4.10) and for $s>\frac N2+1$, proceeding as [20, Lemma 2.3] we obtain that $\frac{\partial(u_n\varphi)}{\partial t}$ is also bounded in $L^1(0,T;H^{-s})$. Consequently, since $s>\frac N2$, we obtain

$$V\subset L^p(\Omega)\subset H^{-s}(\Omega),$$

and the embedding $V \subset L^p(\Omega)$ is compact. Applying [24, Corollary 4], by (4.10) and the compactness results we deduce that $u_n\varphi$ is relatively compact in $L^p(Q_T)$. Hence, up to a subsequences, convergences (4.5) and (4.7) are satisfied. Reasoning in the same way for the sequence $\{v_n\}$, we obtain (4.6) and (4.8). □

Proposition 4.2. *Assume* (A1)–(A7). *Then*

$$\lim_{n\to\infty}\iint_{Q_T}|\nabla(u_n-u)|^p=0, \tag{4.11}$$

$$\lim_{n\to\infty}\iint_{Q_T}|\nabla(v_n-v)|^p=0. \tag{4.12}$$

Therefore,

$$\nabla u_n\to\nabla u \quad \text{a.e. in } Q_T, \tag{4.13}$$

$$\nabla v_n\to\nabla v \quad \text{a.e. in } Q_T. \tag{4.14}$$

The proofs of (4.11) and (4.12) follow directly from [10, Proposition 2.22].

5. The set $\{(x,t)\in Q_T: u(x,t)=0 \text{ a.e. in } Q_T\}$

As a consequence of the uniform estimate near the singularity (3.10), we have the following result.

Proposition 5.1. *The couple* (u,v) *as a solution to* (1.1)*, in the sense of Definition 1.1, satisfies*

$$\iint_{Q_T\cap\{u=0\}}\frac{v}{u^\gamma}\psi=0 \tag{5.1}$$

for every $\psi\in C_0^\infty(\Omega\times[0,T))$, $\psi\geq 0$ *Moreover, it holds*

$$\iint_{Q_T}\frac{v}{u^\gamma}\psi=\iint_{Q_T\cap\{u>0\}}\frac{v}{u^\gamma}\psi. \tag{5.2}$$

Proof. Following the line of the proof of [10, Proposition 2.23], we consider a function $\psi\in C_0^\infty(\Omega\times[0,T))$, $\psi\geq 0$, with $supp\,\psi=C\times[0,T_1]$, $T_1<T$, $C\subset\subset E\subset\subset\Omega$ and $\varphi\in C_0^1(\Omega)$ with $\varphi(x)=1$ over C, $\varphi\geq 0$ with $supp\,\varphi=E$. By the uniform estimate (3.10), we obtain

$$\begin{aligned}
&\iint_{Q_T\cap\{u_n<\delta\}}\frac{v_n}{(u_n+\frac1n)^\gamma}\psi(x,t)\\
&\leq\|\psi\|_\infty\iint_{C\times[0,T]}\frac{v_n}{(u_n+\frac1n)^\gamma}\chi_{\{u_n<\delta\}}\\
&\leq\|\psi\|_\infty\iint_{Q_T}\frac{v_n}{(u_n+\frac1n)^\gamma}\varphi^p(x)\chi_{\{u_n<\delta\}}\leq C\delta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\iint_{Q_T}\frac{v_n}{(u_n+\frac1n)^\gamma}\chi_{\{u_n<\delta\}}\psi(x,t)\\
&=\iint_{Q_T}\frac{v_n}{(u_n+\frac1n)^\gamma}\chi_{\{u_n<\delta\}}\chi_{\{u=\delta\}}\psi(x,t)\\
&\quad+\iint_{Q_T}\frac{v_n}{(u_n+\frac1n)^\gamma}\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}}\psi(x,t)\leq C\delta.
\end{aligned} \tag{5.3}$$

We now observe that there exists at most a countable set D such that meas$\{(x,t):u(x,t)=\delta\}>0$. We take δ outside of this set D, so that, in (5.3), the integral

$$\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n<\delta\}}\chi_{\{u=\delta\}}\psi(x,t)=0.$$

So, we have

$$\begin{aligned}&\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n<\delta\}}\psi(x,t)\\&=\iint_{Q_T}\frac{v_n}{(u_n+\frac{1}{n})^\gamma}\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}}\psi(x,t)\le C\delta.\end{aligned}\tag{5.4}$$

Since by (4.7),

$$\chi_{\{u_n<\delta\}}\chi_{\{u\neq\delta\}}\to\chi_{\{u<\delta\}}\quad\text{a.e. in } Q_T$$

applying Fatou's Lemma in (5.4) for δ fixed, leads to

$$\iint_{Q_T}\frac{v}{u^\gamma}\chi_{\{u<\delta\}}\psi(x,t)\le C\delta.$$

Using again Fatou's Lemma in the last inequality for $\delta\to 0$, we obtain

$$\iint_{Q_T}\frac{v}{u^\gamma}\chi_{\{u=0\}}\psi(x,t)=\iint_{Q_T\cap\{u=0\}}\frac{v}{u^\gamma}\psi(x,t)=0.\tag{5.5}$$

That implies that

$$\iint_{Q_T}\frac{v}{u^\gamma}\psi(x,t)=\iint_{Q_T\cap\{u>0\}}\frac{v}{u^\gamma}\psi(x,t),\tag{5.6}$$

which is the desired identity. $\square$

6. Proof of Theorem 1.2

In this section, we give the proof of the main result of our paper. Since $(u_n,v_n)\ge(0,0)$ a.e. in Q_T, thanks to (4.3) and (4.4) we obtain $(u,v)\ge(0,0)$. Thanks to the convergences (4.5) and (4.6), we can pass to the limit in the parts involving the time derivatives of (2.4) and (2.5).

Concerning to the principal parts we have

$$a(x,t,u_n,\nabla u_n)\to a(x,t,u,\nabla u)\quad\text{in } L^{p'}(Q_T),\tag{6.1}$$

$$b(x,t,v_n,\nabla v_n)\to b(x,t,v,\nabla v)\quad\text{in } L^{p'}(Q_T).\tag{6.2}$$

In fact, we observe that for any measurable set D, the assumptions (A2) and (A5) guarantee

$$\iint_D|a(x,t,u_n,\nabla u_n|^{p'}\le C\iint_D|\nabla u_n|^p,$$
$$\iint_D|b(x,t,v_n,\nabla v_n|^{p'}\le C\iint_D|\nabla v_n|^p.$$

By (4.11) and (4.12), the sequences $\{|a(x,t,u_n,\nabla u_n)|\}$ and $\{|b(x,t,v_n,\nabla v_n)|\}$ are equintegrable. By (4.7), (4.8), (4.13) and (4.14), thanks to Vitali's Theorem (see [7, Theorem 1.0.16]), we obtain (6.1) and (6.2).

We deal now with the singular lower order term. Let be $D=K\times[0,T_1]$, $T_1<T$, such that $K\subset\subset E\subset\subset\Omega$ and $\psi\in C_0^\infty(\Omega\times[0,T))$ with $supp\,\psi=D$. Let φ be a

function such that $\varphi(x) = 1$ on the set K, $0 \le \varphi \le 1$ and $supp(\varphi) = E$. For any $\delta > 0$ we have

$$\begin{aligned} &\iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t)\,dx\,dt \\ &= \iint_{Q_T \cap \{0 \le u_n < \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t)\,dx\,dt \\ &\quad + \iint_{Q_T \cap \{u_n \ge \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t)\,dx\,dt = A + B. \end{aligned} \tag{6.3}$$

To estimate the term A, we proceed as follows:

$$\begin{aligned} A &\le \|\psi\|_\infty \iint_{\{0 \le u_n < \delta\} \cap D} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x)\,dx\,dt \\ &\le \|\psi\|_\infty \iint_{Q_T \cap \{0 \le u_n < \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \varphi^p(x). \end{aligned}$$

By (3.10), we deduce that

$$A \le C\delta, \tag{6.4}$$

where C is a constant independent of n. For handling the term B, we see that

$$\begin{aligned} B &= \iint_{Q_T \cap \{u_n \ge \delta\}} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t)\,dx\,dt \\ &= \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \ge \delta\}} \chi_{\{u \ne \delta\}} \psi(x,t)\,dx\,dt \\ &\quad + \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \ge \delta\}} \chi_{\{u = \delta\}} \psi(x,t)\,dx\,dt = B_1 + B_2. \end{aligned}$$

For the term B_2, we observe that there is at most a countable set $\mathcal{C}$ such that $\text{meas}\{(x,t) : u(x,t) = \delta\} > 0$. We take δ outside of this set $\mathcal{C}$, so that the term B_2 is zero. Since (4.7) holds, for the term B_1 we have that

$$\chi_{\{u_n \ge \delta\}} \chi_{\{u \ne \delta\}} \to \chi_{\{u > \delta\}} \quad \text{a.e. in } Q_T,$$

$$\frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \ge \delta\}} \chi_{\{u \ne \delta\}} \psi(x,t) \le \frac{v_n \psi(x,t)}{\delta^\gamma} \in L^1(Q_T).$$

Thanks to (4.7) and (4.8), the Lebesgue Dominate Convergence Theorem ensures that

$$\lim_{n \to \infty} \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \chi_{\{u_n \ge \delta\}} \chi_{\{u \ne \delta\}} \psi(x,t)\,dx\,dt = \iint_{Q_T} \frac{v}{u^\gamma} \chi_{\{u > \delta\}} \psi(x,t)\,dx\,dt$$

i.e.

$$\lim_{n \to \infty} B = \iint_{Q_T} \frac{v}{u^\gamma} \chi_{\{u > \delta\}} \psi(x,t)\,dx\,dt. \tag{6.5}$$

By (6.3), (6.4), (6.5) and (5.6), we deduce that

$$\begin{aligned} \lim_{n \to \infty} \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t) &= \lim_{\delta \to 0} \lim_{n \to \infty} \iint_{Q_T} \frac{v_n}{(u_n + \frac{1}{n})^\gamma} \psi(x,t) \\ &= \iint_{Q_T \cap \{u > 0\}} \frac{v}{u^\gamma} \psi(x,t)\,dx\,dt \end{aligned}$$

$$= \iint_{Q_T} \frac{v}{u^\gamma} \psi(x,t)\, dx\, dt$$

for every $\psi \in C_0^\infty(\Omega \times [0,T))$. Repeating the same argument for u_n to deal with the case of v_n, completes the proof of Theorem 1.2.

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