EXISTENCE AND NONEXISTENCE OF SOLUTIONS FOR
SUBLEAR EQUATIONS ON EXTERIOR DOMAINS

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Abstract. In this article we study radial solutions of \( \Delta u + K(r)f(u) = 0 \)
on the exterior of the ball of radius \( R > 0 \), \( B_R \), centered at the origin in \( \mathbb{R}^N \)with \( u = 0 \) on \( \partial B_R \) where \( f \) is odd with \( f < 0 \) on \((0, \beta)\), \( f > 0 \) on \((\beta, \infty)\),\( f(u) \sim u^p \) with \( 0 < p < 1 \) for large \( u \) and \( K(r) \sim r^{-\alpha} \) for large \( r \). We prove that if \( N > 2 \) and \( K(r) \sim r^{-\alpha} \) with \( 2 < \alpha < 2(N-1) \) then there are no solutions with \( \lim_{r \to \infty} u(r) = 0 \) for sufficiently large \( R > 0 \). On the otherhand, if \( 2 < N - p(N-2) < \alpha < 2(N-1) \) and \( k, n \) are nonnegative integers with \( 0 \leq k \leq n \) then there exist solutions, \( u_k \), with \( k \) zeros on \((R, \infty)\) and \( \lim_{r \to \infty} u_k(r) = 0 \) if \( R > 0 \) is sufficiently small.

1. Introduction

In this article we study radial solutions of
\[ \Delta u + K(r)f(u) = 0 \quad \text{in} \quad \mathbb{R}^N \setminus B_R, \]
\[ u = 0 \quad \text{on} \quad \partial B_R, \]
\[ u \to 0 \quad \text{as} \quad |x| \to \infty \]
where \( B_R \) is the ball of radius \( R > 0 \) centered at the origin in \( \mathbb{R}^N \) and \( K(r) > 0 \).We assume:

(H1) \( f \) is odd and locally Lipschitz, \( f < 0 \) on \((0, \beta)\), \( f > 0 \) on \((\beta, \infty)\), and \( f'(0) < 0 \).

(H2) There exists \( p \) with \( 0 < p < 1 \) such that \( f(u) = |u|^{p-1}u + g(u) \) where\( \lim_{u \to \infty} \frac{g(u)}{|u|^p} = 0 \).

We let \( F(u) = \int_0^u f(s) \, ds \). Since \( f \) is odd it follows that \( F \) is even and from (H1) itfollows that \( F \) is bounded below by \( -F_0 < 0 \), \( F \) has a unique positive zero, \( \gamma \), with \( 0 < \beta < \gamma \), and

(H3) \( -F_0 < F < 0 \) on \((0, \gamma)\), \( F > 0 \) on \((\gamma, \infty)\).

When \( f \) grows superlinearly at infinity - i.e. \( \lim_{u \to \infty} \frac{f(u)}{u} = \infty \), \( \Omega = \mathbb{R}^N \),and \( K(r) \equiv 1 \) then the problem (1.1), (1.3) has been extensively studied \([1, 3, 10, 12, 14]\).
Interest in the topic for this paper comes from recent papers [5, 11, 13] about solutions of differential equations on exterior domains. In [7-9] we studied (1.1)-(1.3) with \( K(r) \sim r^{-\alpha} \), \( f \) superlinear, and \( \Omega = \mathbb{R}^N \setminus B_R \) with various values for \( \alpha \). In those papers we proved existence of an infinite number of solutions - one with exactly \( n \) zeros for each nonnegative integer \( n \) such that \( u \to 0 \) as \( |x| \to \infty \) for all \( R > 0 \). In [6] we studied (1.1)-(1.3) with \( K(r) \sim r^{-\alpha} \), \( f \) bounded, and \( \Omega = \mathbb{R}^N \setminus B_R \). In this paper we consider the case where \( f \) grows sublinearly at infinity - i.e. \( \lim_{u \to \infty} \frac{f(u)}{u^p} = c_0 > 0 \) with \( 0 < p < 1 \).

Since we are interested in radial solutions of (1.1)-(1.3) we assume that we use \( u(\alpha) = u(r) \) where \( x \in \mathbb{R}^N \) and \( r = |x| = \sqrt{x_1^2 + \cdots + x_N^2} \), so that \( u \) solves
\[
\nu''(r) + \frac{N-1}{r}\nu'(r) + K(r)f(u(r)) = 0 \quad \text{on } (R, \infty) \text{ where } R > 0, \\
u(R) = 0, \nu'(R) = b \in \mathbb{R}.
\]

We will also assume that

(H4) there exist constants \( k_1 > 0, k_2 > 0 \), and \( \alpha \) with \( 0 < \alpha < 2(N-1) \) such that
\[
k_1 r^{-\alpha} \leq K(r) \leq k_2 r^{-\alpha} \quad \text{on } [R, \infty).
\]

(H5) \( K \) is differentiable, on \([R, \infty)\), \( \lim_{r \to \infty} \frac{rK'}{K} = -\alpha \), and \( \frac{K'}{K} + 2(N-1) > 0 \). Note that (H5) implies \( r^{2(N-1)} K(r) \) is increasing. In this article we prove the following result.

**Theorem 1.1.** Let \( N > 2, 0 < p < 1 \), and \( 2 < N - p(N-2) < \alpha < 2(N-1) \). Assuming (H1)-(H5) then given nonnegative integers \( k, n \) with \( 0 \leq k \leq n \) then there exist solutions, \( u_k \), of (1.4)-(1.5) with \( k \) zeros on \((R, \infty)\) and \( \lim_{r \to \infty} u_k(r) = 0 \) if \( R > 0 \) is sufficiently small.

In addition we also prove:

**Theorem 1.2.** Let \( N > 2, 0 < p < 1 \) and \( 2 < \alpha < 2(N-1) \). Assuming (H1)-(H5), there are no solutions of (1.4)-(1.5) such that \( \lim_{r \to \infty} u(r) = 0 \) if \( R > 0 \) is sufficiently large.

Note that for the superlinear problems studied in [7-9] we were able to prove existence for any \( R > 0 \) whereas in the sublinear case and in [6] we only get solutions if \( R \) is sufficiently small.

2. Preliminaries and proof of Theorem 1.2

From the standard existence-uniqueness theorem for ordinary differential equations [4] it follows there is a unique solution of (1.4)-(1.5) on \([R, R + \epsilon]\) for some \( \epsilon > 0 \). We then define
\[
E = \frac{1}{2} \nu'^2 + F(u).
\]
Using (H5) we see that
\[
E'' = -\frac{\nu'^2}{2rK} \left( 2(N-1) + \frac{rK'}{K} \right) \leq 0 \quad \text{for } 0 < \alpha < 2(N-1).
\]
Thus \( E \) is nonincreasing. Hence it follows that
\[
\frac{1}{2} \frac{\nu'^2}{K} + F(u) = E(r) \leq E(R) = \frac{1}{2} \frac{b^2}{K(R)} \quad \text{for } r \geq R
\]
and so we see from (H2)–(H4) that \( u \) and \( u' \) are uniformly bounded wherever they are defined from which it follows that the solution of (1.4)–(1.5) is defined on \([R, \infty)\).

**Lemma 2.1.** Let \( N > 2 \), \( 0 < p < 1 \), and \( 0 < \alpha < 2(N-1) \). Assume (H1)–(H5) and suppose \( u \) satisfies (1.4)–(1.5) with \( b > 0 \). If \( u \) has a zero, \( z_b \), with \( u > 0 \) on \((R, z_b)\) or if \( u > 0 \) for \( r > R \) and \( \lim_{r \to \infty} u = 0 \) then \( u \) has a local maximum, \( M_b \), with \( R < M_b \), \( u' > 0 \) on \((R, M_b)\), \( M_b \to \infty \) as \( b \to \infty \), and \( u(M_b) \to \infty \) as \( b \to \infty \).

**Proof.** Since \( u(R) = 0 \) and \( u'(R) = b > 0 \) we see that \( u \) gets positive for \( r > R \) and if \( u \) has a zero, \( z_b \), or if \( u > 0 \) and \( \lim_{r \to \infty} u(r) = 0 \) then \( u \) has a critical point, \( M_b \), such that \( u' > 0 \) on \((R, M_b)\). Then \( u'(M_b) = 0 \) and \( u''(M_b) \leq 0 \). By uniqueness of solutions of initial value problems it follows that \( u''(M_b) < 0 \) and thus \( M_b \) is a local maximum. Next suppose there exists \( M_0 > R \) such that \( M_b \leq M_0 \) for all \( b > 0 \). Letting \( v_b(r) = \frac{u(r)}{b} \) then from (1.5) we have \( v_b(R) = 0 \), \( v'_b(R) = 1 \) and

\[
v''_b(r) + \frac{N-1}{r} v'_b(r) + K(r) \frac{f(bv_b(r))}{b} = 0 \quad \text{for } r \geq R. \tag{2.4}
\]

It follows from (2.1)–(2.2) that

\[
\left( \frac{1}{2} \frac{v''_b}{K} + \frac{F(bv_b)}{b^2} \right)' \leq 0 \quad \text{for } r \geq R
\]

and thus

\[
\frac{1}{2} \frac{v''_b}{K} + \frac{F(bv_b)}{b^2} \leq \frac{1}{2K(R)} \quad \text{for } r \geq R. \tag{2.5}
\]

It then follows from (2.5) and (H2)–(H4) that \( |v'_b| \) is uniformly bounded for large \( b > 0 \) on \([R, \infty)\). So there is a constant \( C_1 > 0 \) such that

\[
|v'_b| \leq C_1 \text{ for large } b > 0 \quad \text{and all } r \geq R. \tag{2.6}
\]

We now fix a compact set \([R, R_0]\). Then on \([R, R_0]\) we have by (2.6)

\[
|v_b| = |(r-R) + \int_R^r v'_b(t) \, dt| \leq (1 + C_1)(R_0 - R)
\]

so we see that \( |v_b| \) is uniformly bounded for large \( b \) on \([R, R_0]\).

In addition from (H1)–(H2) it follows there is a constant \( C_2 > 0 \) such that

\[
|f(u)| \leq C_2 |u|^p \quad \text{for all } u \tag{2.7}
\]

and therefore since the \( v_b \) are uniformly bounded on \([R, R_0]\) and \( 0 < p < 1 \) it follows that

\[
|\frac{f(bv_b)}{b}| \leq C_2 |v_b|^p \quad \text{as } b \to \infty. \tag{2.8}
\]

Then from (2.4) and (2.8) we see that \( |v''_b| \) is uniformly bounded on \([R, R_0]\). So by the Arzela-Ascoli theorem there is a subsequence of \( v_b \) (still denoted \( v_b \)) such that \( v_b \to v_0 \) and \( v'_b \to v'_0 \) uniformly on \([R, R_0]\) as \( b \to \infty \). It then follows from (2.4) that \( v''_0 \) converges uniformly to \( v''_0 \) on \([R, R_0]\) and \( v''_0 + \frac{N-1}{r} v'_0 = 0 \). Since \( R_0 \) is arbitrary we see that \( v''_0 + \frac{N-1}{r} v'_0 = 0 \) on \([R, \infty)\). Thus, \( v''_0 v_0 = R^{N-1} \) and \( v_0 = R^{N-1} |v''_0|^r/\left(\frac{N-1}{r}\right)^{N-2} \). Now since \( M_b \leq M_0 \) for all \( b > 0 \) then a subsequence of \( M_b \) converges to some \( M \) and since \( v'_b(M_b) = 0 \) it follows that \( v'_0(M) = 0 \). However this contradicts that \( v'_0 = \frac{R^{N-1}}{r^{N-2}} > 0 \). Therefore our assumption that the \( M_b \) are bounded is false and so we see \( M_b \to \infty \) as \( b \to \infty \).

Next we see that since \( M_b \to \infty \) then \( M_b > 2R \) if \( b \) is sufficiently large and since \( u \) is increasing on \([R, M_b]\) then \( \frac{u(M_b)}{b} \geq \frac{u(2R)}{b} = v_b(2R) \to v_0(2R) > 0 \) for
sufficiently large \( b \). Thus \( u(M_b) > \frac{v_0(R_1)}{2} b \) for sufficiently large \( b \) and so we see that \( u(M_b) \to \infty \) as \( b \to \infty \). This completes the proof. \( \square \)

**Lemma 2.2.** Let \( N > 2 \), \( 0 < p < 1 \), \( 2 < \alpha < 2(N - 1) \), and assume (H1)–(H5). If \( u(z_b) = 0 \) with \( u > 0 \) on \((R, z_b)\) or \( u > 0 \) on \((R, \infty)\) with \( \lim_{r \to \infty} u = 0 \) then

\[
[u(M_b)]^\frac{1}{p} M_b^{\frac{p}{2} - 1} \leq \frac{k_2}{2 - 1} \sqrt[2-1]{\frac{1}{p + 1} + \frac{F_0}{\gamma^{p + 1}}}. \tag{2.9}
\]

**Proof.** We first show that if \( u(z_b) = 0 \) with \( u > 0 \) on \((M_b, z_b)\) then \( u' < 0 \) on \((M_b, z_b)\) and if \( u > 0 \) on \((M_b, \infty)\) with \( \lim_{r \to \infty} u(r) = 0 \) then \( u' < 0 \) on \((M_b, \infty)\). In the first case, if \( u \) has a positive local minimum, \( m_b \), with \( M_b < m_b < z_b \) then \( u'(m_b) = 0 \), \( u''(m_b) \leq 0 \), so \( f(u(m_b)) \geq 0 \) which implies \( 0 < u(m_b) \leq \beta \). On the other hand, since \( E \) is nonincreasing \( 0 > F(u(m_b)) = E(m_b) \geq E(z_b) = \frac{1}{2} \frac{u^2(z_b)}{K(z_b)} \geq 0 \) which is impossible. Secondly, suppose \( u > 0 \) on \((R, \infty)\) and \( \lim_{r \to \infty} u(r) = 0 \). Since \( E \) is nonincreasing it follows that \( \lim_{r \to \infty} E(r) \) exists and since \( \frac{1}{2} \frac{u^2}{K} \geq 0 \) and \( F(u(r)) \to 0 \) as \( r \to \infty \) we see that \( \lim_{r \to \infty} E(r) \geq 0 \). Thus \( E(r) \geq 0 \) for all \( r \geq R \).

Next, it follows from (2.1)-(2.2) that \( E(t) \leq E(M_b) \) for \( t \geq M_b \). Rewriting this inequality we obtain

\[
\frac{|u'(t)|}{\sqrt{2} \sqrt{F(u(M_b)) - F(u(t))}} \leq \sqrt{K} \tag{2.10}
\]

If \( u(z_b) = 0 \) then integrating (2.10) on \((M_b, z_b)\) and using that \( u' < 0 \) on \((M_b, z_b)\) gives

\[
\int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} = \int_{z_b}^{M_b} \frac{-u'(t)}{2 \sqrt{F(u(M_b)) - F(u(t))}} dt \\ \leq \int_{M_b}^{z_b} \sqrt{K} dt \\ \leq \frac{k_2}{2 - 1} \left(M_b^{1 - \frac{2}{p}} - z_b^{1 - \frac{2}{p}}\right) \\ \leq \frac{k_2}{2 - 1} M_b^{1 - \frac{2}{p}}. \tag{2.11}
\]

Similarly if \( u(r) > 0 \) and \( \lim_{r \to \infty} u = 0 \) then integrating (2.10) on \((M_b, \infty)\) and using that \( u' < 0 \) on \((M_b, \infty)\) we again obtain

\[
\int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} \leq \frac{k_2}{2 - 1} M_b^{1 - \frac{2}{p}}. \tag{2.12}
\]

Next from (H2), (H3) and (2.7) it follows that \(-F_0 \leq F(u) \leq \frac{C_2 \gamma}{p + 1} \) for all \( u \). Therefore estimating the left-hand side of (2.11) gives

\[
\int_0^{u(M_b)} \frac{dt}{\sqrt{F(u(M_b)) - F(t)}} \geq \frac{u(M_b)}{\sqrt{\frac{C_2}{p + 1} \gamma} + \frac{F_0}{\gamma^{p + 1}}} = \frac{\left[u(M_b)\right]^{\frac{1}{p}}}{\gamma^{p + 1} + \frac{p}{\gamma}}. \tag{2.12}
\]

Also from (2.1)-(2.2) if \( u(z_b) = 0 \) then we have \( F(u(M_b)) = E(M_b) \geq E(z_b) = \frac{1}{2} \frac{u^2(z_b)}{K(z_b)} \geq 0 \) and so \( u(M_b) \geq \gamma \). On the other hand, if \( u > 0 \) and \( \lim_{r \to \infty} u = 0 \)
then as we saw earlier $E(r) \geq 0$ for all $r \geq R$. Thus $F(u(M_b)) = E(M_b) \geq 0$ and again we see $u(M_b) \geq \gamma$. Now using (2.12) and rewriting gives

$$1 - \frac{p}{2} M_b^2 - 1 \leq \frac{k_2}{\alpha - 1} \sqrt{\frac{C_2}{p + 1} + \frac{F_0}{|u(M_b)|^{p+1}}} \leq \frac{k_2}{\alpha - 1} \sqrt{\frac{1}{p + 1} + \frac{F_0}{\gamma^{p+1}}}. \quad (2.13)$$

This completes the proof. \hfill \Box

Proof of Theorem 1.2. If $u$ has a zero, $z_b$, with $u > 0$ on $(R, z_b)$ or $u > 0$ on $(R, \infty)$ with $\lim_{r \to \infty} u(r) = 0$ then by Lemmas 2.1 and 2.2 we know that $u$ has a local maximum, $M_b$, with $R < M_b$ and $u' > 0$ on $(R, M_b)$. In addition, from the proof of Lemma 2.2 we have $u(M_b) \geq \gamma$. Combining this with (2.13) and the fact that $\alpha > 2$ and $0 < p < 1$ we obtain

$$\gamma^{\frac{1-p}{2}} R^{2q-1} \leq |u(M_b)|^{1-p} M_b^{2q-1} \leq \frac{k_2}{\alpha - 1} \sqrt{\frac{1}{p + 1} + \frac{F_0}{\gamma^{p+1}}}. \quad (2.14)$$

Thus we see that if $R$ is sufficiently large then (2.14) is violated and so we obtain a contradiction. This completes the proof of Theorem 1.2. \hfill \Box

### 3. PROOF OF THEOREM 1.1

We now turn to the proof of existence for $N > 2, 0 < p < 1, 2 < N - p(N-2) < \alpha < 2(N-1)$ and $R > 0$ sufficiently small. First we make the change of variables:

$$u(r) = u_1(r^{\alpha-2}).$$

Using (1.4) we see that $u_1$ satisfies

$$u_1'' + h(t) f(u_1) = 0 \quad (3.1)$$

where it follows from (H4)–(H5) that:

$$0 < h(t) = \frac{t^{2(N-1)\alpha}}{(N-2)^2} K(t^{\frac{1-\alpha}{\alpha}}) \quad \text{and} \quad h'(t) < 0 \quad \text{for} \quad t > 0, \quad (3.2)$$

$$u_1(R^{2-N}) = 0 \quad \text{and} \quad u_1'(R^{2-N}) = -\frac{bR^{N-1}}{N-2} < 0. \quad (3.3)$$

In addition, from (H4) we have

$$\frac{k_1}{(N-2)^2 t^q} \leq h(t) \leq \frac{k_2}{(N-2)^2 t^q} \quad \text{for all} \quad t > 0, \quad \text{where} \quad q = \frac{2(N-1) - \alpha}{N-2}. \quad (3.4)$$

**Note:** Since $2 < \alpha < 2(N-1), N > 2$, and $q = \frac{2(N-1) - \alpha}{N-2}$ it follows that $0 < q < 2$.

Now instead of considering (3.1) with (3.3) we consider (3.1) with

$$u_1(0) = 0, \quad u_1'(0) = b_1 > 0. \quad (3.5)$$

Integrating (3.1) twice on $(0, t)$ and using (3.5) we see that a solution of (3.1), (3.5) is equivalent to a solution of:

$$u_1 = b_1 t - \int_0^t \int_0^s h(x) f(u_1) \, dx \, ds. \quad (3.6)$$
Letting \( u_1 = tv_1 \) we see that a solution of (3.6) is equivalent to a solution of

\[
v_1 = b_1 - \frac{1}{t} \int_0^t \int_0^s h(x)f(xv_1) \, dx \, ds.
\]

(3.7)

Now we define

\[
Tv_1 = b_1 - \frac{1}{t} \int_0^t \int_0^s h(x)f(xv_1) \, dx \, ds.
\]

(3.8)

Let \( 0 < \epsilon < 1 \). Denoting \( \|w\| = \sup_{[0, \epsilon]} |w(x)| \) we let

\[
B = \{ v \in C[0, \epsilon] \mid \|v - b_1\| \leq 1 \}
\]

where \( C[0, \epsilon] \) is the set of continuous functions on \([0, \epsilon]\). It follows from (H1)–(H2) that there exists \( L > 0 \) such that

\[
|f(u)| \leq L|u| \quad \text{for all } u.
\]

(3.9)

Then by (3.4), (3.8)-(3.9), and since \( q < 2 \) as well as \( |v_1| \leq 1 + b_1 \):

\[
|Tv_1 - b_1| \leq \frac{Lk_2}{(N-2)^2} \int_0^t \int_0^s x^{-q}x|v_1| \, dx \, ds
\]

\[
\leq \frac{Lk_2(1 + b_1)^2}{(2-q)(3-q)(N-2)^2}
\]

\[
\leq \frac{Lk_2}{(2-q)(3-q)(N-2)^2}.
\]

Thus for sufficiently small \( \epsilon > 0 \) we have \( T : B \rightarrow B \). Next we see by the mean value theorem, (3.4), and (3.9) that we have

\[
|Tv_1 - Tv_2| = \frac{1}{t} \int_0^t \int_0^s h(x)[f(xv_1) - f(xv_2)] \, dx \, ds
\]

\[
\leq \frac{L}{t} \int_0^t \int_0^s xh(x)|v_1 - v_2| \, dx \, ds
\]

\[
\leq \frac{Lk_2}{(N-2)^2} \|v_1 - v_2\| \frac{1}{t} \int_0^t \int_0^s xx^{-q} \, dx \, ds
\]

\[
\leq \frac{Lk_2}{(2-q)(3-q)(N-2)^2} \|v_1 - v_2\|.
\]

Thus for small enough \( \epsilon > 0 \) we see that \( T \) is a contraction for any \( b_1 > 0 \) and so by the contraction mapping principle there is a solution of (3.7) and hence of (3.1), (3.5) on \([0, \epsilon]\) for some \( \epsilon > 0 \).

Next from (3.7) and (3.9) we have

\[
\frac{|u_1|}{t} = |v_1| \leq b_1 + \frac{L}{t} \int_0^t \int_0^s xh(x)|v_1(x)| \, dx \, ds
\]

(3.10)

\[
\leq b_1 + \frac{Lk_2}{(N-2)^2} \int_0^t \int_0^s x^{-q}|v_1(x)| \, dx \, ds
\]

\[
\leq b_1 + \frac{k_2 L}{(N-2)^2} \int_0^t x^{-q}|v_1(x)| \, dx.
\]

(3.11)

Now let \( w_1 = \int_0^s s^{-q}|v_1(s)| \, ds \). Then

\[
w_1' = t^{1-q}|v_1(t)| = t^{-q}|u_1(t)|
\]

(3.12)
and from (3.10)–(3.12) we obtain
\[
 w_1' - \frac{k_2 L}{(N - 2)^2} t^{1-q} w_1 \leq b_1 t^{1-q}. \tag{3.13}
\]
Multiplying (3.13) by \(\mu(t) = e^{-\frac{k_2 L t^{2-q}}{(2-q)(N-2)^2}} \leq 1\), integrating on \([0, t]\), and rewriting gives
\[
w_1 \leq \frac{b_1}{\mu(t)} \int_0^t s^{1-q} \mu(s) \, ds \leq \frac{b_1}{(2-q)} t^{2-q}. \tag{3.14}
\]
Then from (3.12)–(3.14) we obtain
\[
u_1 \leq \left(\frac{k_2 L}{(2-q)(N-2)^2}\right) \frac{b_1 t^{3-q}}{\mu(t)} + b_1 t = b_1 (t + B(t)t^{3-q}) \tag{3.15}
\]
where
\[
B(t) = \left(\frac{k_2 L}{(2-q)(N-2)^2}\right) \frac{1}{\mu(t)}. \tag{3.16}
\]
Note that \(\mu(t)\) is decreasing and continuous hence \(B(t)\) is increasing and continuous.

Next it follows from (3.6) that
\[
u_1' = b_1 - \int_0^t h(x) f(u_1) \, dx \tag{3.17}
\]
and thus from (3.4), (3.15), (3.17), and since \(B(t)\) is increasing:
\[
|u_1'| \leq b_1 + \frac{k_2 L}{(N-2)^2} \int_0^t x^{-q} b_1 (x + B(x)x^{3-q}) \, dx \\
\leq b_1 + \frac{k_2 L b_1}{2(N-2)^2(2-q)} (2t^{2-q} + B(t)t^{4-2q}). \tag{3.18}
\]

Thus from (3.15) and (3.18) we see that \(u_1\) and \(u_1'\) are bounded on \([0, t]\) and so it follows that the solution of (3.1), (3.5) exists on \([0, t]\). Since \(t\) is arbitrary it follows that the solution of (3.1), (3.5) exists on \([0, \infty)\).

**Lemma 3.1.** Let \(N \geq 2\), \(0 < p < 1\), and \(2 < \alpha < 2(N-1)\). Assuming (H1)–(H5) and that \(u_1\) solves (3.1), (3.5) then there exists \(t_{b_1} > 0\) such that \(u_1(t_{b_1}) = \beta\) and \(0 < u_1 < \beta\) on \((0, t_{b_1})\). In addition, \(u_1'(t) > 0\) on \([0, t_{b_1}]\).

**Proof.** Since \(u_1'(0) = b_1 > 0\) we see that \(u_1\) is initially increasing, positive, and less than \(\beta\). On this set \(f(u_1) < 0\) and so by (3.1) we have \(u_1'' > 0\). Thus by (3.5) we have \(u_1' > b_1 > 0\) when \(0 < u_1 < \beta\) and so on this set we have \(u_1 > b_1 t\). Since \(b_1 t\) exceeds \(\beta\) for sufficiently large \(t\) we see then that there exists \(t_{b_1} > 0\) such that \(u_1(t_{b_1}) = \beta\) and \(0 < u_1 < \beta\) on \((0, t_{b_1})\). This completes the proof. \(\square\)

**Lemma 3.2.** Let \(N \geq 2\), \(0 < p < 1\), and \(2 < \alpha < 2(N-1)\). Assuming (H1)–(H5) and that \(u_1\) solves (3.1), (3.5) then \(t_{b_1} \to \infty\) as \(b_1 \to 0^+\).

**Proof.** Evaluating (3.15) at \(t = t_{b_1}\) gives:
\[
\beta = u_1(t_{b_1}) \leq b_1 (t_{b_1} + B(t_{b_1})t_{b_1}^{3-q}). \tag{3.19}
\]
Since \(2 < \alpha < 2(N-1)\) it then follows from the note after (3.4) that \(0 < q < 2\). Now if \(t_{b_1}\) is bounded as \(b_1 \to 0^+\) then the right-hand side of (3.19) goes to 0 as \(b_1 \to 0^+\) which violates (3.19). Thus we obtain a contradiction and so we see that \(t_{b_1} \to \infty\) as \(b_1 \to 0^+\). This completes the proof. \(\square\)
Lemma 3.3. Let $N > 2$, $0 < p < 1$, and $N - p(N - 2) < \alpha < 2(N - 1)$. Assuming (H1)--(H5) and that $u_1$ solves (3.1)--(3.5) then $u_1$ has a local maximum, $M_{b_1}$, on $(0, \infty)$.

Proof. From Lemma 3.1 it follows that there exists $t_{b_1} > 0$ such that $u_1(t_{b_1}) = \beta$ and $u_1' > 0$ on $[0, t_{b_1})$. Now if $u_1$ does not have a local maximum then $u_1' \geq 0$ for $t > t_{b_1}$ and so $u_1 \geq u_1(t_{b_1} + \delta) > \beta > 0$ for $t > t_{b_1} + \delta$ and some $\delta > 0$. Then from (H2) we see that there is a $C_3 > 0$ such that $f(u_1) \geq C_3$ on $|t_{b_1} + \delta, \infty)$. Thus

$$-u_1'' = h(t)f(u_1) \geq C_3 h(t) \text{ for } t > t_{b_1} + \delta.$$  (3.20)

We now divide the rest of the proof into 3 cases.

Case 1: $N < \alpha < 2(N - 1)$ In this case we see from (3.4) that $0 < q < 1$ so integrating (3.20) on $(t_{b_1} + \delta, t)$ and using (3.4) gives

$$u_1' \leq u_1'(t_{b_1} + \delta) - \frac{k_1 C_3}{(1 - q)(N - 2)^2} (t^{1-q} - (t_{b_1} + \delta)^{1-q}) \to -\infty \text{ as } t \to \infty.$$  (3.21)

Thus $u_1'$ gets negative which contradicts that $u_1' \geq 0$ for $t > 0$ and so $u_1$ must have a local maximum.

Case 2: $\alpha = N$ In this case we have $q = 1$ by (3.4) and so again integrating (3.20) on $(t_{b_1} + \delta, t)$ we obtain

$$u_1' \leq u_1'(t_{b_1} + \delta) - \frac{k_1 C_3}{(N - 2)^2} (\ln(t) - \ln(t_{b_1} + \delta)) \to -\infty \text{ as } t \to \infty.$$  (3.22)

which again contradicts that $u_1' \geq 0$ for $t > 0$. Thus $u_1$ must have a local maximum.

Case 3: $N - p(N - 2) < \alpha < N$ We denote

$$E_1 = \frac{1}{2} \frac{u_1''}{h(t)} + F(u_1)$$  (3.23)

and observe from (3.1)--(3.2) that

$$E_1' = \left(\frac{1}{2} \frac{u_1''}{h(t)} + F(u_1)\right)' = -\frac{u_1'^2}{2h^2} \geq 0.$$  (3.24)

In addition we see from (3.4) that $E_1(0) = 0$ and so $E_1(t) \geq 0$ for $t \geq 0$.

We suppose now that $u_1$ is increasing for $t > t_{b_1}$. We first show that there exists $t_{b_2} > t_{b_1}$ such that $u(t_{b_2}) = \gamma$. So we suppose by the way of contradiction that $0 < u_1 < \gamma$ and $u_1' \geq 0$ for $t > t_{b_1}$.

Then from (3.1)--(3.2) and (H3) we have

$$\left(\frac{1}{2} u_1'' + h(t) F(u_1)\right)' = h'(t) F(u_1) \geq 0 \text{ when } 0 \leq u_1 \leq \gamma.$$  (3.25)

Now we recall from (H1) that $\lim_{u \to 0} \frac{F(u)}{u^2} = \frac{f''(0)}{2}$. Also since $u_1(0) = 0$ and $u_1'(0) = b_1$ then $\lim_{u \to 0^+} \frac{u}{t} = b_1$. Therefore for small positive $t$ and (3.4) we have

$$0 \leq h(t) |F(u_1)| = t^2 h(t) \frac{|F(u_1)|}{u_1^2} \leq \frac{|f''(0)| k_2 b_1^2 t^{2-q}}{(N - 2)^2} \to 0$$  (3.26)

as $t \to 0^+$ since $q < 2$. Therefore, integrating (3.23) on $(0, t)$ and using (3.24) we obtain

$$\frac{1}{2} u_1'' + h(t) F(u_1) \geq \frac{1}{2} b_1^2 \text{ when } 0 \leq u_1 \leq \gamma.$$  (3.27)
In addition, since $0 \leq u_1 \leq \gamma$ it follows that $h(t)F(u_1) \leq 0$ and thus from (3.28),
\[ u_1' \geq b_1 \quad \text{when } 0 \leq u_1 \leq \gamma. \] (3.26)

Integrating on $(0, t)$ we obtain
\[ u_1 \geq b_1 t \rightarrow \infty \text{ as } t \rightarrow \infty \]
- a contradiction since we assumed $u_1 < \gamma$. Thus there exists $t_{b_2} > t_{b_1}$ such that $u(t_{b_2}) = \gamma$ and $u_1' \geq b_1 > 0$ on $[0, t_{b_2}]$ by (3.26).

We show now that $u_1(t) \rightarrow \infty$ as $t \rightarrow \infty$. If not then $u_1$ is bounded from above and so there exists $Q > \gamma$ such that $\lim_{t \rightarrow \infty} u_1(t) = Q$. Returning to (3.1) we see that this implies:
\[ \lim_{t \rightarrow \infty} \frac{u_1''}{h(t)} = -f(Q) < 0. \] (3.27)

In particular, $u_1'' < 0$ for large $t$ and so $u_1'$ is decreasing for large $t$. Since $u_1' > 0$ for large $t$ it follows that $\lim_{t \rightarrow \infty} u_1'$ exists. This limit must be zero otherwise this would imply $u_1 \rightarrow \infty$ as $t \rightarrow \infty$ contradicting the assumption that $u_1$ is bounded. Thus $\lim_{t \rightarrow \infty} u_1' = 0$. Next denoting $H(t) = \int_t^\infty h(s) \, ds$ we see that since $N - p(N - 2) < \alpha < N$ and $q = \frac{2(N-1)-\alpha}{N-2}$ this implies:
\[ 1 < q < 1 + p < 2. \] (3.28)

Therefore by (3.4) we see that $h(t)$ is integrable at infinity so $H(t)$ is defined. Then by (3.27) and L’Hôpital’s rule we see that
\[ \lim_{t \rightarrow \infty} \frac{u_1'}{H(t)} = \lim_{t \rightarrow \infty} -\frac{u_1''}{h(t)} = f(Q) > 0. \] (3.29)

Then from (3.4) and (3.28)-(3.29) we see
\[ u_1' \geq f(Q)H(t) \geq \frac{k_1f(Q)}{2(q-1)(N-2)} t^{1-q} \quad \text{for large } t. \] (3.30)

Now integrating (3.30) on $(t_0, t)$ where $t_0$ and $t$ are sufficiently large gives
\[ u_1 \geq u_1(t_0) + \frac{k_1f(Q)}{2(q-1)} \frac{t^{2-q}}{(2-q)(N-2)^2} \rightarrow \infty \quad \text{as } t \rightarrow \infty \text{ since } q < 2 \]
- a contradiction since we assumed $u_1$ was bounded. Thus if $u_1' > 0$ for $t > 0$ then it must be that $u_1 \rightarrow \infty$ as $t \rightarrow \infty$.

Next recalling (3.23) we have
\[ \left( \frac{1}{2} u_1'^2 + h(t)F(u_1) \right)' = h'(t)F(u_1) < 0 \quad \text{when } u_1 > \gamma. \] (3.31)

Integrating this on $(t_{b_2}, t)$ gives
\[ \frac{1}{2} u_1'^2 + h(t)F(u_1) \leq \frac{1}{2} u_1'^2(t_{b_2}) \quad \text{for } t > t_{b_2}. \] (3.32)

On $(t_{b_2}, t)$ we have $h(t)F(u_1) > 0$ and thus from (3.32):
\[ |u_1'| < |u_1'(t_{b_2})| \quad \text{for } t > t_{b_2}. \] (3.33)

We claim now that
\[ \lim_{t \rightarrow \infty} \frac{t^2 h(t)F(u_1)}{u_1} = \infty. \] (3.34)

Integrating (3.33) on $(t_{b_2}, t)$ gives
\[ u_1 < \gamma + (t-t_{b_2}) |u_1'(t_{b_2})| \leq C_4 t \quad \text{for some } C_4 > 0 \text{ for large } t. \] (3.35)
Next from (H2) we have
\[
\frac{f(u_1)}{u_1^p} \geq 1 - \epsilon \text{ for large } u_1.
\]
Thus by (3.35),
\[
\frac{f(u_1)}{u_1} \geq \frac{(1-\epsilon)u_1^p}{u_1} = \frac{(1-\epsilon)}{u_1^{1-p}} \geq \frac{(1-\epsilon)}{C_4^{1-p}t^{1-p}} \text{ for large } t. \tag{3.36}
\]
Therefore by (3.4), (3.28), and (3.36):
\[
\frac{t^2 h(t) f(u_1)}{u_1} \geq \frac{k_1(1-\epsilon)}{C_4^{1-p}(N-2)^2} \frac{t^{1-p}}{t^{1-p}} \to \infty,
\]
since \(1 + p > q\). This establishes (3.34).

Next let (3.4) and that \(u_1\) solves (3.34), (3.28), and (3.36):\]
\[
\frac{t^2 h(t) f(u_1)}{u_1} = \frac{k_1(1-\epsilon)}{C_4^{1-p}(N-2)^2} \frac{t^{1-p}}{t^{1-p}} \to \infty,
\]
since \(1 + p > q\). This establishes (3.34).

Next we rewrite (3.1) as
\[
u_1'' + \frac{t^2 h(t) f(u_1) u_1}{t^2} = 0. \tag{3.37}
\]
Now it follows from (3.34) that we may choose \(t_0\) sufficiently large so that
\[
\frac{t^2 h(t) f(u_1)}{u_1} \geq A > \frac{1}{4} \text{ on } [t_0, \infty).
\]

Next let \(y_1\) be the solution of
\[
y_1'' + \frac{A y_1}{t^2} = 0 \tag{3.38}
\]
with \(y_1(t_0) = u_1(t_0) = \gamma\) and \(y_1'(t_0) = u_1'(t_0) > 0\). It follows then for some constants \(d_1 \neq 0\) and \(d_2\) that
\[
y_1 = d_1 \sqrt{t} \left( \sin \left( \ln \left( \sqrt{A} - \frac{1}{4} \right) + d_2 \right) \right)
\]
and so clearly \(y_1\) has an infinite number of local extrema on \([t_0, \infty)\). Consider now the interval \([t_0, M]\) such that \(y_1 > 0, y_1' > 0\) on \([t_0, M]\) and \(y_1'(M) = 0\). We claim now that \(u_1\) must get negative on \([t_0, M]\). So suppose not. Then \(u_1' \geq 0\) on \([t_0, M]\).

Then multiplying (3.37) by \(y_1\), multiplying (3.38) by \(u_1\), and subtracting we obtain
\[
(y_1 u_1' - y_1'u_1)' + \left( \frac{t^2 h(t) f(u_1)}{u_1} - A \right) \frac{y_1 u_1}{t^2} = 0.
\]
Integrating this on \([t_0, M]\) gives
\[
y_1(M)u_1'(M) + \int_{t_0}^{M} \left( \frac{t^2 h(t) f(u_1)}{u_1} - A \right) \frac{y_1 u_1}{t^2} \, dt = 0. \tag{3.39}
\]
The integral term in (3.39) is positive by (3.34) and also \(y_1(M)u_1'(M) \geq 0\) yielding a contradiction. Therefore we see that \(u_1\) must have a maximum, \(M_{b_1} > 0\), and \(u_1' > 0\) on \([0, M_{b_1}]\). This completes the proof. \(\square\)

**Lemma 3.4.** Let \(N > 2, 0 < p < 1\), and \(N - p(N - 2) < \alpha < 2(N - 1)\). Assuming (H1)-(H5) and that \(u_1\) solves (3.1), (3.5), then there exists \(t_0 > M_{b_1}\) such that \(u_1(t_0) = \frac{2 + \epsilon}{2}^\frac{1}{2}\) and \(u_1' < 0\) on \([M_{b_1}, t_0]\).
Proof. If \( u_1 \geq \frac{\beta + \gamma}{2} \) for all \( t \geq M_{b_i} \), then \( f(u_1) > 0 \) for \( t \geq M_{b_i} \). Then from (3.1) it follows that \( u''_1 < 0 \) and thus \( u'_1(t) \leq u'_1(t_0) < 0 \) for \( t > t_0 > M_{b_i} \). Integrating this inequality on \((t_0,t)\) gives

\[
u_1(t) \leq u_1(t_0) + u'_1(t_0)(t - t_0) \to -\infty \quad \text{as } t \to \infty
\]

which gives a contradiction since we assumed \( u_1 \geq \frac{\beta + \gamma}{2} \) for all \( t \geq M_{b_i} \). Thus there exists \( t_{b_i} > M_{b_i} \) such that \( u_1(t_{b_i}) = \frac{\beta + \gamma}{2}, u_1 > \frac{\beta + \gamma}{2} \), and \( u'_1 < 0 \) on \((M_{b_i}, t_{b_i})\). □

Lemma 3.5. Let \( N > 2, 0 < p < 1, \) and \( N - p(N - 2) < \alpha < 2(N - 1) \). Assuming (H1)–(H5) and that \( u_1 \) solves (3.1), (3.5) then there exists \( z_{1,b_i} > M_{b_i} \) such that \( u_1(z_{1,b_i}) = 0 \). In fact, \( u_1 \) has an infinite number of zeros on \((0, \infty)\).

Proof. Suppose now by the way of contradiction that \( 0 < u_1 < \gamma \) and thus \( F(u_1) < 0 \) for \( t > t_{b_i} \). Then from (3.21)–(3.22) we have

\[rac{1}{2} u_1^2 + F(u_1) \geq F(u_1(M_{b_i})) > 0 \quad \text{for } t \geq M_{b_i}.
\]

Therefore by (3.4) and (3.40) we have

\[u'_1^2 \geq 2h(t)F(u_1(M_{b_i})) \geq \frac{2k_1F(u_1(M_{b_i}))}{(N-2)^2}\]

for \( t > t_{b_i} \). Thus:

\[-u'_1 \geq C_5 t^{-q/2} \quad \text{where } C_5 = \frac{\sqrt{2k_1F(u_1(M_{b_i}))}}{N-2} > 0 \quad \text{for } t > t_{b_i}.
\]

Integrating (3.41) on \((t_{b_i}, t)\) gives

\[u_1 \leq \frac{\beta + \gamma}{2} - C_5(t^{-\frac{q}{2}} - t_{b_i}^{-\frac{q}{2}}) \to -\infty \quad \text{as } t \to \infty \text{ since } q < 2.
\]

Thus \( u_1 \) gets negative contradicting that \( u_1 > 0 \) on \((0, \infty)\). Hence there exists \( z_{1,b_i} > M_{b_i} \) such that \( u_1(z_{1,b_i}) = 0 \) and \( u'_1 < 0 \) on \((M_{b_i}, z_{1,b_i})\).

In a similar way to Lemma 3.3 we can show that \( u_1 \) has a negative local minimum, \( m_{b_i} > z_{1,b_i} \), and similar to Lemma 3.5 we can show that \( u_1 \) has a second zero \( z_{2,b_i} > m_{b_i} \). It then in fact follows that \( u_1 \) has an infinite number of zeros \( z_{n,b_i} \). This completes the proof. □

Proof of Theorem 1.1. By continuous dependence on initial conditions it follows that \( z_{1,b_i} \) is a continuous function of \( b_i \). In addition, by Lemma 3.2 it follows that \( t_{b_i} \to 0 \) as \( b_i \to 0^+ \) and since \( z_{1,b_i} > t_{b_i} \) it follows that \( z_{1,b_i} \to 0^+ \) as \( b_i \to 0^+ \).

So now let \( k,n \) be nonnegative integers with \( 0 \leq k \leq n \). Choose \( R > 0 \) sufficiently small so that \( z_{1,b_i} < \cdots < z_{n,b_i} < R^{2-N} \). Then by the intermediate value theorem there exists a smallest value of \( b_i > 0 \), say \( b_{1,k} \), such that \( z_{k,b_{1,k}} = R^{2-N} \). Then \( u_1(t,b_{1,k}) \) is a solution of (3.1) and (3.5) such that \( u_1(t,b_{1,k}) \) has \( k \) zeros on \((0, R^{2-N})\).

Finally defining

\[U_k(r) = (-1)^k u_1(r^{2-N}, b_{1,k})
\]

we see that \( U_k \) solves (1.4), \( U_k \) has \( k \) zeros on \((R, \infty)\), and \( \lim_{r \to \infty} U_k(r) = 0 \). This completes the proof. □
Note: A crucial step in proving Theorem 1.1 is Lemma 3.3 which says that if $N - p(N - 2) < \alpha < 2(N - 1)$ then every solution of (3.1), (3.5) must have a local maximum. We conjecture that a similar lemma does not hold for $2 < \alpha < N - p(N - 2)$ because for an appropriate constant $c > 0$ the function $ct^{(N-2)(1-p)}$ is a monotonically increasing solution of the model equation

$$u'' + \frac{1}{t^q}u^p = 0$$

with $q = \frac{2(N-1)-\alpha}{N-2}$ and $0 < p < 1$.

References


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