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MULTIPLICITY OF SOLUTIONS FOR NON-HOMOGENEOUS NEUMANN PROBLEMS IN ORLICZ-SOBOLEV SPACES

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Abstract. This article concerns the existence of non-trivial weak solutions for a class of non-homogeneous Neumann problems. The approach is through variational methods and critical point theory in Orlicz-Sobolev spaces. We investigate the existence of two solutions for the problem under some algebraic conditions with the classical Ambrosetti-Rabinowitz condition on the nonlinear term and using a consequence of the local minimum theorem due to Bonanno and mountain pass theorem. Furthermore, by combining two algebraic conditions on the nonlinear term and employing two consequences of the local minimum theorem due Bonanno we ensure the existence of two solutions, by applying the mountain pass theorem of Pucci and Serrin, we set up the existence of the third solution for the problem.

1. Introduction

In this paper we consider the non-homogeneous Neumann problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) + \alpha(|u(x)|)u(x) = \lambda f(x,u(x)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.1}$$

Here, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, ν is the outer unit normal to $\partial\Omega$, $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function such that $f(x,0) \neq 0$ for all $x \in \Omega$, λ is a positive parameter and $\alpha : (0,\infty) \to \mathbb{R}$ is such that the mapping $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(t) = \begin{cases} \alpha(|t|)t, & \text{for } t \neq 0, \\ 0, & \text{for } t = 0, \end{cases}$$

is an odd, strictly increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$.

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It should be noticed that if $\varphi(t) = |t|^{p-2}t$, then problem (1.1) becomes the well-known Neumann boundary value problem involving the p-Laplacian equation

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = \lambda f(x, u(x)) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{1.2}$$

This problem arises in the study of mathematical models in biological formation theory governed by diffusion and cross-diffusion systems [37]. We refer to the recent monograph by Kristály et al. [31] for several related results and examples.

In recent years, quasilinear elliptic partial differential equations involving non-homogeneous differential operators are becoming increasingly important in applications in many fields of mathematics, such as approximation theory, mathematical physics (electrorheological fluids, nonlinear elasticity and plasticity), calculus of variations, nonlinear potential theory, the theory of quasi-conformal mappings, differential geometry, geometric function theory, probability theory (for instance see [19, 24, 32, 41, 43, 46]). Another recent application which uses non-homogeneous differential operators can be found in the framework of image processing (see [14]). The study of nonlinear elliptic equations involving quasilinear homogeneous type operators is based on the theory of Sobolev spaces $W^{m,p}(\Omega)$ in order to find weak solutions. In the case of non-homogeneous differential operators, the natural setting for this approach is the use of Orlicz-Sobolev spaces. These spaces consist of functions that have weak derivatives and satisfy certain integrability conditions. Many properties of Orlicz-Sobolev spaces can be found in [1, 18, 20, 38]. Due to these, many researchers have studied the existence of solutions for eigenvalue problems involving non-homogeneous operators in the divergence form in Orlicz-Sobolev spaces by means of variational methods and critical point theory, monotone operator methods, fixed point theory and degree theory (for instance, see [2, 3, 5, 8, 9, 10, 11, 13, 15, 16, 22, 23, 26, 30, 33, 34, 35, 36, 45]). For example, Clément et al. in [15] established the existence of weak solutions in an Orlicz-Sobolev space for the Dirichlet problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = g(x, u(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$, $g \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$, and the function $\varphi(s) = s\alpha(|s|)$ is an increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. Under appropriate conditions on φ, g and the Orlic–Sobolev conjugate Φ^* of $\Phi(s) = \int_0^s \varphi(t)\,\mathrm{d}t$, they obtained the existence of non-trivial solutions of mountain pass type. Moreover Clément et al. in [16] used Orlicz-Sobolev spaces theory and a variant of the Mountain–Pass Lemma of Ambrosetti-Rabinowitz to obtain the existence of a (positive) solution to a semi-linear system of elliptic equations. In addition, by an interpolation theorem of Boyd, they established an elliptic regularity result in Orlicz-Sobolev spaces. Halidias and Le in [23], by a Brezis-Nirenberg's local linking theorem, investigated the existence of multiple solutions for the problem (1.3). Mihăilescu and Rădulescu in [34], by adequate variational methods in Orlicz-Sobolev spaces, studied the boundary value problem

$$\begin{gathered} -\operatorname{div}(\log(1 + |\nabla u|^q)|\nabla u|^{p-2}\nabla u) = f(u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary. They distinguished the cases where either $f(u) = -\lambda|u|^{p-2}u + |u|^{r-2}u$ or $f(u) = \lambda|u|^{p-2}u - |u|^{r-2}u$, with $p,\ q > 1$, $p + q < \min\{N, r\}$, and $r < (Np - N + p)/(N - p)$. In the first case they showed the existence of infinitely many weak solutions for any $\lambda > 0$ and in the second case they proved the existence of a non-trivial weak solution if λ is sufficiently large, while in [33] they considered the boundary value problem

$$\begin{gathered} -\operatorname{div}\left((a_1(|\nabla u|) + a_2(|\nabla u|)\nabla u\right) = \lambda |u|^{q(x)-2}u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary, λ is a positive real number, q is a continuous function and $a_1,\ a_2$ are two mappings such that $a_1(|t|)t,\ a_2(|t|)t$ are increasing homeomorphisms from $\mathbb{R}$ to $\mathbb{R}$. They established the existence of two positive constants λ_0 and λ_1 with $\lambda_0 \leq \lambda_1$ such that any $\lambda \in [\lambda_1, \infty)$ is an eigenvalue, while any $\lambda \in (0, \lambda_1)$ is not an eigenvalue of the problem (1.4). Kristály et al. in [30] by using a recent variational principle of Ricceri, established the existence of at least two non-trivial solutions for the problem (1.1) in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$. Mihăilescu and Repovš in [36], by combining Orlicz-Sobolev spaces theory with adequate variational methods and a variant of Mountain Pass Lemma, proved the existence of at least two non-negative and non-trivial weak solutions for the problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u(x)|)\nabla u(x)) = \lambda f(x, u(x)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered}$$

where α is the same as in the problem (1.1), $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and λ is a positive parameter. In [10] Bonanno et al. studied the problem (1.1) and established that for all λ in a prescribed open interval, the problem has infinitely many solutions that converge to zero in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$. In [9] they also established a multiplicity result for (1.1). In fact, they employed a recent critical points result for differentiable functionals in order to prove the existence of a determined open interval of positive eigenvalues for which the problem (1.1) admits at least three weak solutions in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$, while in [8] under an appropriate oscillating behavior of the nonlinear term, they proved the existence of a determined open interval of positive parameters for which (1.1) admits infinitely many weak solutions that strongly converges to zero, in the same Orlicz-Sobolev space. In [2] employing variational methods and critical point theory, in an appropriate Orlicz-Sobolev setting, the existence of infinitely many solutions for Steklov problems associated to non-homogeneous differential operators was established.

In [21] the authors considered eigenvalue problems involving non-homogeneous differential operators and as an application of their results, they proved the existence of solutions for non-homogeneous Dirichlet problem. In [12] the authors analyzed a class of quasilinear elliptic problems involving a $p(\cdot)$-Laplace-type operator on a bounded domain $\Omega \subseteq \mathbb{R}^N$, $N \geq 2$ dealing with nonlinear conditions on the boundary. In fact, working on the variable exponent Lebesgue-Sobolev spaces, they followed the steps described by the fountain theorem and they established the existence of a sequence of weak solutions for the problem. In [25] using variational methods and critical point theory the existence of infinitely many solutions for perturbed Kirchhoff-type non-homogeneous Neumann problems involving two parameters in Orlicz-Sobolev spaces was discussed.

To the best of our knowledge, for the non-homogeneous Neumann problem, there has so far been few papers concerning its multiple solutions.

Motivated by the above facts, in the present paper, we are interested in investigating the existence of solutions for the non-homogeneous Neumann problem (1.1). First using a consequence of the local minimum theorem due Bonanno and mountain pass theorem we obtain the existence of two non-trivial solutions for the problem (1.1) in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$, by combining an algebraic condition on f with the classical Ambrosetti-Rabinowitz (AR) condition ([4]) (see Theorem 3.1). The role of (AR) is to ensure the boundedness of the Palais-Smale sequences for the Euler-Lagrange functional associated with the problem. This is very crucial in the applications of critical point theory. Then, combining two algebraic conditions employing two consequences of the local minimum theorem due Bonanno we guarantee the existence of two local minima for the Euler-Lagrange functional and applying the mountain pass theorem as given by Pucci and Serrin (see [39]), we ensure the existence of the third critical point for the corresponding functional which is the third weak solution of our problem in the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ (see Theorems 3.13 and 3.14).

Our approach is variational and the main tool is a local minimum theorem for differentiable functionals established in [6], two of whose consequences are here applied (see Theorems 2.1 and 2.2).

We should emphasize that in the present paper the method used for analyzing the multiplicity and existence of solutions for the problem (1.1) differs completely from all the methods used in [3, 8, 9] for ensuring the solution of the problem and similar ones so far. In fact, we establish the existence of two weak solutions for the problem (1.1) employing a local minimum theorem and the classical theorem of Ambrosetti and Rabinowitz under an algebraic condition on the nonlinear part with the classical Ambrosetti-Rabinowitz (AR) condition on the nonlinear term, which is extremely fundamental in critical point theory. Moreover, by combining two algebraic conditions on the nonlinear term which guarantee the existence of two weak solutions, applying the mountain pass theorem given by Pucci and Serrin we established the existence of third weak solution for the problem (1.1), while in [3, 8, 9] the existence of multiple solutions have been established directly using multiple critical point theorems.

Here, we state two special cases of our results when the Orlicz-Sobolev space $W^1L_\Phi(\Omega)$ coincides with the Sobolev space $W^{1,p}(\Omega)$.

Theorem 1.1. *Let $p > N$ and $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that $g(0) \neq 0$ and*

$$\lim_{\xi \to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty.$$

Putting

$$G(t) = \int_0^t g(\xi)\,\mathrm{d}\xi, \quad \forall\ t \in \mathbb{R},$$

suppose that

(AR) *there exist constants $\nu > p$ and $R > 0$ such that, for all $\xi \geq R$,*

$$0 < \nu G(\xi) \leq \xi g(\xi).$$

Then, for each

$$\lambda \in \Big]0, \frac{1}{(2\kappa)^p \operatorname{meas}(\Omega)} \sup_{\gamma>0} \frac{\gamma^p}{G(\gamma)}\Big[,$$

where κ is a constant such that

$$\|u\|_\infty \leq \kappa \|u\|_{W^{1,p}(\Omega)}\,,$$

for every $u \in W^{1,p}(\Omega)$ and

$$\|u\|_{W^{1,p}(\Omega)} := \Big(\int_\Omega |\nabla u(x)|^p dx + \int_\Omega |u(x)|^p dx \Big)^{1/p},$$

the problem

$$\begin{gathered} -\Delta_p u + |u|^{p-2}u = \lambda g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.5}$$

admits at least two positive weak solutions in $W^{1,p}(\Omega)$.

Theorem 1.2. *Let $p > N$. Assume that Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$ such that* $\operatorname{meas}(\Omega) > \frac{p}{(4\kappa)^p}$*, where κ is the same constant as in Theorem 1.1. Let $g : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that $g(0) \neq 0$,*

$$\lim_{\xi\to 0^+} \frac{g(\xi)}{\xi^{p-1}} = +\infty, \quad \lim_{\xi\to +\infty} \frac{g(\xi)}{\xi^{p-1}} = 0$$

and

$$\int_0^1 g(t)\,\mathrm{d}t < \frac{p}{(4\kappa)^p \operatorname{meas}(\Omega)} \int_0^2 g(t)\,\mathrm{d}t.$$

Then, for each

$$\lambda \in \Big] \frac{2^p}{\int_0^2 g(t)\,\mathrm{d}t}, \frac{1}{(2\kappa)^p \operatorname{meas}(\Omega) \int_0^1 g(t)\,\mathrm{d}t} \Big[$$

problem (1.5) *admits at least three positive weak solutions in $W^{1,p}(\Omega)$.*

For a thorough study on the subject, we also refer the reader to [7, 17, 27, 28].

2. Preliminaries

Our main tools are the following theorems, that are consequences of the existence result of a local minimum theorem for differentiable functionals [6, Theorem 3.1], which is inspired by Ricceri's variational principle (see [42]).

For a given non-empty set X, and two functionals $J, I : X \to \mathbb{R}$, we define the following functions

$$\vartheta(r_1, r_2) = \inf_{v\in J^{-1}(r_1,r_2)} \frac{\sup_{u\in J^{-1}(r_1,r_2)} I(u) - I(v)}{r_2 - J(v)},$$

$$\rho_1(r_1, r_2) = \sup_{v\in J^{-1}(r_1,r_2)} \frac{I(v) - \sup_{u\in J^{-1}(-\infty,r_1]} I(u)}{J(v) - r_1}$$

for all $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, and

$$\rho_2(r) = \sup_{v\in J^{-1}(r,\infty)} \frac{I(v) - \sup_{u\in J^{-1}(-\infty,r]} I(u)}{J(v) - r}$$

for all $r \in \mathbb{R}$.

Theorem 2.1 ([6, Lemma 5.1])**.** *Let X be a real Banach space, $J : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Assume that there are $r_1, r_2 \in \mathbb{R}$, $r_1 < r_2$, such that*

$$\vartheta(r_1, r_2) < \rho_1(r_1, r_2).$$

Then, setting $\Gamma_\lambda := J - \lambda I$, for each $\lambda \in (\frac{1}{\rho_1(r_1,r_2)}, \frac{1}{\vartheta(r_1,r_2)})$, there is $u_{0,\lambda} \in J^{-1}(r_1, r_2)$ such that $\Gamma_\lambda(u_{0,\lambda}) \le \Gamma_\lambda(u)$ for all $u \in J^{-1}(r_1, r_2)$ and $\Gamma'_\lambda(u_{0,\lambda}) = 0$.

Theorem 2.2 ([6, Lemma 5.3])**.** *Let X be a real Banach space, $J : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative admits a continuous inverse on X^*, and $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable function whose Gâteaux derivative is compact. Fix $\inf_X J < r < \sup_X J$, and assume that*

$$\rho_2(r) > 0,$$

and for each $\lambda > \frac{1}{\rho_2(r)}$, the functional $\Gamma_\lambda := J - \lambda I$ is coercive. Then for each $\lambda \in (\frac{1}{\rho_2(r)}, +\infty)$, there is $u_{0,\lambda} \in J^{-1}(r, +\infty)$ such that $\Gamma_\lambda(u_{0,\lambda}) \le \Gamma_\lambda(u)$ for all $u \in J^{-1}(r, +\infty)$ and $\Gamma'_\lambda(u_{0,\lambda}) = 0$.

Since the operator in the divergence form is non-homogeneous, we introduce an Orlicz-Sobolev space setting for problems of this type. We first recall some basic facts about Orlicz-Sobolev spaces.

Set

$$\Phi(t) = \int_0^t \varphi(s)\,\mathrm{d}s, \quad \Phi^\star(t) = \int_0^t \varphi^{-1}(s)\,\mathrm{d}s, \quad \text{for all } t \in \mathbb{R}.$$

We observe that Φ is a Young function, that is, $\Phi(0) = 0$, Φ is convex, and

$$\lim_{t\to\infty} \Phi(t) = +\infty.$$

Furthermore, since $\Phi(t) = 0$ if and only if $t = 0$,

$$\lim_{t\to 0} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = +\infty,$$

then Φ is called an N-function. The function $\Phi^\star$ is called the complementary function of Φ and it satisfies

$$\Phi^\star(t) = \sup\{st - \Phi(s);\ s \ge 0\}, \quad \text{for all } t \ge 0\,.$$

We observe that $\Phi^\star$ is also an N-function and the following Young's inequality holds true:

$$st \le \Phi(s) + \Phi^\star(t), \quad \text{for all}\, s, t \ge 0\,.$$

Assume that Φ satisfies the following structural hypotheses

$$1 < \liminf_{t\to\infty} \frac{t\varphi(t)}{\Phi(t)} \le p^0 := \sup_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \infty; \tag{2.1}$$

$$N < p_0 := \inf_{t>0} \frac{t\varphi(t)}{\Phi(t)} < \liminf_{t\to\infty} \frac{\log(\Phi(t))}{\log(t)}. \tag{2.2}$$

The Orlicz space $L_\Phi(\Omega)$ defined by the N-function Φ (see for instance [1] and [29]) is the space of measurable functions $u : \Omega \to \mathbb{R}$ such that

$$\|u\|_{L_\Phi} := \sup\Big\{ \int_\Omega u(x)v(x)\,\mathrm{d}x;\ \int_\Omega \Phi^\star(|v(x)|)\,\mathrm{d}x \le 1 \Big\} < \infty\,.$$

Then $(L_\Phi(\Omega), \|\cdot\|_{L_\Phi})$ is a Banach space whose norm is equivalent to the Luxemburg norm

$$\|u\|_\Phi := \inf\Big\{k > 0;\ \int_\Omega \Phi\Big(\frac{u(x)}{k}\Big)\,\mathrm{d}x \le 1\Big\}.$$

We denote by $W^1L_\Phi(\Omega)$ the corresponding Orlicz-Sobolev space for problem (1.1), defined by

$$W^1L_\Phi(\Omega) = \Big\{u \in L_\Phi(\Omega);\ \frac{\partial u}{\partial x_i} \in L_\Phi(\Omega),\ i = 1, \dots, N\Big\}.$$

This is a Banach space with respect to the norm

$$\|u\|_{1,\Phi} = \|\nabla u\|_\Phi + \|u\|_\Phi,$$

see [1] and [15].

As mentioned in [8, 10], Assumption (Φ_0) is equivalent with the fact that Φ and $\Phi^\star$ both satisfy the Δ_2 condition (at infinity), see [1, p. 232]. In particular, (Φ, Ω) and $(\Phi^\star, \Omega)$ are $\Delta-$regular, see [1, p.232]. Consequently, the spaces $L_\Phi(\Omega)$ and $W^1L_\Phi(\Omega)$ are separable, reflexive Banach spaces, see [1, p. 241 and p. 247].

These spaces generalize the usual spaces $L^p(\Omega)$ and $W^{1,p}(\Omega)$, in which the role played by the convex mapping $t \mapsto |t|^p/p$ is assumed by a more general convex function $\Phi(t)$.

We recall the following useful lemma regarding the norms on Orlicz-Sobolev spaces.

Lemma 2.3 ([30, Lemma 2.2])**.** *On $W^1L_\Phi(\Omega)$ the norms*

$$\|u\|_{1,\Phi} = \||\nabla u|\|_\Phi + \|u\|_\Phi,$$
$$\|u\|_{2,\Phi} = \max\{\||\nabla u|\|_\Phi, \|u\|_\Phi\},$$
$$\|u\| = \inf\Big\{\mu > 0 : \int_\Omega \big[[\Phi\big(\frac{|u(x)|}{\mu}\big) + \Phi\big(\frac{|\nabla u(x)|}{\mu}\big)\big]\,\mathrm{d}x \le 1\Big\},$$

are equivalent. More precisely, for every $u \in W^1L_\Phi(\Omega)$ we have

$$\|u\| \le 2\|u\|_{2,\Phi} \le 2\|u\|_{1,\Phi} \le 4\|u\|.$$

We also recall the following lemmas which will be used in the proofs.

Lemma 2.4 ([25, Lemma 2.3])**.** *Let $u \in W^1L_\Phi(\Omega)$. Then*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \ge \|u\|^{p^0}, \quad \text{if } \|u\| < 1,$$
$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \ge \|u\|^{p_0}, \quad \text{if } \|u\| > 1,$$
$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \le \|u\|^{p_0}, \quad \text{if } \|u\| < 1,$$
$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \le \|u\|^{p^0}, \quad \text{if } \|u\| > 1.$$

Lemma 2.5 ([25, Lemma 2.5])**.** *Let $u \in W^1L_\Phi(\Omega)$ and assume that $\|u\| = 1$. Then*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x = 1.$$

Lemma 2.6 ([9, Lemma 2.2]). *Let $u \in W^1L_\Phi(\Omega)$ and assume that*

$$\int_\Omega [\Phi(|u(x)|) + \Phi(|\nabla u(x)|)]\,\mathrm{d}x \le r,$$

for some $0 < r < 1$. Then, one has $\|u\| < 1$.

Now from hypothesis (2.2), by Lemma D.2 in [15] it follows that $W^1L_\Phi(\Omega)$ is continuously embedded in $W^{1,p_0}(\Omega)$. On the other hand, since we assume $p_0 > N$ we deduce that $W^{1,p_0}(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$. Thus, one has that $W^1L_\Phi(\Omega)$ is compactly embedded in $C^0(\overline{\Omega})$ and there exists a constant $c > 0$ such that

$$\|u\|_\infty \le c\,\|u\|_{1,\Phi}, \quad \text{for all } u \in W^1L_\Phi(\Omega) \tag{2.3}$$

where $\|u\|_\infty := \sup_{x\in\overline{\Omega}} |u(x)|$. A concrete estimation of a concrete upper bound for the constant c remains an open question.

Let

$$F(x,\xi) = \int_0^\xi f(x,t)\,\mathrm{d}t \quad \text{for } (x,\xi) \in \Omega \times \mathbb{R}.$$

Now for every $u \in W^1L_\Phi(\Omega)$, we define $\Gamma_\lambda(u) := J(u) - \lambda I(u)$ where

$$J(u) = \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]\,\mathrm{d}x, \tag{2.4}$$

$$I(u) = \int_\Omega F(x,u(x))\,\mathrm{d}x. \tag{2.5}$$

Standard arguments show that $\Gamma_\lambda \in C^1(W^1L_\Phi(\Omega),\mathbb{R})$. In fact, one has

$$\begin{aligned}
\Gamma_\lambda'(u)(v) &= \lim_{h\longrightarrow 0} \frac{\Gamma_\lambda(u+hv) - \Gamma_\lambda(u)}{h} \\
&= \int_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega \alpha(|u(x)|)u(x)v(x)\,\mathrm{d}x \\
&\quad - \lambda \int_\Omega f(x,u(x))v(x)\,\mathrm{d}x.
\end{aligned}$$

for all $u, v \in W^1L_\Phi(\Omega)$ (see [30] for more details).

A function $u : \Omega \to \mathbb{R}$ is a weak solution for problem (1.1) if

$$\begin{aligned}
&\int_\Omega \alpha(|\nabla u(x)|)\nabla u(x)\cdot\nabla v(x)\,\mathrm{d}x + \int_\Omega \alpha(|u(x)|)u(x)v(x)\,\mathrm{d}x \\
&- \lambda \int_\Omega f(x,u(x))v(x)\,\mathrm{d}x = 0,
\end{aligned}$$

for every $v \in W^1L_\Phi(\Omega)$.

3. Main results

For a non-negative constant γ and a positive constant δ with

$$\gamma \ne 2c\,(\Phi(\delta)\,\mathrm{meas}(\Omega))^{1/p^0},$$

we set

$$a_\gamma(\delta) := \frac{\int_\Omega \sup_{|t|\le\gamma} F(x,t)\,\mathrm{d}x - \int_\Omega F(x,\delta)\,\mathrm{d}x}{\gamma^{p^0} - (2c)^{p^0}\Phi(\delta)\,\mathrm{meas}(\Omega)}.$$

Theorem 3.1. *Assume that there exist a non-negative constant γ_1 and two positive constants γ_2 and δ, with $\gamma_2 < 2c$ and*
$$\frac{\gamma_1^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)} < \Phi(\delta) < \frac{\gamma_2^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)}, \tag{3.1}$$
where c is defined in (2.3), such that

(A1) *$a_{\gamma_2}(\delta) < a_{\gamma_1}(\delta)$;*

(A2) *there exist $\nu > p^0$ and $R > 0$ such that for all $|\xi| \geq R$ and for all $x \in \Omega$,*
$$0 < \nu F(x,\xi) \leq \xi f(x,\xi). \tag{3.2}$$

Then, for each $\lambda \in]\frac{1}{(2c)^{p^0}}\frac{1}{a_{\gamma_1}(\delta)}, \frac{1}{(2c)^{p^0}}\frac{1}{a_{\gamma_2}(\delta)}[$, problem (1.1) admits at least two non-trivial weak solutions u_1 and u_2 in $W^1L_\Phi(\Omega)$, such that
$$\frac{\gamma_1^{p^0}}{(2c)^{p^0}} < J(u_1) < \frac{\gamma_2^{p^0}}{(2c)^{p^0}}.$$

Proof. Take $X := W^1L_\Phi(\Omega)$. For $u \in X$, put $\Gamma_\lambda(u) = J(u) - \lambda I(u)$ where J and I are given as in (2.4) and (2.5), respectively. Moreover, owing that Φ is convex, it follows that J is a convex functional, hence one has that J is sequentially weakly lower semicontinuous. We see that J is a coercive functional. Indeed, by Lemma 2.4, we deduce that for any $u \in X$ with $\|u\| > 1$ we have $J(u) \geq \|u\|^{p_0}$ which follows $\lim_{\|u\|\to+\infty} J(u) = +\infty$. Finally we observe that the functional $J : X \to \mathbb{R}$ is continuously Gâteaux differentiable while Lemma 2.3 of [30] gives that its Gâteaux derivative admits a continuous inverse on X^*. On the other hand, the fact that X is compactly embedded into $C^0(\overline{\Omega})$ implies that the operator $I' : X \to X^*$ is compact. Note that the critical points of Γ_λ are the weak solutions of the problem (1.1). Choose
$$r_1 = \Big(\frac{\gamma_1}{2c}\Big)^{p^0}, \quad r_2 = \Big(\frac{\gamma_2}{2c}\Big)^{p^0}$$
and $w(x) := \delta$ for all $x \in \Omega$. Clearly $w \in X$. Hence
$$J(w) = \int_\Omega [\Phi(|\nabla w(x)|) + \Phi(|w(x)|)]\,\mathrm{d}x = \int_\Omega \Phi(\delta)\,\mathrm{d}x = \Phi(\delta)\operatorname{meas}(\Omega).$$
From condition (3.1), we obtain $r_1 < \Phi(w) < r_2$. For all $u \in X$, by (2.3) and Lemma 2.3, we have
$$|u(x)| \leq \|u\|_\infty \leq c\|u\|_{1,\Phi} \leq 2c\|u\|, \quad \text{for all } x \in \Omega.$$
Hence, since $\gamma_2 < 2c$, taking Lemmas 2.4 and 2.6 into account one has
$$J^{-1}(-\infty, r_2) \subseteq \{u \in X; \|u\| \leq \frac{\gamma_2}{2c}\} \subseteq \{u \in X; |u(x)| \leq \gamma_2 \text{ for all } x \in \Omega\},$$
and it follows that
$$\sup_{u\in J^{-1}(-\infty,r_2)} I(u) \leq \int_\Omega \sup_{|t|\leq\gamma_2} F(x,t)\,\mathrm{d}x.$$
Therefore, one has
$$\begin{aligned}\vartheta(r_1,r_2) &\leq \frac{\sup_{u\in J^{-1}(-\infty,r_2)} I(u) - I(w)}{r_2 - J(w)}\\ &\leq (2c)^{p^0}\frac{\int_\Omega \sup_{|t|\leq\gamma_2} F(x,t)\,\mathrm{d}x - \int_\Omega F(x,\delta)\,\mathrm{d}x}{\gamma_2^{p^0} - (2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}\end{aligned}$$

$$= (2c)^{p^0} a_{\gamma_2}(\delta).$$

On the other hand, one has

$$\begin{aligned}
\rho_1(r_1, r_2) &\geq \frac{I(w) - \sup_{u \in J^{-1}(-\infty, r_1]} I(u)}{J(w) - r_1} \\
&\geq (2c)^{p^0} \frac{\int_\Omega \sup_{|t| \leq \gamma_1} F(x,t)\,\mathrm{d}x - \int_\Omega F(x,\delta)\,\mathrm{d}x}{(2c)^{p^0} \Phi(\delta)\operatorname{meas}(\Omega) - \gamma_1^{p^0}} \\
&= (2c)^{p^0} a_{\gamma_1}(\delta).
\end{aligned}$$

Hence, from (A1), one has $\vartheta(r_1, r_2) < \rho_1(r_1, r_2)$. Therefore, from Theorem 2.1, for each

$$\lambda \in \Big]\frac{1}{(2c)^{p^0}} \frac{1}{a_{\gamma_1}(\delta)}, \frac{1}{(2c)^{p^0}} \frac{1}{a_{\gamma_2}(\delta)}\Big[,$$

the functional Γ_λ admits at least one non-trivial critical point u_1 such that

$$r_1 < J(u_1) < r_2,$$

that is

$$\frac{\gamma_1^{p^0}}{(2c)^{p^0}} < J(u_1) < \frac{\gamma_2^{p^0}}{(2c)^{p^0}}.$$

Now, we prove the existence of the second critical point distinct from the first one. To this purpose, we verify the hypotheses of the mountain-pass theorem for the functional Γ_λ. Clearly, the functional Γ_λ is of class C^1 and $\Gamma_\lambda(0) = J(0) - \lambda I(0) = 0$. The first part of proof guarantees that $u_1 \in X$ is a local non-trivial local minimum for Γ_λ in X. We can assume that u_1 is a strict local minimum for Γ_λ in X. Therefore, there is $\rho > 0$ such that $\inf_{\|u-u_1\|=\rho} \Gamma_\lambda(u) > \Gamma_\lambda(u_1)$, so condition [40, (I_1), Theorem 2.2] is verified. By integrating the condition (3.2) there exist constants $a_1, a_2 > 0$ such that

$$F(x,t) \geq a_1|t|^\nu - a_2$$

for all $x \in \Omega$ and $t \in \mathbb{R}$. Now, choosing any $u \in X \setminus \{0\}$, and for convenience, let

$$p^\star = \begin{cases} p^0, & \text{if } \|u\| > 1, \\ p_0, & \text{if } \|u\| < 1. \end{cases}$$

One has

$$\begin{aligned}
\Gamma_\lambda(\tau u) &= (J - \lambda I)(\tau u) \\
&= \int_\Omega (\Phi(|\tau \nabla u(x)|) + \Phi(|\tau u(x)|))\,\mathrm{d}x - \lambda \int_\Omega F(x, \tau u(x))\,\mathrm{d}x \\
&\leq \tau^{p^\star} \|u\|^{p^\star} - \lambda \tau^\nu a_1 \int_\Omega |u(t)|^\nu\,\mathrm{d}t + \lambda a_2 \to -\infty
\end{aligned}$$

as $\tau \to +\infty$, so condition [40, (I_2), Theorem 2.2] is satisfied. So, the functional Γ_λ satisfies the geometry of mountain pass. Moreover, Γ_λ satisfies the Palais-Smale condition. Indeed, assume that $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $\{\Gamma_\lambda(u_n)\}_{n\in\mathbb{N}}$ is bounded and

$$\Gamma'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.3}$$

Then, there exists a positive constant C_0 such that

$$|\Gamma_\lambda(u_n)| \leq C_0, \quad |\Gamma'_\lambda(u_n)| \leq C_0, \quad \forall n \in \mathbb{N}.$$

Therefore, since
$$p^0 \geq \frac{t\varphi(t)}{\Phi(t)}, \quad \forall t > 0,$$
we deduce from the definition of Γ'_λ and the assumption (A2) that
$$\begin{aligned}
C_0 + C_1\|u_n\| &\geq \nu\Gamma_\lambda(u_n) - \Gamma'_\lambda(u_n)(u_n)\\
&= \nu \int_\Omega (\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|))\,\mathrm{d}x\\
&\quad - \int_\Omega \varphi(|\nabla u_n(x)|)\nabla u_n(x)\,\mathrm{d}x - \int_\Omega \varphi(|u_n(x)|)u_n(x)\,\mathrm{d}x\\
&\quad - \lambda \int_\Omega (\nu F(x,u_n(x)) - f(x,u_n(x))(u_n(x)))\,\mathrm{d}x\\
&\geq \begin{cases} (\nu - p^0)\|u_n\|^{p_0}, & \text{if } \|u_n\| \geq 1,\\ (\nu - p^0)\|u_n\|^{p^0}, & \text{if } \|u_n\| < 1, \end{cases}
\end{aligned}$$
for some $C_1 > 0$. Since $\nu > p^0$ this implies that (u_n) is bounded. Consequently, since X is a reflexive Banach space there exists a subsequence, still denoted by $\{u_n\}$, and $u \in X$ such that $\{u_n\}$ converges weakly to u in X. Now, arguing as in [34], from the continuity of f, we have that
$$\lim_{n\to\infty} I(u_n) = I(u), \quad \lim_{n\to\infty} I'(u_n) = I'(u). \tag{3.4}$$
Since
$$J(u) = \Gamma_\lambda(u) - \lambda I(u), \quad \forall u \in X\,,$$
relations (3.3) and (3.4) imply
$$\lim_{n\to\infty} J'(u_n) = -\lambda I'(u), \quad \text{in } X^*. \tag{3.5}$$
By the convexity of Φ we have the convexity of J and thus
$$J(u_n) \leq J(u) + (J'(u_n), u_n - u).$$
Passing to the limit as $n \to \infty$ and using (3.5) we deduce that
$$\limsup_{n\to\infty} J(u_n) \leq J(u). \tag{3.6}$$
Since J is weakly lower semi-continuous we have
$$\liminf_{n\to\infty} J(u_n) \geq J(u). \tag{3.7}$$
By (3.6) and (3.7) we have
$$\lim_{n\to\infty} J(u_n) = J(u)$$
or
$$\lim_{n\to\infty} \int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]\,\mathrm{d}x = \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]\,\mathrm{d}x\,. \tag{3.8}$$
Since Φ is increasing and convex, it follows that
$$\begin{aligned}
&\Phi\Big(\frac{1}{2}|\nabla u_n(x) - \nabla u(x)|\Big) + \Phi\Big(\frac{1}{2}|u_n(x) - u(x)|\Big)\\
&\leq \Phi\Big(\frac{1}{2}\,(|\nabla u_n(x)| + |\nabla u(x)|)\Big) + \Phi\Big(\frac{1}{2}\,(|u_n(x)| + |u(x)|)\Big)\\
&\leq \frac{\Phi(|\nabla u_n(x)|) + \Phi(|\nabla u(x)|)}{2} + \frac{\Phi(|u_n(x)|) + \Phi(|u(x)|)}{2}\,,
\end{aligned}$$

for all $x \in \Omega$ and all n. Integrating the above inequalities over Ω we find

$$\begin{aligned}
0 &\leq \int_\Omega \Big[\Phi\Big(\frac{1}{2}|\nabla(u_n-u)(x)|\Big) + \Phi\Big(\frac{1}{2}|(u_n-u)(x)|\Big)\Big]\,\mathrm{d}x \\
&\leq \frac{\int_\Omega \Phi(|\nabla u_n(x)|)\,\mathrm{d}x + \int_\Omega \Phi(|\nabla u(x)|)\,\mathrm{d}x}{2} + \frac{\int_\Omega \Phi(|u_n(x)|)\,\mathrm{d}x + \int_\Omega \Phi(|u(x)|)\,\mathrm{d}x}{2} \\
&= \frac{\int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]\,\mathrm{d}x + \int_\Omega [\Phi(|\nabla u(x)|) + \Phi(|u(x)|)]\,\mathrm{d}x}{2},
\end{aligned}$$

for all n. We point out that Lemma 2.4 implies

$$\int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]\,\mathrm{d}x \leq \|u_n\|^{p_0} < 1,$$

provided that $\|u_n\| < 1$, and

$$\int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]\,\mathrm{d}x \leq \|u_n\|^{p^0},$$

provided that $\|u_n\| > 1$. Since $\{u_n\}$ is bounded in X, the above inequalities prove the existence of a positive constant M_1 such that

$$\int_\Omega [\Phi(|\nabla u_n(x)|) + \Phi(|u_n(x)|)]\,\mathrm{d}x \leq M_1,$$

for all n. So, there exists a positive constant M_2 such that

$$0 \leq \int_\Omega \Big[\Phi\Big(\frac{1}{2}|\nabla(u_n-u)(x)|\Big) + \Phi\Big(\frac{1}{2}|(u_n-u)(x)|\Big)\Big]\,\mathrm{d}x \leq M_2, \tag{3.9}$$

for all n. On the other hand, since $\{u_n\}$ converges weakly to u in X, Theorem 2.1 in [21] implies

$$\int_\Omega \frac{\partial u_n}{\partial x_i} v\,\mathrm{d}x \to \int_\Omega \frac{\partial u}{\partial x_i} v\,\mathrm{d}x, \quad \forall v \in L_{\Phi^\star}(\Omega),\ i = 1,\dots,N.$$

In particular this holds for all $v \in L^\infty(\Omega)$. Hence $\{\frac{\partial u_n}{\partial x_i}\}$ converges weakly to $\frac{\partial u}{\partial x_i}$ in $L^1(\Omega)$ for all $i = 1,\dots,N$. Thus we deduce that

$$\nabla u_n(x) \to \nabla u(x) \quad \text{a.e. } x \in \Omega. \tag{3.10}$$

Relations (3.8), (3.9) and (3.10) and Lebesgue's dominated convergence theorem imply

$$\lim_{n\to\infty} \int_\Omega \Big[\Phi\Big(\frac{1}{2}|\nabla(u_n-u)(x)|\Big) + \Phi\Big(\frac{1}{2}|(u_n-u)(x)|\Big)\Big]\,\mathrm{d}x = 0. \tag{3.11}$$

On the other hand, the assumption (Φ_0) implies that Φ satisfies Δ_2-condition. Thus, by (3.11) and [16, Lemma A.4] (see also [1, p. 236]) we have

$$\lim_{n\to\infty} \|\frac{1}{2}(u_n - u)\| = 0.$$

So $\|u_n - u\| \to 0$ as $n \to \infty$, which implies that $\{u_n\}$ converges strongly to u in X. Therefore, Γ_λ satisfies the Palais-Smale condition. Hence, the classical theorem of Ambrosetti and Rabinowitz ensures a critical point u_2 of Γ_λ such that $\Gamma_\lambda(u_2) > \Gamma_\lambda(u_1)$. Since $f(x,0) \neq 0$ for all $x \in \Omega$, u_1 and u_2 are two distinct non-trivial solutions of (1.1) and the proof is complete. $\square$

Remark 3.2. In Theorem 3.1 we ensured the existence of at least two non-trivial weak solutions u_1 and u_2 for (1.1), with u_2 obtained in association with the classical Ambrosetti-Rabinowitz condition on the data by assuming $f(x,0) \neq 0$ for all $x \in \Omega$. If $f(x,0) = 0$ for all $x \in \Omega$, u_2 may be trivial.

Now, we point out an immediate consequence of Theorem 3.1.

Theorem 3.3. *Assume that there exist two positive constants δ and γ, with $\gamma < 2c$ and*

$$\Phi(\delta) < \frac{\gamma^{p^0}}{(2c)^{p^0} \operatorname{meas}(\Omega)},$$

such that (A2) in Theorem 3.1 holds. Furthermore, suppose that

$$\frac{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x}{\gamma^{p^0}} < \frac{\int_\Omega F(x,\delta)\,\mathrm{d}x}{(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}. \tag{3.12}$$

Then, for each

$$\lambda \in \Big]\frac{\Phi(\delta)\operatorname{meas}(\Omega)}{\int_\Omega F(x,\delta)\,\mathrm{d}x}, \frac{\gamma^{p^0}}{(2c)^{p^0}\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x}\Big[,$$

problem (1.1) *admits at least two non-trivial weak solutions u_1 and u_2 in $W^1L_\Phi(\Omega)$ such that*

$$0 < J(u_1) < \frac{\gamma^{p^0}}{(2c)^{p^0}}.$$

Proof. The conclusion follows from Theorem 3.1, by taking $\gamma_1 = 0$ and $\gamma_2 = \gamma$. Indeed, owing to the inequality (3.12), one has

$$\begin{aligned}
a_\gamma(\delta) &= \frac{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x - \int_\Omega F(x,\delta)\,\mathrm{d}x}{\gamma^{p^0} - (2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)} \\
&< \frac{(1 - \frac{(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}{\gamma^{p^0}})\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x}{\gamma^{p^0} - (2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)} \\
&= \frac{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x}{\gamma^{p^0}} \\
&< \frac{\int_\Omega F(x,\delta)\,\mathrm{d}x}{(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)} \\
&= a_0(\delta).
\end{aligned}$$

In particular, one has

$$a_\gamma(\delta) < \frac{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x}{\gamma^{p^0}},$$

which follows

$$\frac{1}{(2c)^{p^0}}\frac{\gamma^{p^0}}{\int_\Omega \sup_{|t|\leq\gamma} F(x,t)\,\mathrm{d}x} < \frac{1}{(2c)^{p^0}}\frac{1}{a_{\gamma_2}(\delta)}.$$

Hence, Theorem 3.1 concludes the result. □

Now, we give an application of Theorem 2.2 which will be used later to ensure the existence of multiple solutions for non-homogeneous Neumann problems.

Theorem 3.4. *Assume that there exist two positive constants $\bar{\gamma}$ and $\bar{\delta}$ with $\bar{\gamma} < 2c$ and*

$$\Phi(\bar{\delta}) > \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)},$$

such that

$$\int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x < \int_\Omega F(x,\bar{\delta})\,\mathrm{d}x,$$

$$\limsup_{|\xi|\to+\infty} \frac{F(x,\xi)}{|\xi|^{p_0}} \le 0 \quad \textit{uniformly in } \mathbb{R}. \tag{3.13}$$

Then, for each $\lambda > \tilde{\lambda}$, where

$$\tilde{\lambda} := \frac{(2c)^{p^0}\Phi(\bar{\delta})\operatorname{meas}(\Omega) - \bar{\gamma}^{p^0}}{(2c)^{p^0}\Big(\int_\Omega F(x,\bar{\delta})\,\mathrm{d}x - \int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x\Big)},$$

problem (1.1) *admits at least one non-trivial weak solution $\bar{u} \in W^1L_\Phi(\Omega)$ such that*

$$J(\bar{u}) > \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0}}.$$

Proof. Take the real Banach space X as defined in Theorem 3.3, and for $u \in X$ put $\Gamma_\lambda(u) = J(u) - \lambda I(u)$ where J and I are given as in (2.4) and (2.5), respectively. Our aim is to apply Theorem 2.2 to function Γ_λ. The functionals J and I satisfy all required assumptions in Theorem 2.2. Moreover, for $\lambda > 0$, the functional Γ_λ is coercive. Indeed, fix $0 < \epsilon < \frac{1}{c^{p_0}\operatorname{meas}(\Omega)\lambda}$. From (3.13) there is a function $\rho_\epsilon \in L^1(\Omega)$ such that

$$F(x,t) \le \epsilon|t|^{p_0} + \rho_\epsilon(x),$$

for every $x \in \Omega$ and $t \in \mathbb{R}$. Taking (2.3) into account, it follows that, for each $u \in X$ with $\|u\| > 1$,

$$\begin{aligned} J(u) - \lambda I(u) &= \int_\Omega (\Phi[|\nabla u(x)|) + \Phi(|u(x)|)]\,\mathrm{d}x - \lambda\int_\Omega F(x,u(x))\,\mathrm{d}x \\ &\ge \|u\|^{p_0} - \lambda\epsilon\int_\Omega |u(x)|^{p_0}\,\mathrm{d}x - \lambda\|\rho_\epsilon\|_{L^1(\Omega)} \\ &\ge (1 - \lambda\epsilon c^{p_0}\operatorname{meas}(\Omega))\|u\|^{p_0} - \lambda\|\rho_\epsilon\|_{L^1(\Omega)}, \end{aligned}$$

and thus

$$\lim_{\|u\|\to+\infty}(J(u) - \lambda I(u)) = +\infty,$$

which means the functional Γ_λ is coercive. Choosing $\bar{r} = \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0}}$ and $\bar{w}(x) = \bar{\delta}$ for all $x \in \Omega$, and arguing as in the proof of Theorem 3.1, we obtain that

$$\rho(\bar{r}) \ge (2c)^{p^0}\frac{\int_\Omega F(x,\bar{\delta})\,\mathrm{d}x - \int_\Omega \sup_{|t|\le\bar{\gamma}} F(x,t)\,\mathrm{d}x}{(2c)^{p^0}\Phi(\bar{\delta})\operatorname{meas}(\Omega) - \bar{\gamma}^{p^0}}.$$

So, from our assumption it follows that $\rho(\bar{r}) > 0$. Hence, from Theorem 2.2 for each $\lambda > \tilde{\lambda}$, the functional Γ_λ admits at least one local minimum $\bar{u}$ such that $J(\bar{u}) > \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0}}$. The conclusion is achieved. $\square$

Now, we point out some results in which the function f has separated variables. To be precise, consider the problem

$$\begin{gathered} -\operatorname{div}(\alpha(|\nabla u|)\nabla u)+\alpha(|u|)u=\lambda\theta(x)g(u) \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega \end{gathered} \tag{3.14}$$

where $\theta:\Omega\to\mathbb{R}$ is a non-negative and non-zero function such that $\theta\in L^1(\Omega)$ and $g:\mathbb{R}\to\mathbb{R}$ is a non-negative continuous function.

Put $G(t)=\int_0^t g(\xi)\,\mathrm{d}\xi$ for all $t\in\mathbb{R}$.

Since the nonlinear term is supposed to be non-negative, the following results give the existence of multiple positive solutions. To justify this, we point out the following weak maximum principle.

Lemma 3.5. *Suppose that $u_*\in W^1L_\Phi(\Omega)$ is a non-trivial weak solution of the problem* (3.14)*. Then, u_* is positive.*

Proof. Arguing by a contradiction, assume that the set $\mathcal{A}=\{x\in\Omega;\ u_*(x)<0\}$ is non-empty and of positive measure. Put $u_*^-(x)=\min\{u_*(x),0\}$. By [22, Remark 5] we deduce that $u_*^-\in W^1L_\Phi(\Omega)$. Suppose that $\|u_*\|<1$. Using this fact that u_* also is a weak solution of (3.14) and by choosing $v=u_*^-$, since

$$p_0\le\frac{t\varphi(t)}{\Phi(t)}\,,\quad \forall\, t>0,$$

and using the first inequality of Lemma 2.4 and recalling our sign assumptions on the data, we have

$$\begin{aligned} \|u_*\|^{p^0}_{W^1L_\Phi(\mathcal{A})} &\le \int_{\mathcal{A}}[\Phi(|\nabla u_*(x)|)+\Phi(|u_*(x)|)]\,\mathrm{d}x \\ &\le \frac{1}{p_0}\int_{\mathcal{A}}[\varphi(|\nabla u_*(x)|)|\nabla u_*(x)|+\varphi(|u_*(x)|)|u_*(x)|]\,\mathrm{d}x \\ &= \frac{1}{p_0}\int_{\mathcal{A}}[\alpha(|\nabla u_*(x)|)|\nabla u_*(x)|^2+\alpha(|u_*(x)|)|u_*(x)|^2]\,\mathrm{d}x \\ &= \frac{1}{p_0}\int_{\mathcal{A}}\theta(x)g(u_*(x))u_*(x)\,\mathrm{d}x\le 0, \end{aligned}$$

i.e.,

$$\|u_*\|^{p^0}_{W^1L_\Phi(\mathcal{A})}\le 0,$$

which contradicts that u_* is a non-trivial weak solution. Hence, the set $\mathcal{A}$ is empty, and u_* is positive. The proof of the case $\|u_*\|>1$ is similar to the case $\|u_*\|<1$ (use the second part of Lemma 2.4 instead). For the case $\|u_*\|=1$, we may assume $\|u_*\|_{W^1L_\Phi(\mathcal{A})}=1$, and arguing as for the case $\|u_*\|<1$, and using Lemma 2.5, we have

$$\begin{aligned} \|u_*\|_{W^1L_\Phi(\mathcal{A})} &= \int_{\mathcal{A}}[\Phi(|\nabla u_*(x)|)+\Phi(|u_*(x)|)]\,\mathrm{d}x \\ &\le \frac{1}{p_0}\int_{\mathcal{A}}\theta(x)g(u_*(x))u_*(x)\,\mathrm{d}x\le 0, \end{aligned}$$

which also contradicts with the fact that u_* is a non-trivial weak solution. Therefore, we deduce u_* is positive. $\square$

Setting $f(x,t)=\theta(x)g(t)$ for every $(x,t)\in\Omega\times\mathbb{R}$, the following existence results are consequences of Theorems 3.1-3.4, respectively.

Theorem 3.6. *Assume that $g(0)\neq 0$ and there exist a non-negative constant γ_1 and two positive constants γ_2 and δ, with $\gamma_2<2c$ and*
$$\frac{\gamma_1^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)}<\Phi(\delta)<\frac{\gamma_2^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)},$$
such that
$$\frac{G(\gamma_1)-G(\delta)}{\gamma_1^{p^0}-(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}<\frac{G(\gamma_2)-G(\delta)}{\gamma_2^{p^0}-(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}.$$
Furthermore, suppose that

(AR) *there exist constants $\nu>p^0$ and $R>0$ such that, for all $\xi\geq R$,*
$$0<\nu G(\xi)\leq \xi g(\xi).$$

Then, for each $\lambda\in]\lambda_1,\lambda_2[$, where
$$\lambda_1=\frac{1}{(2c)^{p^0}}\frac{\gamma_1^{p^0}-(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}{\|\theta\|_{L^1(\Omega)}\left(G(\gamma_1)-G(\delta)\right)},$$
$$\lambda_2=\frac{1}{(2c)^{p^0}}\frac{\gamma_2^{p^0}-(2c)^{p^0}\Phi(\delta)\operatorname{meas}(\Omega)}{\|\theta\|_{L^1(\Omega)}\left(G(\gamma_2)-G(\delta)\right)},$$
problem (3.14) admits at least two positive weak solutions u_1 and u_2 in $W^1L_\Phi(\Omega)$ such that
$$\frac{\gamma_1^{p^0}}{(2c)^{p^0}}<J(u_1)<\frac{\gamma_2^{p^0}}{(2c)^{p^0}}.$$

Theorem 3.7. *Assume that $g(0)\neq 0$ and there exist two positive constants δ and γ, with $\gamma<2c$ and*
$$\Phi(\delta)<\frac{\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)},$$
such that
$$\frac{G(\gamma)}{\gamma^{p^0}}<\frac{1}{(2c)^{p^0}\operatorname{meas}(\Omega)}\frac{G(\delta)}{\Phi(\delta)}. \tag{3.15}$$
Furthermore, suppose that (AR) holds. Then, for every
$$\lambda\in\left]\frac{\Phi(\delta)\operatorname{meas}(\Omega)}{\|\theta\|_{L^1(\Omega)}G(\delta)},\frac{\gamma^{p^0}}{(2c)^{p^0}\|\theta\|_{L^1(\Omega)}G(\gamma)}\right[,$$
problem (3.14) admits at least two positive weak solutions u_1 and u_2 in $W^1L_\Phi(\Omega)$ such that
$$0<J(u_1)<\frac{\gamma^{p^0}}{(2c)^{p^0}}.$$

Theorem 3.8. *Assume that $g(0)\neq 0$ and there exist two positive constants $\bar\gamma$ and $\bar\delta$ with $\bar\gamma<2c$ and*
$$\Phi(\bar\delta)>\frac{\bar\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)}, \tag{3.16}$$
such that
$$G(\bar\gamma)<G(\bar\delta) \tag{3.17}$$

and

$$\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^{p_0}}\le 0.$$

Then, for each $\lambda>\bar\lambda$, *where*

$$\bar\lambda:=\frac{(2c)^{p^0}\Phi(\bar\delta)\operatorname{meas}(\Omega)-\bar\gamma^{p^0}}{(2c)^{p^0}\|\theta\|_{L^1(\Omega)}\big(G(\bar\delta)-G(\bar\gamma)\big)},$$

problem (3.14) *admits at least one positive weak solution* $\bar{u_1}\in W^1L_\Phi(\Omega)$ *such* $J(\bar{u_1})>\frac{\bar\gamma^{p^0}}{(2c)^{p^0}}$.

Now we illustrate Theorem 3.8 by presenting the following example.

Example 3.9. Let $3\le N<p$, and let $\Omega\subset\mathbb{R}^N$ be a domain such that

$$\operatorname{meas}(\Omega)>\frac{p(p+2)}{(2\sqrt{3})^{p+2}c^p\left[(p+2)\log(1+c^2)-c^2\right]},\tag{3.18}$$

and let

$$\varphi(t)=\log(1+|t|^2)|t|^{p-2}t,\quad t\in\mathbb{R}.$$

It is easy to see that, $\varphi:\mathbb{R}\to\mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$, and one has

$$p_0=p\quad\text{and}\quad p^0=p+2.$$

Thus the relations (2.1) and (2.2) are satisfied (see [16, Example 2] for the details). Now we define the function $g:\mathbb{R}\to\mathbb{R}$ by

$$g(t)=\frac{c}{c^2+t^2}\,e^{\arctan(t/c)}.$$

Clearly, g is a non-negative continuous function, $g(0)\ne 0$ and

$$G(t)=e^{\arctan(t/c)}-1,\quad\forall t\in\mathbb{R}.$$

Thus

$$\limsup_{|\xi|\to+\infty}\frac{G(\xi)}{|\xi|^{p_0}}=\limsup_{|\xi|\to+\infty}\frac{e^{\arctan(\xi/c)}-1}{|\xi|^p}=0.$$

By choosing $\bar\delta=c$ and $\bar\gamma=\sqrt{3}c/3<2c$ we clearly observe that (3.16) and (3.17) are satisfied. Indeed,

$$G(\bar\gamma)=e^{\pi/6}-1<e^{\pi/4}-1=G(\bar\delta)$$

and by (3.18) we have

$$\begin{aligned}
\Phi(\bar\delta)=\Phi(c)&=\frac{c^p}{p}\log(1+c^2)-\frac{2}{p}\int_0^c\frac{s^{p+1}}{1+s^2}\,ds\\
&>\frac{c^p}{p}\log(1+c^2)-\frac{2}{p}\int_0^c s^{p+1}\,ds=\frac{c^p}{p}\log(1+c^2)-\frac{c^{p+2}}{p(p+2)}\\
&>\frac{(\frac{\sqrt{3}}{3}c)^{p+2}}{(2c)^{p+2}\operatorname{meas}(\Omega)}=\frac{\bar\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)}.
\end{aligned}$$

Hence, by applying Theorem 3.8, for every

$$\lambda>\frac{(2c)^{p+2}\Phi(c)\mathrm{meas}(\Omega)-(\frac{\sqrt{3}}{3}c)^{p+2}}{(2c)^{p+2}\mathrm{meas}(\Omega)(e^{\pi/4}-e^{\pi/6})},$$

the problem

$$-\operatorname{div}\left(\log(1+|\nabla u|^2)|\nabla u|^{p-2}\nabla u\right)+\log(1+|u|^2)|u|^{p-2}u=\frac{\lambda c}{c^2+u^2}e^{\arctan\frac{u}{c}} \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu}=0 \quad \text{on } \partial\Omega$$

has at least one positive weak solution.

A further consequence of Theorem 3.1 is the following existence result.

Theorem 3.10. *Assume that $g(0)\neq 0$ and*

$$\lim_{\xi\to 0^+}\frac{G(\xi)}{\Phi(\xi)}=+\infty. \tag{3.19}$$

Furthermore, suppose that (AR) holds. Then, for every $\lambda\in]0,\lambda^\star_\gamma[$ where

$$\lambda^\star_\gamma:=\frac{1}{(2c)^{p^0}\|\theta\|_{L^1(\Omega)}}\sup_{0<\gamma<2c}\frac{\gamma^{p^0}}{G(\gamma)},$$

problem (3.14) *admits at least two positive weak solutions in $W^1L_\Phi(\Omega)$.*

Proof. Fix $\lambda\in]0,\lambda^\star_\gamma[$. Then there is $0<\gamma<2c$ such that $\lambda<\frac{\gamma^{p^0}}{(2c)^{p^0}\|\theta\|_{L^1(\Omega)}G(\gamma)}$. From (3.19) there exists a positive constant δ with

$$\Phi(\delta)<\frac{\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)},$$

such that

$$\frac{1}{\lambda}<\frac{\|\theta\|_{L^1(\Omega)}G(\delta)}{\Phi(\delta)\operatorname{meas}(\Omega)}.$$

Therefore, the conclusion follows from Theorem 3.3. $\square$

Remark 3.11. Theorem 1.1 immediately follows from Theorem 3.10 by setting $\alpha(|t|)=|t|^{p-2}$ (for details about this choice of $\alpha(|t|)$, see [9, Remark 3.4]).

Now we illustrate Theorem 3.10 by presenting the following example.

Example 3.12. Let $N=3$, $\Omega\subset\mathbb{R}^3$, $p=5$ and define

$$\varphi(t)=\begin{cases}\frac{|t|^{p-2}t}{\log(1+|t|)}, & \text{if } t\neq 0,\\ 0, & \text{if } t=0.\end{cases}$$

It is easy to see that $\varphi:\mathbb{R}\to\mathbb{R}$ is an odd, increasing homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. By [16, Example 3] one has

$$p_0=p-1<p^0=p=\liminf_{t\to\infty}\frac{\log(\Phi(t))}{\log(t)}.$$

Thus the relations (2.1) and (2.2) are satisfied. Now let

$$g(t)=\begin{cases}1+t^6, & |t|\geq 1,\\ 3-t^2, & |t|<1.\end{cases}$$

In this case, g is non-negative, continuous, $g(0)=3\neq 0$ and the condition (3.19) holds. Moreover, taking into account that

$$\lim_{|\xi|\to+\infty}\frac{\xi g(\xi)}{G(\xi)}=\lim_{|\xi|\to+\infty}\frac{\xi+\xi^7}{\xi+\frac{1}{7}\xi^7}=7>p$$

by choosing $\nu = 7 > p$, there exists $R > 1$ such that the assumptions (AR) fulfilled. Hence, by applying Theorem 3.10, for every $\lambda > 0$ the problem

$$-\operatorname{div}\Big(\frac{|\nabla u(x)|^3}{\log(1+|\nabla u(x)|)}\nabla u(x)\Big) + \frac{|u(x)|^3}{\log(1+|u(x)|)}u(x) = \lambda g(u(x)) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

has at least two positive weak solutions.

Next, as a consequence of Theorems 3.7 and 3.8 we obtain the following result on the existence of three solutions.

Theorem 3.13. *Suppose that $g(0) \neq 0$ and*

$$\limsup_{|\xi|\to+\infty} \frac{G(\xi)}{|\xi|^{p_0}} \leq 0. \tag{3.20}$$

Moreover, assume that there exist four positive constants γ, δ, $\bar\gamma$ and $\bar\delta$ with $\bar\gamma < 2c$ and

$$\frac{\bar\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)} < \Phi(\bar\delta) \leq \Phi(\delta) < \frac{\gamma^{p^0}}{(2c)^{p^0}\operatorname{meas}(\Omega)},$$

such that (3.15) *and* (3.17) *hold, and*

$$\frac{G(\gamma)}{\gamma^{p^0}} < \frac{G(\bar\delta) - G(\bar\gamma)}{(2c)^{p^0}\Phi(\bar\delta)\operatorname{meas}(\Omega) - \bar\gamma^{p^0}} \tag{3.21}$$

is satisfied. Then for each

$$\lambda \in \Lambda = \Big]\max\big\{\bar\lambda, \frac{\Phi(\delta)\operatorname{meas}(\Omega)}{\|\theta\|_{L^1(\Omega)}G(\delta)}\big\}, \frac{\gamma^{p^0}}{(2c)^{p^0}\|\theta\|_{L^1(\Omega)}G(\gamma)}\Big[,$$

problem (3.14) *admits at least three positive weak solutions u_1^*, u_2^* and u_3^* such that*

$$J(u_1^*) < \frac{\gamma^{p^0}}{(2c)^{p^0}}, \quad J(u_2^*) > \frac{\bar\gamma^{p^0}}{(2c)^{p^0}}.$$

Proof. First, in view of (3.15) and (3.21), we have $\Lambda \neq \emptyset$. Next, fix $\lambda \in \Lambda$. Employing Theorem 3.7 there is a positive weak solution u_1^* such that

$$J(u_1^*) < \frac{\gamma^{p^0}}{(2c)^{p^0}}$$

which is a local minimum for the associated functional Γ_λ, as well as Theorem 3.8 ensures a positive weak solution u_2^* such that

$$J(u_2^*) > \frac{\bar\gamma^{p^0}}{(2c)^{p^0}}$$

which is another local minimum for Γ_λ. Arguing as in the proof of Theorem 3.4 from the condition (3.20), we see that the functional Γ_λ is coercive, and then it satisfies the (PS) condition. Hence, the conclusion follows from the mountain pass theorem as given by Pucci and Serrin (see [39]). $\square$

Now we present the following existence result as a consequence of Theorem 3.13.

Theorem 3.14. *Assume that* $g(0) \neq 0$,

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\Phi(\xi)} = +\infty, \tag{3.22}$$

$$\limsup_{\xi \to +\infty} \frac{G(\xi)}{|\xi|^{p_0}} = 0. \tag{3.23}$$

Furthermore, suppose that there exist two positive constants $\bar{\gamma}$ *and* $\bar{\delta}$ *with* $\bar{\gamma} < 2c$ *and*

$$\Phi(\bar{\delta}) > \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0} \operatorname{meas}(\Omega)} \tag{3.24}$$

such that

$$\frac{G(\bar{\gamma})}{\bar{\gamma}^{p^0}} < \frac{G(\bar{\delta})}{(2c)^{p^0} \Phi(\bar{\delta}) \operatorname{meas}(\Omega)}. \tag{3.25}$$

Then for each

$$\lambda \in \Big] \frac{\Phi(\bar{\delta}) \operatorname{meas}(\Omega)}{\|\theta\|_{L^1(\Omega)} G(\bar{\delta})}, \frac{\bar{\gamma}^{p^0}}{(2c)^{p^0} \|\theta\|_{L^1(\Omega)} G(\bar{\gamma})} \Big[,$$

problem (3.14) *admits at least three positive weak solutions.*

Proof. We easily observe that from (3.23) the condition (3.20) is satisfied. Moreover, by choosing δ small enough and $\gamma = \bar{\gamma}$, one can derive the condition (3.15) from (3.22) as well as the conditions (3.17) and (3.21) from (3.25). Hence, the conclusion follows from Theorem 3.13. □

Remark 3.15. Theorem 1.2 immediately follows from Theorem 3.14 by setting $\alpha(|t|) = |t|^{p-2}$.

Finally, we present an application of Theorem 3.14 as follows.

Example 3.16. *Let* $N = 3$, $3 < p < 4$, *and*

$$\varphi(t) = \log(1 + |t|^2)|t|^{p-2}t, \quad t \in \mathbb{R}.$$

It is easy to see that $\varphi : \mathbb{R} \to \mathbb{R}$ *is an odd, increasing homeomorphism from* $\mathbb{R}$ *onto* $\mathbb{R}$, *and one has* $p_0 = p$ *and* $p^0 = p + 2$. *Thus relations* (2.1) *and* (2.2) *are satisfied (see* [16, Example 2] *for the details). Now let* $g : \mathbb{R} \to \mathbb{R}$ *be the function defined by*

$$g(t) = 1 + \frac{t^2}{1 + t^2}.$$

Thus g *is non-negative and continuous,* $g(0) \neq 0$ *and*

$$G(t) = 2t - \arctan t \text{ for every } t \in \mathbb{R}.$$

Therefore, one has

$$\limsup_{\xi \to 0^+} \frac{G(\xi)}{\Phi(\xi)} = \lim_{\xi \to 0^+} \frac{2\xi - \arctan \xi}{\xi^{p+2}} = +\infty,$$

$$\limsup_{\xi \to +\infty} \frac{G(\xi)}{|\xi|^{p_0}} = \lim_{\xi \to +\infty} \frac{2\xi - \arctan \xi}{|\xi|^p} = 0.$$

Letting $\Omega \subset \mathbb{R}^3$ *be such that*

$$\frac{1}{2^{p+2}\Phi(\pi + c)} < \operatorname{meas}(\Omega) < \frac{1}{2^{p+2}\Phi(\pi + c)} \frac{2(\pi + c) - \arctan(\pi + c)}{2c - \arctan c},$$

where $\Phi(\pi + c) = \int_0^{\pi+c} \log(1+|t|^2)|t|^{p-2}t dt$, *by choosing* $\bar{\gamma} = c$ *and* $\bar{\delta} = \pi + c$, *we observe that* (3.24) *and* (3.25) *are satisfied. Hence, by applying Theorem 3.14, for every*

$$\lambda \in \Big] \frac{\Phi(\pi + c)}{2(\pi + c) - \arctan(\pi + c)}, \frac{1}{2^{p+2}\,\mathrm{meas}(\Omega)(2c - \arctan c)} \Big[,$$

the problem

$$-\operatorname{div}\big(\log(1+|\nabla u|^2)|\nabla u|^{p-2}\nabla u\big) + \log(1+|u|^2)|u|^{p-2}u = \lambda\Big(1 + \frac{u^2}{1+u^2}\Big) \quad \text{in } \Omega,$$
$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

has at least three positive weak solutions.

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